APPROXIMATION SCHEMES FOR 15-DEGREE EQUATIONS WITH VARIABLE VELOCITY

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In another paper in this report, Bjorn Engquist derived the additional terms needed in order to take first-order amplitude effects into account when doing migration for an inhomogeneous medium. (See also my paper on one-way wave equations for variable velocity media). The resulting migration equation in un-retarded variables is:

$$P_{tz} - \left(\frac{1}{v}\right)P_{tt} + \left(\frac{v}{2}\right)P_{xx} - \frac{1}{2}\left(\frac{v_z}{v}\right)P_t - \frac{1}{2}v_x P_x = 0, \tag{1}$$

where v is a function of both x and z. To use this for migration, it is more convenient to transform to some kind of retarded time frame. For the constant-velocity case, or even when the velocity depends on depth, it is possible to find a transformation which will eliminate the P_{tt} term in the differential equation altogether. This has the obvious advantage that is allows the resulting differential equation to be approximated with only two-time levels, rather than the three-time levels that the double-t-derivative implies. When the velocity is a function of both space variables, however, no transformation can be found which will eliminate the P_{tt}-term without at the same time introducing a plethora of annoying other terms. The next best compromise is to retard to some constant velocity, v_o, with the idea in mind of making the coefficient of P_{tt} as small as possible. The approximate transformation for upcoming waves is:

$$x' = x$$

$$z' = z$$

$$t' = t + z/v \tag{2}$$

which →

$$\partial_t = \partial_{t'}$$

$$\partial_x = \partial_{x'}$$

and
$$\partial_z = \partial_{z'} + \frac{1}{v_o} \partial_{t'} \qquad (3)$$

Equation (1) becomes in this retarded-time frame:

$$P_{zt} = \left(\frac{1}{v} - \frac{1}{v_o}\right)P_{tt} - \frac{v}{2} P_{xx} + \frac{v_z}{2v} P_t + \frac{v_x}{2} P_x \qquad (4)$$

It is now desired to find a stable differencing scheme for this equation, Although implicit schemes can also be found, we will derive an explicit scheme so as to be able to easily include 4th-order x-differences. A theorem by Gustafsson (1972) says that if we can find a stable scheme for the differential equation without the extra first derivative terms ($\sim P_t$ and $\sim P_x$), that any scheme for adding those terms on afterwards will also be stable, i.e., we need only find a stable differencing scheme for the constant-velocity 15-degree equation with $v \neq v_o$, then we can add on the extra terms in some reasonable fashion without having to prove stability. Equation (4) with constant velocity becomes:

$$P_{zt} = \left(\frac{1}{v} - \frac{1}{v_o}\right) P_{tt} - \frac{v}{2} P_{xx} \qquad (4)$$

The following schemes are conditionally stable when solving for $P^{n+1}_{j-1,k}$:

$$D_+^{\ z} D_0^{\ t} P^{\ n}_{j,k} = \left(\frac{1}{v} - \frac{1}{v_o}\right) D_+^{\ t} D_-^{\ t} \frac{1}{2}(P^n_{j,k} + P^{n+1}_{j,k}) -$$

$$\left(\frac{v}{2}\right) D_+^{\ x} D_-^{\ x} \frac{1}{2}(P^{n+1}_{j+1,k} + P^n_{j-1,k}) \qquad (5a)^*$$

* $P^n_{j,k} = P(_j\Delta t,\ n\Delta z,\ k\Delta x)$

and

$$D_+^{z} D_0^{t} P^n_{j,k} = \left(\frac{1}{v} - \frac{1}{v_o}\right) D_+^{t} D_-^{t} \frac{1}{2}\left(P^n_{j,k} + P^{n+1}_{j,k}\right) -$$

$$\left(\frac{v}{2}\right) D_+^{x} D_-^{x} \left(1 - \frac{\Delta x^2}{12} D_+^{x} D_-^{x}\right) \frac{1}{2}\left(P^{n+1}_{j+1,k} + P^{n}_{j-1,k}\right) \tag{5b}$$

Equation (5b) is the same as equation (5a), except that the x-derivative is 4th-order accurate in (5b) and only second order accurate in (5a). Schematically, the difference stars look like:

$$\begin{array}{c|cc|} \downarrow t \;\; \overrightarrow{\;\;Z\;\;} & n & n+1 \\ \hline j-1 & 1 & -1 \\ j & & \\ j+1 & -1 & 1 \end{array} P \; - \; b \begin{array}{|cc|} 1 & 1 \\ -2 & -2 \\ 1 & 1 \end{array} P \; + \; a_o \begin{array}{|cc|} 1 & \\ & \\ & 1 \end{array} \delta_{xx} P = 0 \tag{6}$$

where $\delta_{xx} = \Delta x^2 D_+^{x} D_-^{x}$ for (5a) and $\delta_{xx} = \Delta x^2 D_+^{x} D_-^{x} \left(1 + \frac{\Delta x^2}{12} D_+^{x} D_-^{x}\right)$ for (5b). Here, $a_0 \equiv \frac{v \Delta z \Delta t}{2\Delta x^2}$ and $b = \left(\frac{1}{v} - \frac{1}{v_o}\right)\frac{\Delta z}{\Delta t}$, where $v = \frac{1}{2}\left(v_k^{n+1} + v_k^{n}\right)$

$D_+^{x} P_k = (P_{k+1} - P_k)/\Delta x$, $D_0^{x} P_k = (P_{k+1} - P_{k-1})/2\Delta x$, and

$$D_+^{x} D_-^{x} = \frac{1}{\Delta x^2}\left(P_{k+1} - 2P_k + P_{k-1}\right).$$

In order to derive the stability conditions for these schemes, we first Fourier-transform the difference equations (5a) and (5b) in x and t letting all coefficients be constant for the moment. The result is:

$$\left[(1 + b) + 2bZ - (4a + b - 1)Z^2\right] P^{n+1}(Z)$$

$$= \left[(4a + b - 1) - 2bZ + (b + 1)\right] \hat{P}^{n}(Z) \tag{7}$$

Here $Z = e^{+iw\Delta t}$ (the plus sign is taken because we are extrapolating in the $-t$ direction), $a = a_0 \sin^2(\frac{k_x \Delta x}{2})$ for equation (5a) and $a = a_0 \sin^2(\frac{k_x \Delta x}{2})[1 + \frac{1}{3}\sin^2(\frac{k_x \Delta x}{2})]$ for (5b). Equation (7) can be rewritten as:

$$P^{n+1}(Z) = -\frac{(b+1)Z^2 - 2bZ + (4a+b-1)}{(4a+b-1)Z^2 - 2bZ + (b+1)} P^n(Z)$$

$$= -\frac{A(Z)}{B(Z)} P^n(Z) \qquad (7')$$

For stability in Z, we need only show that $\left|\frac{A(Z)}{B(Z)}\right| \leq 1$, and and for stability in $-t$, $B(Z)$ must be a minimum-phase polynomial.

By inspection,

$$\left| -\frac{A(Z)}{B(Z)} \right| = \left| \frac{Z^2 B(1/Z)}{B(Z)} \right| = \left| \frac{Z^4 B(Z) B(1/Z)}{B(Z) B(1/Z)} \right| = \left| Z^4 \right| = 1,$$

so the first condition is satisfied. The second condition is a bit more difficult to show.

Instead of showing that (B(Z) has all its roots <u>outside</u> the unit circle, we will show that $A(Z) = Z^2 B(1/Z)$ has its roots <u>inside</u> the unit circle. (This is fully equivalent). To find the roots, we write

$$A(Z) = (b-1)Z^2 - 2bZ + (4a+b-1) = 0 \qquad (8)$$

The roots of $A(Z)$ are given by:

$$Z = \frac{1}{b+1}[b \pm \sqrt{1 - 4ab - 4a}].$$

We must consider all possible cases:

Case 1: $b \leq 0$; $1 > 4ab - 4a$.

In this case, the root with largest absolute value will be

$Z = \frac{1}{b+1} [B - \sqrt{1 - 4ab - 4a}]$ and is negative. We want

$Z \geq -1$, or equivalently,

$$\frac{1}{1+b} [b - \sqrt{1 - 4ab - 4a}] \geq -1$$

If $b \geq -1$, then this implies

$$-(4a + 4b) \leq b(4b + 4a)$$

$$\rightarrow \quad b \geq -1 \quad \text{if} \quad a + b \geq 0 \tag{9}$$

$$\text{or} \quad b \leq -1 \quad \text{if} \quad a + b \leq 0 \tag{10}$$

Since $a \geq 0$, both of these statements contradict the original assumptions. If $b \leq -1$, the resulting conditions also contradict the assumption that $b \leq 0$. Hence we must require that

$$b = (\frac{1}{v} - \frac{1}{v_o})\frac{\Delta Z}{\Delta t} \geq 0, \text{ and hence } v_o \geq v. \tag{11}$$

Case 2: $b \geq 0$, $1 > 4a - 4b$

Now $Z = \frac{1}{1+b} [b + \quad 1-4ab - 4a]$ is the largest root and is positive.

We get

$$\sqrt{1 - 4ab - 4a} \leq 1 \Rightarrow b \geq -1,$$

which is trivially satisfied by the assumptions.

Case 3: $1 - 4ab - 4a < 0$

Then $$Z^2 = \frac{1}{(1+b)^2} [b^2 + 4a + 4ab - 1] \leq 1$$

$$\Rightarrow b^2 + 4a + 4ab - 1 \leq b^2 + 2b + 1$$

$$\Rightarrow a \leq \frac{1}{2} . \qquad (12)$$

For the second-order-in-Δx scheme, this means that

$$\frac{v\Delta z\Delta t}{2\Delta x^2} \sin^2 \frac{(k_x \Delta x)}{2} \leq \frac{1}{2} , \text{ or } \frac{v\Delta z\Delta t}{x^2} \leq 1. \qquad (13a)$$

For the fourth order scheme,

$$\frac{v\Delta z\Delta t}{2\Delta x^2} \sin^2 (\frac{k_x \Delta x}{2}) [1 + \frac{1}{3} \sin^2 (\frac{k_x \Delta x}{2})] \leq \frac{1}{2} ,$$

$$\text{or } \frac{v\Delta z\Delta t}{\Delta x^2} \leq \frac{3}{4} . \qquad (13b)$$

Case 4: Double roots.

If both roots are the same, then, if they are both on the unit circle, linear growth of the L_2 - norm is possible. Double roots are possible if $b = (1 + 4a)/4a$. Then, $Z_1 = Z_2 = \frac{b}{1 + b} = 1 - 4a$. Clearly $|1 - 4a| < 1$ unless $a = 0$, in which case linear growth is possible. For real data, this probably won't be a problem.

To summarize, equations (11) and (13) give the stability restrictions on the method. Although we assumed that v was constant when doing stability analysis, the stability requirements for the variable coefficient case are the same except that we require that (11) and (13) be satisfied for all values of v.

Before continuing further, it is worthwhile to mention one other possible scheme which would seem more intuitive, but which introduces impractical stability conditions: If in equation (6), the averaging star

1	
	1

is replaced with

1	1

the stability requirements become $v_o \leq v$ and $\dfrac{v\Delta z\Delta t}{2\ x^2} - (\dfrac{1}{v} - \dfrac{1}{v_o})\dfrac{\Delta z}{\Delta t}$.

Obviously, if v is variable, this can be a severe requirement as v approaches v_o, and since the accuracy of the differential equation deteriorates as $(\frac{1}{v} - \frac{1}{v_o})$ becomes large, this will quite likely happen.

Bjorn Engquist discusses a similar problem in SEP-8 in his article on slant-frame approximations. We can now add on the lower order derivatives. Recall the differential equation:

$$P_{zt} - (\frac{1}{v} - \frac{1}{v_o})\ P_{tt} + \frac{v}{2}\ P_{xx} - \frac{v_z}{2v}\ P_t - \frac{v_x}{2}\ P_x = 0 \qquad (4)$$

The difference approximations will look like

1	-1

$P - b$

1	1
-2	-2

$P + a_o$

1	

$\delta_{xx} P - c$

-1	-1

$P - d$

1	1

$\delta_x P = 0$ (14)

Here $c = \dfrac{v_k^{n+1} - v_k^n}{v_k^{n+1} + v_k^n}\ \Delta z$, and $d\delta_x = \dfrac{\Delta z\Delta t}{8\Delta x^2}\ [\Delta x D_0{}^x(v_k^{n+1})][\Delta x D_0{}^x]$

for the second-order scheme. A scheme accurate to fourth-order in Δx can be obtained by replacing $D_0^{\ x}$ with $D_0^{\ x}(1 - \frac{\Delta x}{6} D_+^{\ x} D_-^{\ x})$. Due to the theorem mentioned above, the resulting second and fourth-order schemes will have the stability requirements given by equations (11) and (13).

The scheme given in equation (14) was tested for random initial data, for a_0, b, c and d constant. The behavior of the discrete L_2 - norm was observed as a function of n (number of z-steps). For c = d = 0, the L_2 - norm remained approximately constant, when a and b were restricted according to (11) and (13) which is consistent with the stability analysis discussed earlier. For c and d, not zero, the L_2 - norm was found to exhibit <u>bounded</u> exponential growth or decay. It is relatively simple to show that this behavior is consistent with the differential equation (4): Fourier-transforming this equation with respect to x and t, assuming for the moment that the coefficients are constant,

$$\hat{P}_z = [\frac{i\omega}{v} + \frac{ik_x^{\ 2}}{\omega} + \frac{v_z}{2v} - (\frac{vk_x}{\omega})\frac{v_x}{2v}]\ P$$

$$= [i(\frac{\omega}{v} + \frac{k_x^{\ 2}}{\omega}) + (\frac{v_z}{2v} - \sin\phi\ \frac{v_x}{2v})]\ P\ , \qquad (15)$$

where ϕ is the dip angle. The solution to this O.D.E. is

$$\hat{P}(k_x,z,\omega) = P(k_x, z = 0,\omega)\ e^{i(\frac{\omega}{v} + \frac{k_x^{\ 2}}{\omega})z}\ e^{(\frac{v_z}{2v} - \sin\phi\frac{v_x}{2v})z}\ ,$$

whence

$$\left|P(k_x,z,\omega\right| \leq \left|P(k_x,z = 0,\omega)\right|\ e^{(\alpha + |\beta|)z}\ ,$$

where $\alpha = \frac{v_z}{2v}$ and $\beta = v_x/2v$. By Parseval's relation, this implies

$$\left\|P(x,z,t)\right\|_{l_2} \leq e^{(\alpha + |\beta|)z}\ \ P(x,z = 0,t)\ \ _{L_2},$$

which indicates that bounded exponential growth or decay is possible.