Acoustic Wave Equation

Sjoerd de Ridder (most of the slides) & Biondo Biondi

January 16th 2011
Table of Topics

- Basic Acoustic Equations
- Wave Equation
- Finite Differences
- Finite Difference Solution
- Pseudospectral Solution
- Stability and Accuracy
- Green’s function
- Perturbation Representation
- Born Approximation
Basic linearized acoustic equations in lossless, isotropic, non-flowing media

Linearized - Linear for small perturbation on a static state.
Lossless - Material parameters are independent of time.
Isotropic - Material response independent of direction.
Non flowing - No material derivative

Equation of motion

$$\rho \partial_t v_i + \partial_i p = f_i$$  \hspace{1cm} (1)

(three equations for three components)

Acoustic stress-strain relationship

$$\rho \partial_t p + \partial_i v_i = q$$  \hspace{1cm} (2)

(a pressure-rate strain-rate relation)
Fields

\[ p = p(x, t) \] pressure
\[ v_i = v_i(x, t) \] i – component of velocity

Sources

\[ q = q(x, t) \] volume injection rate
\[ f_i = f_i \] i – component of external force

Medium Parameters

\[ \kappa = \kappa(x) \] compressibility
\[ \rho = \rho(x) \] density
Wave Equation

Solve equations (1) and (2) for pressure

\[ \rho \partial_i \rho^{-1} \partial_i p - \rho \kappa \partial_t^2 p = \rho \partial_i \rho^{-1} f_i - \rho \partial_t q, \]  

(3)

or

\[ \partial_i^2 p - \rho \kappa \partial_t^2 p = \rho \partial_i \rho^{-1} f_i - \rho \partial_t q + \rho^{-1} \partial_i \rho \partial_i p. \]  

(4)

Thus in a constant density and sourceless medium

\[ \partial_i^2 p - c^{-2} \partial_t^2 p = 0, \]  

(5)

with wave velocity \( c = c(x) = \sqrt{\kappa \rho}, \kappa = \kappa(x), \rho = \rho_0. \)
Finite Differences

Derivation of finite difference stencils for $\frac{\partial F(s)}{\partial s}$

Expand $F(s + \Delta s)$ in Taylor series

$$F(s + \Delta s) = F(s) + \frac{1}{1!} \partial_s F(s) \Delta s + \sum_{i=2}^\infty \frac{1}{i!} \partial_s^i F(s) \{\Delta s\}^i$$  \hspace{1cm} (6)

Express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$\partial_s F(s) = \frac{1}{\Delta s} \{F(s + \Delta s) - F(s)\} - \sum_{i=2}^\infty \frac{1}{i!} \partial_s^i F(s) \{\Delta s\}^{i-1}$$  \hspace{1cm} (7)

this is a forward finite difference stencil.
Expand $F(s + \Delta s)$ and $F(s - \Delta s)$ in Taylor series

$$
F(s + \Delta s) = F(s) + \frac{1}{1!} \partial_s F(s) \Delta s + \sum_{i=2}^{\infty} \frac{1}{i!} \partial_s^i F(s) \{\Delta s\}^i
$$

(8)

$$
F(s - \Delta s) = F(s) - \frac{1}{1!} \partial_s F(s) \Delta s + \sum_{i=2}^{\infty} \frac{1}{i!} \partial_s^i F(s) \{-\Delta s\}^i
$$

(9)

Subtract equations (9) from (8), express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$
\partial_s F(s) = \frac{1}{2\Delta s} \{F(s + \Delta s) - F(s - \Delta s)\} - \sum_{i=1}^{\infty} \frac{1}{(1 + 2i)!} \partial_s^{1+2i} F(s) \{\Delta s\}^{2i}
$$

(10)

this is a centered finite difference stencil.
or last, $\partial_s F(s)$ in a backward finite difference stencil from equation (9) as

$$
\partial_s F(s) = \frac{1}{\Delta s} \{F(s) - F(s - \Delta s)\} - \sum_{i=2}^{\infty} \frac{1}{i!} \partial^i_s F(s) \{\Delta s\}^{i-1} \quad (11)
$$
Derivation of finite difference stencils for $\partial_s^2 F(s)$

Add equation (9) and (8), express $\partial_s^2 F(s)$ as a function of ...

$$\partial_s^2 F(s) = \frac{1}{(\Delta s)^2} \left\{ F(s - \Delta s) - 2F(s) + F(s + \Delta s) \right\} +$$

$$\sum_{i=1}^{\infty} \frac{1}{(2 + 2i)!} \partial_s^{2+2i} F(s) \{\Delta s\}^{2i} \quad (12)$$

This is a centered finite difference. Forward and backward finite difference stencils for $\partial_s^2 F(s)$ can be obtained from combinations of Taylor series for $F(s + \Delta s)$ and $F(s + 2\Delta s)$, or $F(s - \Delta s)$ and $F(s - 2\Delta s)$ respectively.
Finite Difference Solution of WE

Wave equation, FD $2^{nd}$-order in space

$\Delta h = \Delta y = \Delta h$

\[
\nabla^2 P(x, t) - \frac{1}{c^2(x)} \frac{\partial^2 P(x, t)}{\partial t^2} = \frac{1}{\Delta h^2} \begin{array}{ccc}
+1 & 0 & +1 \\
+1 & 0 & +1 \\
+1 & 0 & +1 \\
\end{array}

P_x(t) - \frac{1}{c^2(x)} \frac{\partial^2 P_x(t)}{\partial t^2}

(13)

Laplacian, FD $4^{th}$-order in space

\[
\nabla^2 P(x, t) - \frac{1}{c^2(x)} \frac{\partial^2 P(x, t)}{\partial t^2} = \frac{1}{12(\Delta h)^2} \begin{array}{cccc}
-1 & 0 & +16 & 0 \\
-1 & +16 & -30 & +16 \\
0 & +16 & -1 \\
\end{array}

P_x(t) - \frac{1}{c^2(x)} \frac{\partial^2 P_x(t)}{\partial t^2}

(14)
Laplacian, FD $4^{th}$-order in space, isotropic

\[
\nabla^2 P(x, t) = \frac{1}{\alpha \ 12(\Delta h)^2} \begin{bmatrix}
-1 & +16 \\
-1 & +16 & -30 & +16 & -1 \\
+16 \\
-1 & +16 \\
-1 & +16 & -30 & +16 & -1 \\
\end{bmatrix} P_x(t) + \\
\]

\[
\frac{1}{\beta \ 12(\Delta h)^2} \begin{bmatrix}
-1 \\
+16 & +16 \\
-30 \\
+16 & +16 \\
-1 & +16 & -30 & +16 & -1 \\
\end{bmatrix} P_x(t) \\
\]

(15)

\[
\beta = \frac{1-\alpha}{2}, \text{ for example: } \alpha = 1 \rightarrow \beta = 0, \ \alpha = 1/2 \rightarrow \beta = 1/4 \text{ or } \alpha = 2/3 \rightarrow \beta = 1/6.
\]
Wave equation, FD $2^{nd}$-order time stepping

\[ \frac{\partial^2 P(x, t)}{\partial t^2} - c^2(x) \nabla^2 P(x, t) = \frac{1}{\Delta t^2} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} P_t(x) - c^2(x) \nabla^2 P_t(x) \tag{16} \]

Solve for $P_{t+1}(x)$

\[ P_{t+1}(x) = 2P_t(x) - P_{t-1}(x) + \Delta t^2 c^2(x) \nabla^2 P_t(x) \tag{17} \]

Wave equation, FD $4^{th}$-order time stepping

Include the $4^{th}$-order derivative from equation (12), by substituting the wave equation (Dablain, 1986), as

\[ \partial_t^4 P(x, t) = \partial_t^2 \partial_t^2 P(x, t) = \partial_t^2 c^2(x) \nabla^2 P(x, t) \tag{18} \]
Pseudospectral (Fourier) methods

- **Laplacian computed using FFTs:**
  \[ c^2 (x) \nabla^2 P_t (x) \approx c^2 (x) \text{FFT}^{-1} \left\{ -|\vec{k}|^2 \text{FFT} [P_t (x)] \right\} \]

- **Wave equation, FD 2nd-order time stepping and pseudospectral Laplacian:**
  \[
P_{t+1} (x) =
  2P_t (x) - P_{t-1} (x) + \Delta t^2 c^2 (x) \text{FFT}^{-1} \left\{ -|\vec{k}|^2 \text{FFT} [P_t (x)] \right\}
  \]
Stability and accuracy of explicit methods

- **Courant number**: \( \text{Cour} = \frac{c_{\text{max}} \Delta t}{\min(\Delta x, \Delta y, \Delta z)} \) where \( c_{\text{max}} \) is the maximum velocity.

- **Courant-Friedrichs-Lewy (CFL) condition**: \( \text{Cour} \leq 1 \) it is a necessary, but not sufficient condition for a stable explicit extrapolator.

- **Numerical dispersion** causes \( c_P \neq c \), where \( c_P = \frac{\omega}{|k|} \) is the effective phase velocity of numerically propagated waves.
Stability and accuracy analysis of pseudospectral methods

- **Substitute a generic plane wave solution:**
  \[ \exp \left[ i \left( \vec{k} x + \omega t \right) \right] \]

- **Dispersion relation:**
  \[ \omega = \frac{2 \sin^{-1} \left( \pm \frac{c \Delta t |\vec{k}|}{2} \right)}{\Delta t} \]

- **Phase velocity:**
  \[ c_P = \frac{\omega}{|\vec{k}|} = \frac{2 \sin^{-1} \left( \pm \frac{c \Delta t |\vec{k}|}{2} \right)}{\Delta t |\vec{k}|} \]

- **For stability it must be** \( \frac{c \Delta t |\vec{k}|}{2} \leq 1: \)
  - **1D:** Maximum \( k \) equal to Nyquist wavenumber \( k_{Nyq} = \pi / \Delta x \) stability requires \( \text{Cour} \leq 2 / \pi \approx 0.636 \)
  - **2D:** \( k_{\text{max}} = \sqrt{2} k_{Nyq} \) stability requires \( \text{Cour} \leq \sqrt{2} / \pi \approx 0.45 \)
  - **3D:** \( k_{\text{max}} = \sqrt{3} k_{Nyq} \) stability requires \( \text{Cour} \leq 2 / \sqrt{3} \pi \approx 0.367 \)
Stability and accuracy of 2nd-order in time and space

- Substitute a generic plane wave solution:
  \[ \exp \left[ i \left( \vec{k}x + \omega t \right) \right] \]

- Dispersion relation: \[ \omega = \frac{2 \sin^{-1} \left[ \frac{c\Delta t}{\Delta x} \sqrt{\sin^2 \left( \frac{k_x\Delta x}{2} \right) + \sin^2 \left( \frac{k_z\Delta z}{2} \right)} \right]}{\Delta t} \]

- Phase velocity (worst case at \( k_x = 0 \) or \( k_z = 0 \)):
  \[ c_P = \frac{\omega}{k_x} = \frac{2 \sin^{-1} \left[ \frac{c\Delta t}{\Delta x} \sin \left( \frac{k_x\Delta x}{2} \right) \right]}{\Delta t k_x} \]

- For stability the argument of \( \sin^{-1} \) must be between -1 and 1:
  - 1D: \( \text{Cour} \leq 1 \)
  - 2D: Worst case at \( k_x = k_z = k_{\text{Nyq}} \): \( \text{Cour} \leq \sqrt{2}/2 \approx 0.707 \)
Observations

- **Stability**
  - Stability constraint becomes more stringent with higher dimensions
  - FD "more stable" than pseudospectral because errors in the spatial derivatives slows down high frequencies.

- **Dispersion**
  - Pseudospectral
    - High frequencies (wavenumbers) arrive before low frequencies (wavenumbers).
    - Dispersion gets worse as the Courant number increases.
  - FD
    - High frequencies (wavenumbers) "tend" to arrive after low frequencies (wavenumbers).
    - Dispersion gets better as the Courant number increases.
Frequency dispersion with finite-differences schemes
Frequency dispersion with finite-differences schemes
Frequency dispersion with finite-differences schemes

Time [s]

Amplitudes

T: 2nd order - X: 10th order
Frequency dispersion with finite-differences schemes

T: 4\textsuperscript{nd} order - X: 10\textsuperscript{th} order
Frequency dispersion with pseudospectral Laplacian
Wavelength dispersion with finite-differences schemes
Wavelength dispersion with finite-differences schemes
Wavelength dispersion with finite-differences schemes
Wavelength dispersion with finite-differences schemes

T: 4\textsuperscript{nd} order - X: 10\textsuperscript{th} order
Wavelength dispersion with pseudospectral Laplacian
Green’s function

Introduce Green’s function for a constant density and sourceless medium equation (5) by a point source term acting at \( t = 0 \) and \( x = x_s \)

\[
\partial_i^2 G - c^{-2} \partial_t^2 G = -\delta(x - x_s)\delta(t),
\]

where \( G = G(x, x_s, t) \) is the Green’s function.

The solution for pressure to another forcing function for example \( s = s x, t \) can be represented as

\[
p(x, t) = - \int \int G(x, x', t - t')s(x', t')dx'dt'
\]

(20)
Perturbation Representation

Represent the medium velocity as a background velocity and a perturbation

\[ c^{-2}(x) = c^{-2}_b(x) [1 + \alpha(x)] \] (21)

Substitution into equation (19) gives

\[
\partial_i^2 G(x, x_s, t) - c^{-2}_b(x) \partial_t^2 G(x, x_s, t) =
\]

\[
-\delta(x - x_s) \delta(t) + \alpha(x) c^{-2}_b(x) \partial_t^2 G(x, x_s, t),
\]
Introducing $G_b(x, x_s, t)$ as a solution to

$$\partial_i^2 G_b(x, x_s, t) - c_b^{-2}(x) \partial_t^2 G_b(x, x_s, t) = -\delta(x - x_s)\delta(t), \quad (23)$$

we see that if we represent the full solution as a sum of the background solution plus a perturbed solution as

$$G(x, x_s, t) = G_b(x, x_s, t) + G_p(x, x_s, t). \quad (24)$$
Equation (22) can be thus written as

$$\partial_i^2 G_p(x, x_s, t) - c_b^{-2}(x) \partial_t^2 G_p(x, x_s, t) = \alpha(x) c_b^{-2}(x) \partial_t^2 G(x, x_s, t).$$

(25)

Note the forcing function dependent on medium parameter $\alpha$. Thus using a representation as (20) for $G_p(x, x_s, t)$ we find for $G(x, x_s, t)$

$$G(x, x_s, t) = G_b(x, x_s, t) - \int \int G_b(x, x', t - t') \alpha(x') c_b^{-2}(x') \partial_t^2 G(x', x_s, t') dx'dt'.$$

(26)
Born Approximation

The Born approximation is made in the perturbation representation by substituting the total field under the integral for the background field.

\[
G(x, x_s, t) = G_b(x, x_s, t) - \int \oint G_b(x, x', t - t') \alpha(x') c_b^{-2}(x') \partial_t^2 G_b(x', x_s, t') dx' dt'
\]  

(27)

This is an explicit representation for \( G(x, x_s, t) \).

The perturbation can represent a (single additional) scattered wavefield as

\[
G_s(x, x_s, t) = d(x, x_s, t) = - \int \oint G_b(x, x', t - t') \alpha(x') c_b^{-2}(x') \partial_t^2 G_b(x', x_s, t') dx' dt'.
\]

(28)