# Lab 4: Seismic illumination and imaging through inversion

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## ABSTRACT

In this lab, you will perform seismic illumination analysis on a synthetic data set, and get hands-on experience with the scattering wavenumber vectors which are closely related to the subsurface illumination. You will also investigate an efficient method based on phase encoding for Hessian computation. Finally, you need to code the forward operator which convolves the local Hessian operator with the reflectivity model and also its adjoint. Your will use your code to invert for the reflectivity for the Sigsbee2A model.

## INTRODUCTION

In this lab, we formulate the imaging problem as an inverse problem and apply a target-oriented strategy to invert for the reflectivities. To do so, we have to explicitly compute the Hessian operator, also known as the resolution function. It is closely related to the subsurface illumination and contains all possible information for imaging, including wave types, background velocity, acquisition geometry and the bandwidth of the seismic waves. We will study the operator carefully in both the space domain and the local Fourier domain.

We will first review the theory of imaging through inversion, then discuss the subsurface illumination in both space domain and Fourier domain, and investigate an efficient way for generating the Hessian operator. Finally, I provide a skeleton of an inversion code, which you need to modify to run inversion on the Sigsbee2A model. I strongly encourage you read Plessix and Mulder (2004), Valenciano et al. (2006) and Tang (2008) carefully, you will find them very useful.

# LEAST-SQUARES HESSIAN

Seismic imaging can be formulated as an inverse problem by minimizing the least-squares functional $F$ defined as follows:

$$F(\mathbf{m}) = ||\mathbf{Lm} - \mathbf{d}_{\text{obs}}||_2, \tag{1}$$

where $\mathbf{d}_{\text{obs}}$ is the recorded seismic data, $\mathbf{m}$ is the model parameters (reflectivity of the subsurface) that we are trying to recover and $\mathbf{L}$ is the forward Born modeling operator, which defines the mapping from the model space to the data space. $\mathbf{L}$ is derived under the single scattering assumption, thus it only models primaries. $\mathbf{L}$ is defined as follows:

$$d(\mathbf{r}_k, \mathbf{s}_i, \omega) = \omega^2 \sum_{l=1}^{L} f_s(\omega) G_i(\mathbf{s}_i, \mathbf{x}_l, \omega) G_i(\mathbf{x}_l, \mathbf{r}_k, \omega) m(\mathbf{x}_l), \tag{2}$$

where $d(\mathbf{r}_k, \mathbf{s}_i, \omega)$ is the modeled seismic data for receiver location $\mathbf{r}_k$ ($k = 1, \cdots, M$) and shot location $\mathbf{s}_i$ ($i = 1, \cdots, N$) for a single frequency $\omega$, $f_s(\omega)$ is the source signature. $m(\mathbf{x}_l)$ ($l = 1, \cdots, L$) is the reflectivity at $\mathbf{x}_l$ in the subsurface, $G_i(\mathbf{s}_i, \mathbf{x}_l, \omega)$ is the monochromatic Green's function from the source $\mathbf{s}_i$ to the image point $\mathbf{x}_l$ for the $i$th shot, and $G_i(\mathbf{x}_l, \mathbf{r}_k, \omega)$ is the monochromatic Green's function from the image point $\mathbf{x}_l$ to the receiver $\mathbf{r}_k$ for the $i$th shot.

The data-space objective function $F$ can be minimized directly by using the gradient-based method Nemeth et al. (1999); Clapp (2005). However, we can also reformulate it and solve it in the model space. $\mathbf{L}$ is a linear operator, so $F$ is a quadratic function. Its minimum can be obtained when $\mathbf{m}$ satisfies:

$$\mathbf{m} = \mathbf{H}^{-1}\mathbf{L}'\mathbf{d}_{\text{obs}}, \tag{3}$$

where $\mathbf{H} = \mathbf{L}'\mathbf{L}$ is the Hessian operator. The Hessian provides the curvature information of the objective function $F$. We can rewrite equation 3 as follows by recognizing that $\mathbf{L}'\mathbf{d}_{\text{obs}}$ is the migrated image $\mathbf{m}_{\text{mig}}$:

$$\mathbf{Hm} = \mathbf{m}_{\text{mig}}. \tag{4}$$

Now we can formulate our new objective function as follows:

$$J(\mathbf{m}) = ||\mathbf{Hm} - \mathbf{m}_{\text{mig}}||_2. \tag{5}$$

Equation 5 says that once we explicitly compute the Hessian operator $\mathbf{H}$, we can use the linear conjugate-gradient or any other method to minimize the new objective function $J$ defined in equation 5.

The Hessian can be obtained by taking the second derivative of the objective function $F$ with respect to the model parameters, the formula is given as follows (see Appendix A in Tang (2008) for derivation):

$$H(\mathbf{x}_p, \mathbf{x}_q) = \sum_{\omega} \omega^4 \sum_{i=1}^{N} |f_s(\omega)|^2 G_i(\mathbf{s}_i, \mathbf{x}_p, \omega) G_i'(\mathbf{s}_i, \mathbf{x}_q, \omega) \sum_{k=1}^{M} G_i(\mathbf{r}_k, \mathbf{x}_p, \omega) G_i'(\mathbf{r}_k, \mathbf{x}_q, \omega) \tag{6}$$

## Questions

1. Why is migration not good enough for amplitude purposes?

   **Your answer:**

2. What are the advantages and disadvantages for the data-space inversion approach?

   **Your answer:**

3. What are the advantages and disadvantages for the model-space inversion approach?

   **Your answer:**

4. In your opinion, which one is more suitable for large-scale problems?

   **Your answer:**

# SCATTERING WAVENUMBER VECTORS AND SUBSURFACE ILLUMINATION

The scattering wavenumber $\mathbf{k}_{\text{scatter}}$ is defined at a chosen image point and in a background velocity model (Figure 1). $\mathbf{k}_{\text{scatter}}$ is the difference of two local vectors $\mathbf{k}_r$ and $\mathbf{k}_s$, i.e., $\mathbf{k}_{\text{scatter}} = \mathbf{k}_r - \mathbf{k}_s$. $\mathbf{k}_s$ and $\mathbf{k}_r$ are the local wavenumbers for the source wavefield and the receiver wavefield. They define the local phase directions (propagation directions) for the source wavefield and receiver wavefield, respectively. It can be easily shown that the magnitude of the scattering wavenumber is related to the scattering angle $\theta_{\text{scatter}}$ which bisects the propagation directions of source and receiver wavefields at the image point (Figure 1):

$$|\mathbf{k}_{\text{scatter}}| = \frac{2\omega}{v} \sin \theta_{\text{scatter}}. \qquad (7)$$

From the above equation, we find that for a fixed velocity $v$, the length of the vector is dependent on the frequency $\omega$ and the scattering angle $\theta_{\text{scatter}}$.
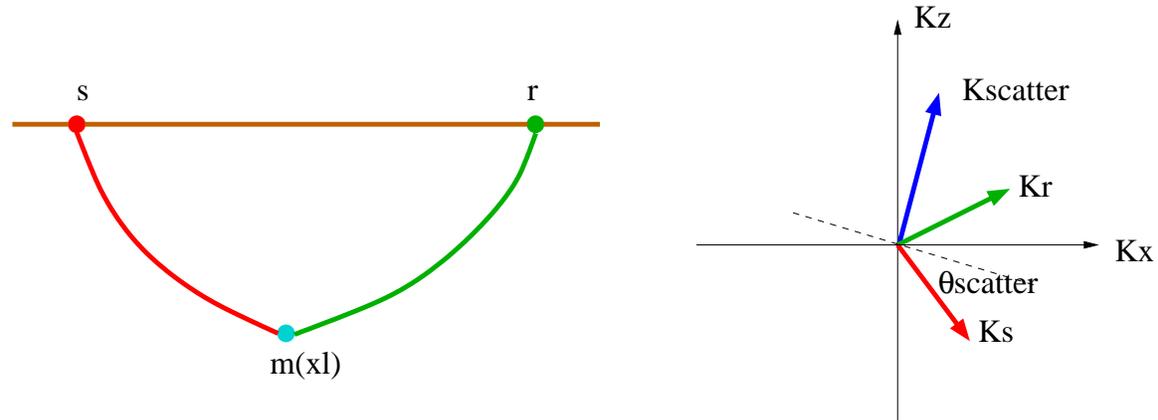
Figure 1: Scattering wavenumber vectors. Raypaths are drawn in the left panel to illustrate the propagation effect in the background model. The right panel shows the local definition of the scattering wavenumber $\mathbf{k}_{\text{scatter}}$ as the difference of the receiver wavenumber $\mathbf{k}_r$ and the source wavenumber $\mathbf{k}_s$.

## Questions

1. Derive equation 7.

   **Your answer:**

2. Describe how the length of the scattering wavenumber $\mathbf{k}_{\text{scatter}}$ changes with the frequency $\omega$ and the scattering angle $\theta_{\text{scatter}}$.

   **Your answer:**

3. Compare the scattering wavenumber for small and large angles of incidence $\theta_i$. Which one has higher resolution? Explain why.

   **Your answer:**

4. Plot the Fourier support area at a single frequency for only one shot but with full aperture recording; i.e., we can record the scattered wavefields all around the scattering object. (**Hint**: use Matlab)

   **Your answer:**

5. Plot the Fourier support area at a single frequency for surface seismic acquisition geometry; i.e., we can only record the scattered wavefields on the surface. For simplicity, we assume infinite source and receiver coverage on the surface.

   **Your answer:**

The scattering wavenumber $\mathbf{k}_{\text{scatter}}$ is an important concept in seismic illumination or resolution analysis, and the overall distribution of $\mathbf{k}_{\text{scatter}}$ at an image point gives information of how many Fourier components of that image point are sampled for a given seismic survey. Therefore, it is a direct measure of the subsurface illumination. The illumination response in the local Fourier domain can be obtained using ray based method, as discussed in Lecomte (2008). It can also be obtained by performing the local Fourier transform on the Hessian operator discussed in the previous section.

Figure 2(a) shows the diagonal of the Hessian, i.e., when $\mathbf{x}_q = \mathbf{x}_p$, for a constant velocity model (2000m/s). The acquisition geometry contains only one shot located at $-600$m and two receivers located at 600m and 1200m on the surface. The frequency band is from 5 Hz to 35 Hz. Figure 3(a) shows the Hessian off-diagonals for the image point $\mathbf{x}_p = (x = 680, z = 800)$. You can run the commands `make const-hess-exact.view` and `make const-hess-exact-offd.view` to look at these figures carefully.

For comparison, Figure 2(b) shows the diagonal of Hessian for a slightly different acquisition geometry. There is one shot located at $-600$ m, but 401 receivers ranging from $-2000$ m to 2000 m with a spacing equals to 10 m. Figure 3(b) shows the the Hessian off-diagonals for the image point $\mathbf{x}_p = (x = 680, z = 800)$. You can run the commands `make const-hess-more-exact.view` and `make const-hess-more-exact-offd.view` to look at these figures carefully.

## Questions

1. In Figure 3(a), do you see two events intersecting at the origin? Explain why. In Figure 3(b), why is the energy more focused at the origin? What do you expect for the Hessian off-diagonals at the same image point $\mathbf{x}_p = (x = 680, z = 800)$ when there are more shots and receivers on the surface?

   **Your answer:**

2. Use `SEPlib` command `Ft3d` to transform the local Hessian operator shown in Figure 3(a) to the local Fourier domain. Plot the amplitude spectrum of your result. Explain what you see and relate it to the acquisition geometry.
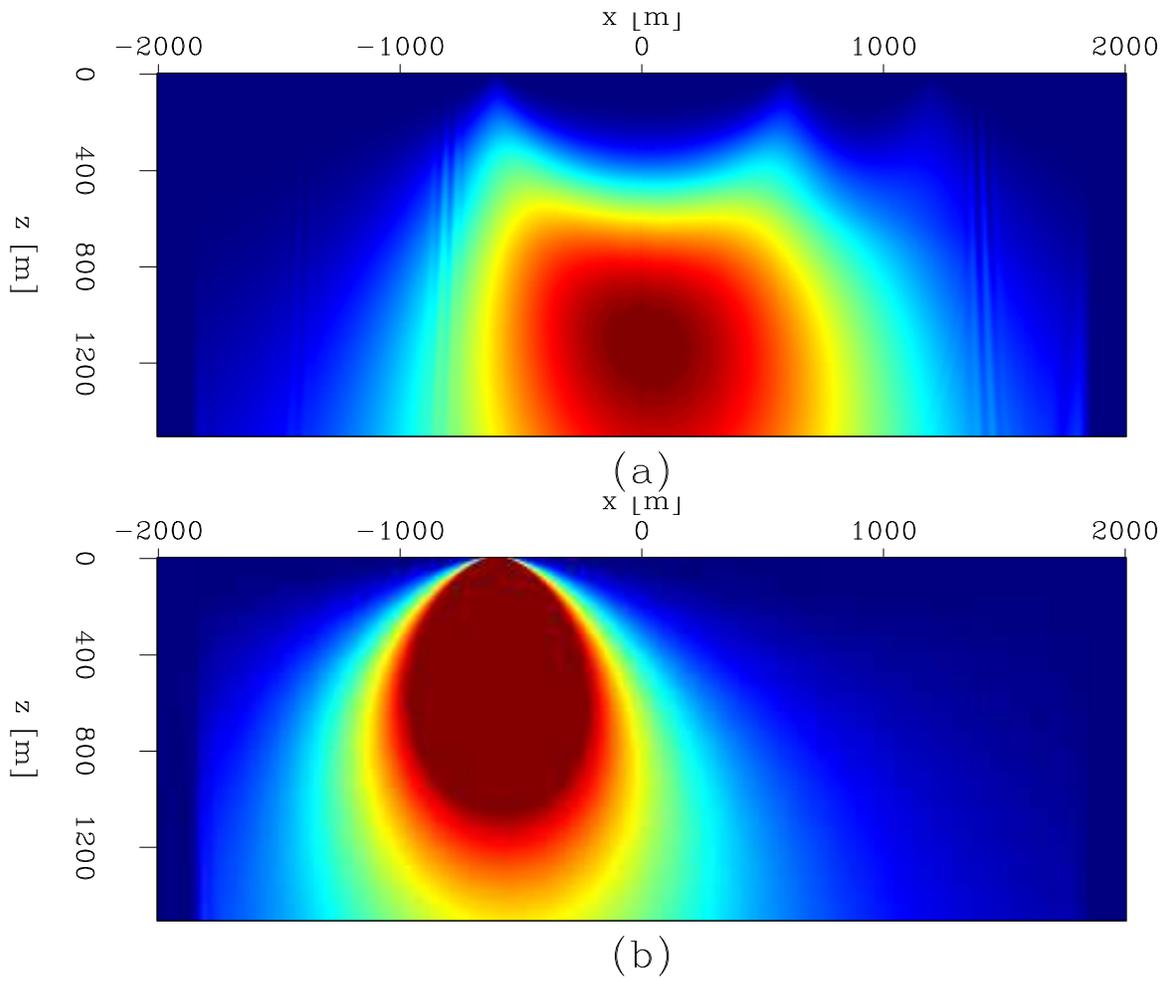
Figure 2: The diagonal of the Hessian. Left panel is obtained with one shot and two receivers. Right panel is obtained with one shot and 401 receivers.
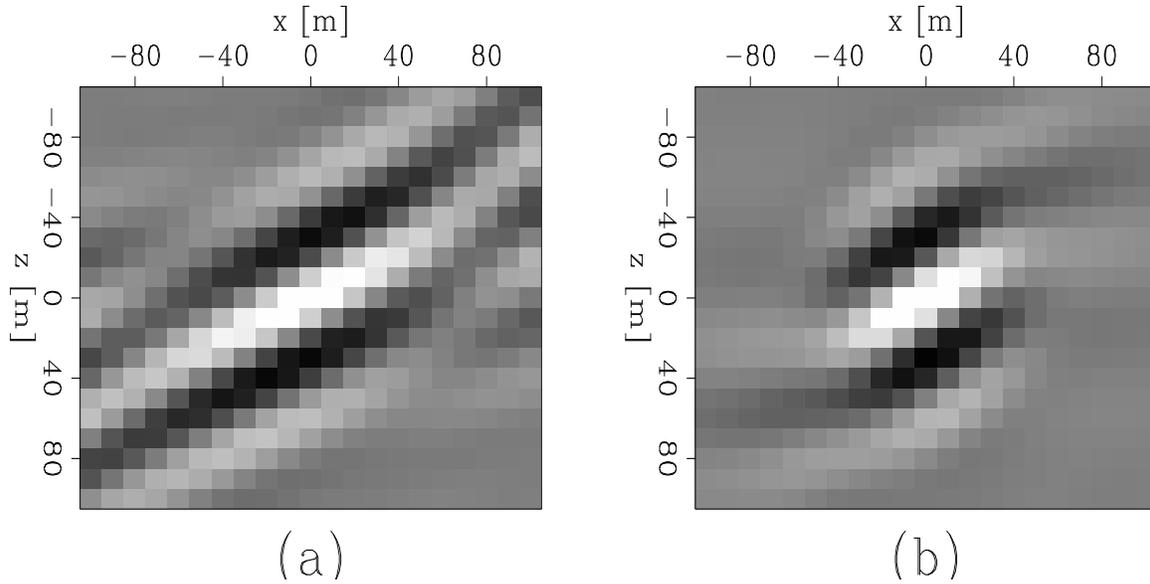
Figure 3: The off-diagonals of Hessian for image point at $(x = 680, z = 800)$. Left panel is obtained with one shot and two receivers. Right panel is obtained with for one shot and 401 receivers.

**Your answer:**

3. Transform the Hessian shown in Figure 3(b), which is obtained for a different acquisition geometry, to the local Fourier domain. Plot your result and compare it with your previous result, and explain what you see.

**Your answer:**

# APPROXIMATIONS TO THE HESSIAN

The Hessian can be obtained by directly implementing equation 6, however, direct implementation of equation 6 is not very practical, because it requires saving the Green's functions on the disk. The size of Green's functions can be huge (easily several or several hundreds terabytes for a typical 3-D survey). Therefore, it is definitely very cumbersome to deal with such a big file. Tang (2008) discusses an alternative way to compute the Hessian operator based on the phase encoding technique, eliminating the need for Green's functions to be saved on the disk. The algorithm is closely related to the conventional shot-profile migration except that the imaging condition is slightly modified. The new algorithm is also efficient since multiple Green's functions are computed at the same time during the wavefield extrapolation. Read Tang (2008) carefully and answer the following questions.

## Questions

1. What are the advantages and disadvantages of phase-encoded Hessian?

    **Your answer:**

2. Write the pseudo code for the direct implementation of equation 6?

    **Your answer:**

3. Write the pseudo code for the random phase encoding algorithm (only encode the receiver-side Green's functions)

    **Your answer:**

# LEAST-SQUARES INVERSION

As discussed in the second section of this lab, we can do target-oriented wave-equation inversion by minimizing the model-space objective function defined in equation 5. In this section, you will experiment with this idea on the well known Sigsbee2A data set. You are provided the migrated image for a particular target area and the corresponding Hessian operator obtained by random phase encoding. Your task is to code the forward and adjoint operator to invert for the reflectivity using the conjugate-gradient method. Figure 4, 5 and 6 show the velocity model, the actual reflectivity model that we are trying to recover and the migrated image, respectively. Issue commands `make wem-sigsb2a-vmod-win-lab.view`, `make sigsb2a-refl-win-lab.view` and `make wem-sigsb2a-imag-fsec-all-lab.view` to look at them carefully. Obviously the migrated image is far from the correct reflectivity model. Now your task is to get a better recovery of the reflectivity model by using the target-oriented inversion approach. Read the codes `hesconv.f90`, `hes_convol_mod.f90` and `hes_reglar_mod.f90` thoroughly. It is critical that you understand those codes very well to finish these questions.

## Questions

1. Run command `make wem-sigsb2a-hess-random-diag-win-lab.view` to bring up the diagonal of Hessian.
    You can also run the command `make wem.sigsb2a.hess.random.z13000.H` which will bring up a movie showing the local Hessian operator for different
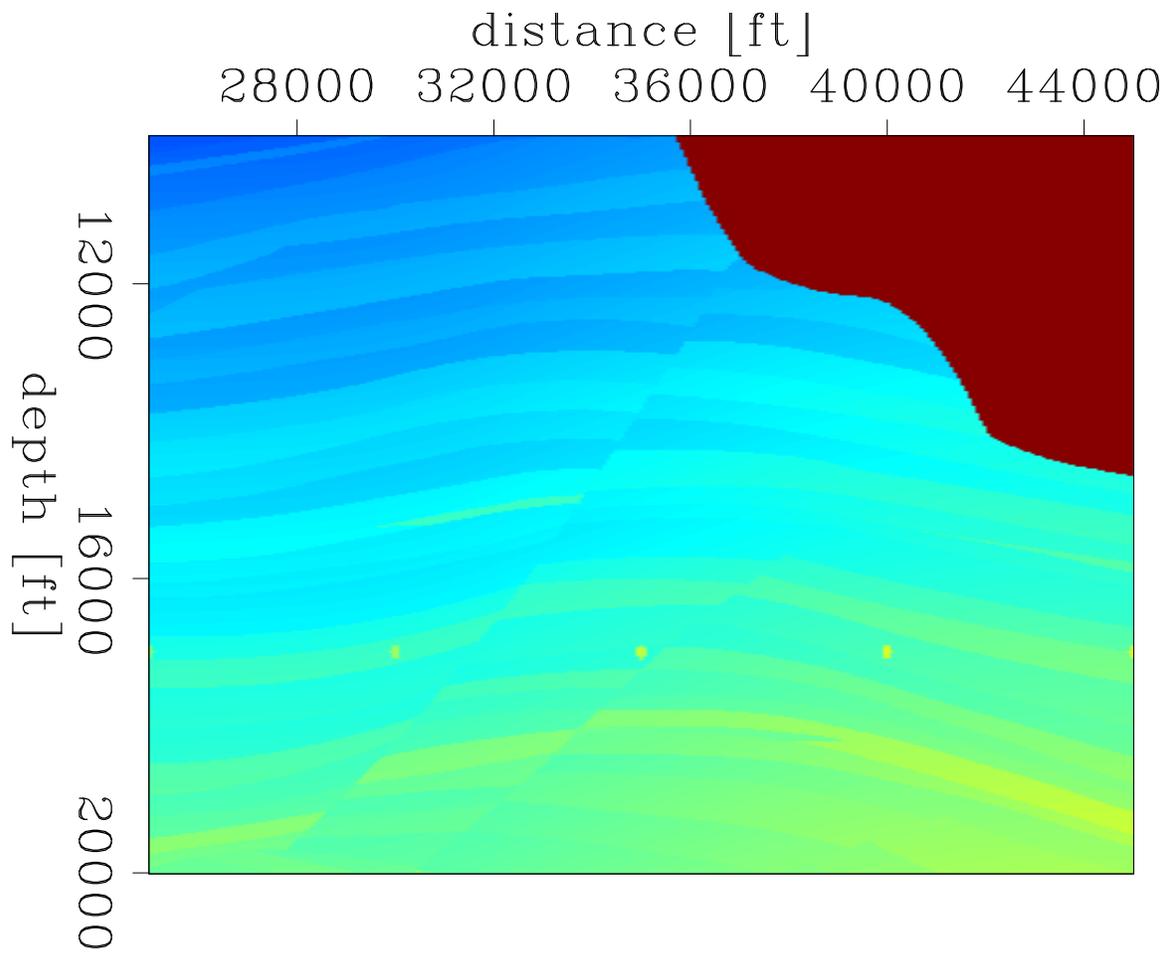
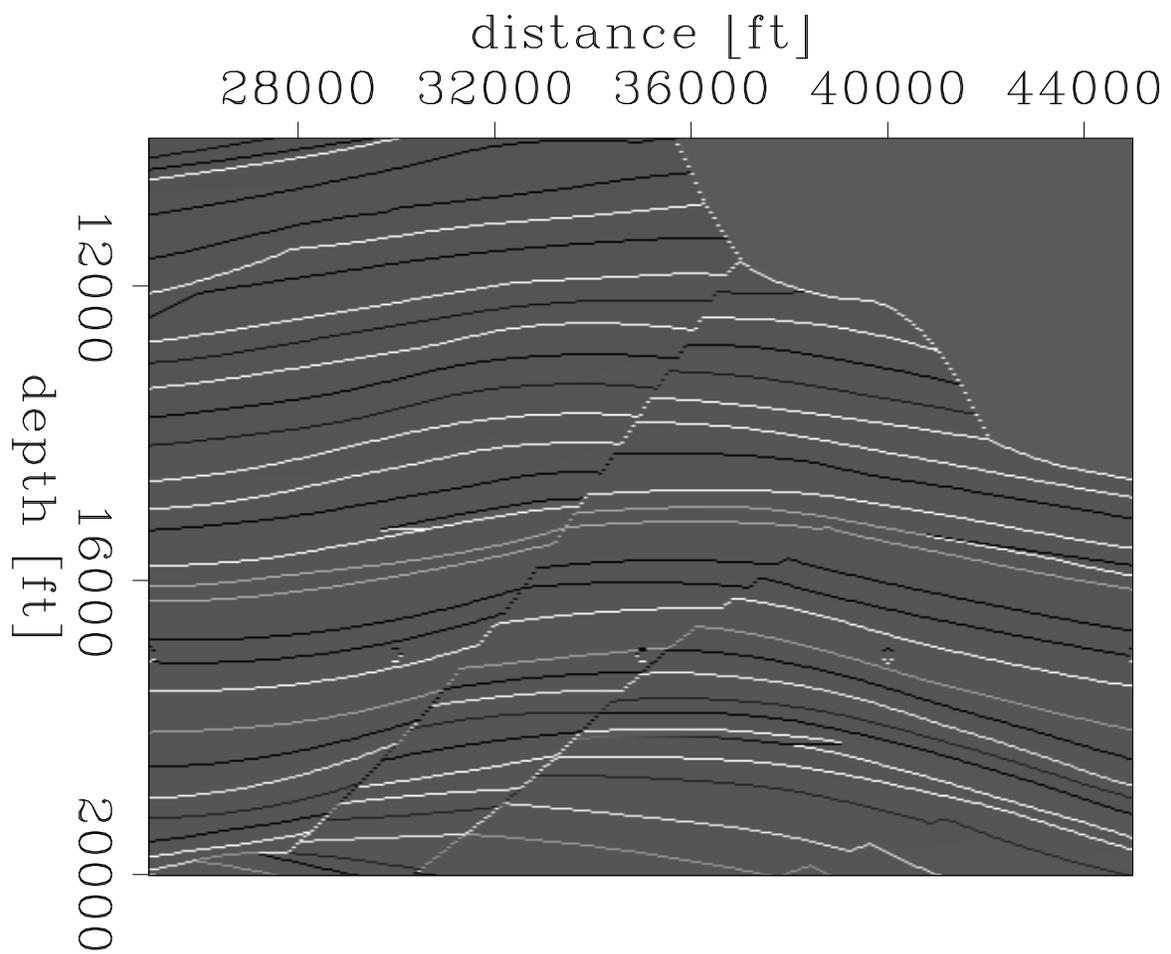Figure 4: Part of the Sigsbee2A velocity model.

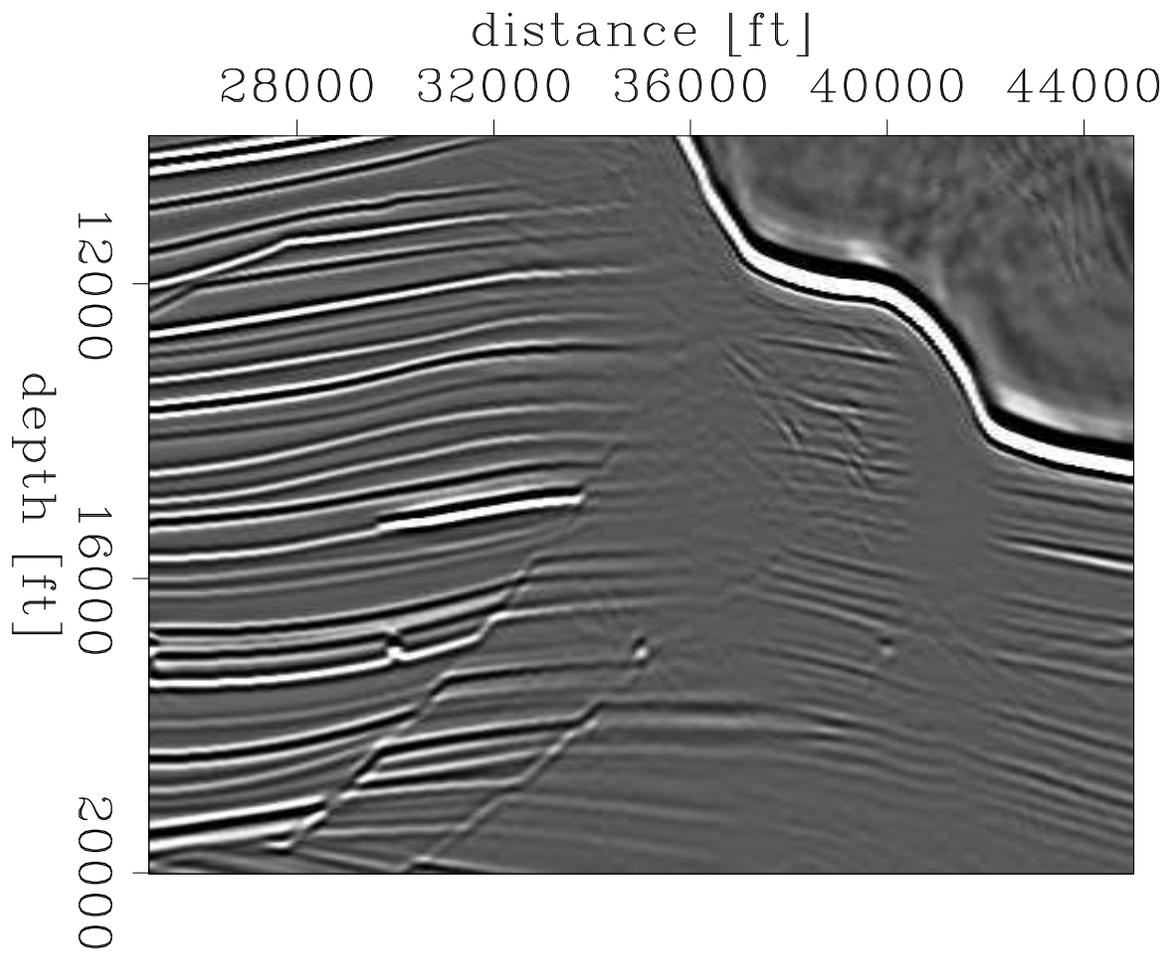Figure 5: The true reflectivity of the Sigsbee2A model

Figure 6: The migrated image.

horizontal locations at the same depth level $z = 13000$ ft. Compare those Hessian operators and explain why they change across the section.

**Your answer:**

2. Open `hes_convol_mod.f90` with your favorite editor and code the forward operator which convolves the reflectivity model with the local Hessian operator and its adjoint. Recall the conjugate-gradient solvers that your learned in `GP211: Image Estimation by Examples` and add one to `hesconv.f90`. Run dot-product test to verify if you code them correctly. Include a copy of your codes in this paper.

**Your answer:**

3. Run command `make wem.sigsb2a.syn.data.lab.H` which will do forward modeling by convolving the true reflectivity model with the Hessian operator ($\mathbf{Hm}_{\text{true}} = \mathbf{m}_{\text{syn}}$). Plot your result $\mathbf{m}_{\text{syn}}$ and compare it with the migrated image $\mathbf{m}_{\text{mig}}$ shown in Figure 6. Do they look similar? Can you find any differences between these two images? Explain the causes of these differences.

**Your answer:**

4. Run command `make wem.sigsb2a.syn.invt.lab.H` to invert for the reflectivity with your modeled data $\mathbf{m}_{\text{syn}}$, experiment with the inversion parameters in the Makefile, and plot your best inversion result. Compare your inverted reflectivity with the true reflectivity shown in Figure 5. Do you get a nearly perfect inversion result?

**Your answer:**

5. Now let us run inversion with the migrated image $\mathbf{m}_{\text{mig}}$ instead of the modeled image $\mathbf{m}_{\text{syn}}$ by issuing command `make wem.sigsb2a.inv.lab.H`. Once again play with the inversion parameters to produce your best inversion result. Try different regularizations as provided in `hes_reglar_mod.f90`. Compare your inversion result with the true reflectivity and the migrated image. Does the inverted image look better than the migration shown in Figure 6? Explain the differences between your inversion result and the migrated image.

**Your answer:**

6. Compare the two inversion results, i.e., the one obtained using your modeled image $\mathbf{m}_{\text{syn}}$ and the one obtained using the migrated image $\mathbf{m}_{\text{mig}}$. Which one looks better? Explain why? (**Hint**: the migration is performed on the data modeled by two-way wave equation, which contains not only primaries but also multiples.)

**Your answer:**

# ALL DONE

When your are all done, upload a copy of this document to the online dropbox, and clean your directory (`make clean`).

# REFERENCES

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