Acoustic Wave Equation

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Basic linearized acoustic equations in lossless, isotropic, non flowing media

Linearized - Linear for small perturbation on a static state.
Lossless - Material parameters are independent of time.
Isotropic - Material response independent of direction.
Non flowing - No material derivative

Equation of motion

$$\rho \partial_t v_i + \partial_i p = f_i \tag{1}$$

(three equations for three components)

Acoustic stress-strain relationship

$$\rho \partial_t p + \partial_i v_i = q \tag{2}$$

(a pressure-rate strain-rate relation)

Fields

$p = p(\mathbf{x}, t)$	pressure
$\upsilon_i = \upsilon_i(\mathbf{x}, t)$	i – component of velocity

Sources

$q = q(\mathbf{x}, t)$	volume injection rate
$\upsilon_i = \upsilon_i(\mathbf{x}, t)$	i – component of externalforce

Medium Parameters

$\kappa = \kappa(\mathbf{x})$	compressibility
$\rho = \rho(\mathbf{x})$	density

Wave Equation

Solve equations (1) and (2) for pressure

$$\rho\partial_i\rho^{-1}\partial_i p - \rho\kappa\partial_t^2 p = \rho\partial_i\rho^{-1} f_i - \rho\partial_t q, \tag{3}$$

or

$$\partial_i^2 p - \rho\kappa\partial_t^2 p = \rho\partial_i\rho^{-1} f_i - \rho\partial_t q + \rho^{-1}\partial_i\rho\partial_i p. \tag{4}$$

Thus in a constant density and sourceless medium

$$\partial_i^2 p - c^{-2}\partial_t^2 p = 0, \tag{5}$$

with wave velocity $c = c(\mathbf{x}) = \sqrt{\kappa\rho}$, $\kappa = \kappa(\mathbf{x})$, $\rho = \rho_0$.

Finite Differences

Derivation of finite difference stencils for $\frac{\partial F(s)}{\partial s}$

Expand $F(s+\Delta s)$ in Taylor series

$$F(s+\Delta s) = F(s) + \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^i \tag{6}$$

Express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$\partial_s F(s) = \frac{1}{\Delta s}\left\{F(s+\Delta s) - F(s)\right\} - \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^{i-1} \tag{7}$$

this is a forward finite difference stencil.

Expand $F(s+\Delta s)$ and $F(s-\Delta s)$ in Taylor series

$$F(s+\Delta s) = F(s) + \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^i \tag{8}$$

$$F(s-\Delta s) = F(s) - \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{-\Delta s\right\}^i \tag{9}$$

Substract equations (9) from (8), express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$\partial_s F(s) = \frac{1}{2\Delta s}\left\{F(s+\Delta s) - F(s-\Delta s)\right\} - \sum_{i=1}^{\infty} \frac{1}{(1+2i)!}\partial_s^{1+2i} F(s)\left\{\Delta s\right\}^{2i} \tag{10}$$

this is a centered finite difference stencil.

or last, $\partial_s F(s)$ in a backward finite difference stencil from equation (9) as

$$\partial_s F(s) = \frac{1}{\Delta s}\left\{F(s) - F(s - \Delta s)\right\} - \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^{i-1} \quad (11)$$

Derivation of finite difference stencils for $\partial_s^2 F(s)$

Add equation (9) and (8), express $\partial_s^2 F(s)$ as a function of ...

$$\partial_s^2 F(s) = \frac{1}{(\Delta s)^2} \{F(s - \Delta s) - 2F(s) + F(s + \Delta s)\} \quad + \sum_{i=1}^{\infty} \frac{1}{(2+2i)!} \partial_s^{2+2i} F(s) \{\Delta s\}^{2i} \qquad (12)$$

This is a centered finite difference. Forward and backward finite difference stencils for $\partial_s^2 F(s)$ can be obtained from combinations of Taylor series for $F(s + \Delta s)$ and $F(s + 2\Delta s)$, or $F(s - \Delta s)$ and $F(s - 2\Delta s)$ respectively.

Finite Difference Solution of WE

Wave equation, FD 2^{nd}-order in space

$\Delta h = \Delta y = \Delta h$

$$\nabla^2 P(x,t) - \frac{1}{c^2(x)}\partial_t^2 P(x,t) = \frac{1}{(\Delta h)^2}\begin{array}{|c|c|c|}\hline & +1 & \\ \hline +1 & -2 & +1 \\ \hline & +1 & \\ \hline\end{array} P_x(t) - \frac{1}{c^2(x)}\partial_t^2 P_x(t) \tag{13}$$

Laplacian, FD 4^{th}-order in space

$$\nabla^2 P(x,t) - \frac{1}{c^2(x)}\partial_t^2 P(x,t) = \frac{1}{12(\Delta h)^2}\begin{array}{|c|c|c|c|c|}\hline & & -1 & & \\ \hline & & +16 & & \\ \hline -1 & +16 & -60 & +16 & -1 \\ \hline & & +16 & & \\ \hline & & -1 & & \\ \hline\end{array} P_x(t) - \frac{1}{c^2(x)}\partial_t^2 P_x(t) \tag{14}$$

Laplacian, FD 4^{th}-order in space, isotropic

$$\nabla^2 P(x,t) = \frac{1}{\alpha\ 12(\Delta h)^2} \begin{array}{|c|c|c|c|c|}\hline & & -1 & & \\ \hline & & +16 & & \\ \hline -1 & +16 & -60 & +16 & -1 \\ \hline & & +16 & & \\ \hline & & -1 & & \\ \hline \end{array}\ P_x(t) +$$

$$\frac{1}{\beta\ 12(\Delta h)^2} \begin{array}{|c|c|c|c|c|}\hline -1 & & & & -1 \\ \hline & +16 & & +16 & \\ \hline & & -60 & & \\ \hline & +16 & & +16 & \\ \hline -1 & & & & -1 \\ \hline \end{array}\ P_x(t) \qquad (15)$$

$\beta = \frac{1-\alpha}{2}$, for example: $\alpha = 1 \rightarrow \beta = 0$, $\alpha = 1/2 \rightarrow \beta = 1/4$ or $\alpha = 2/3 \rightarrow \beta = 1/6$.

Wave equation, FD 2^{nd}-order time stepping

$$\partial_t^2 P(x,t) - c^2(x)\nabla^2 P(x,t) = \frac{1}{\Delta t^2} \begin{array}{|c|c|c|} \hline 1 & -2 & 1 \\ \hline \end{array} \, P_t(x) - c^2(x)\nabla^2 P_t(x) \tag{16}$$

Solve for $P_{t+1}(x)$

$$P_{t+1}(x) = 2P_t(x) - P_{t-1}(x) + \Delta t^2 c^2(x)\nabla^2 P_t(x) \tag{17}$$

Wave equation, FD 4^{th}-order time stepping

Include the 4^{th}-order derivative from equation (12), by substituting the wave equation (Dablain, 1986), as

$$\partial_t^4 P(x,t) = \partial_t^2 \partial_t^2 P(x,t) = \partial_t^2 c^2(x)\nabla^2 P(x,t) \tag{18}$$