Hubbert math

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Abstract

Hubbert fits growth and decay of petroleum production to the logistic function. The concepts may be expressed as four different equations, each offering its own insights. They are all stated here, then derived from one of them, thus showing they are equivalent.

PREFACE

This article was rejected by Wikipedia. It seems an encyclopedia does not contain the derivation of important equations.

THE FOUR FORMS OF HUBBERT'S EQUATION

Over the long haul populations grow and decay. To describe the growth and decay of society's dependence on nuclear and fossil fuels, M. King Hubbert chose an equation that describes many natural processes. Introduce bacteria to food and their population will grow exponentially until there no longer is food. As we catch all the fish in the lake our daily catch will be proportional to the number of remaining fish. Hubbert's equation models both exponential growth and decay with a single equation of three parameters to be chosen from the data. He claimed 52 years ago that worldwide oil production would be peaking about now (2008). It is.

Hubbert's math has four different expressions which we examine before showing they are mathematically equivalent.

Basic definitions

We define:

- t is time in years
- $Q(t)$ is cumulative production in billion barrels at year t.
- Q_∞ is the ultimate recoverable resource.
- $P(t) = dQ/dt$ is production in billion barrels/year at year t.
- τ is the year at which production peaks.
- ω is an inverse decay time (imaginary frequency).

Hubbert's equations can be expressed in four forms. They may all be derived from the mathematical definition of $Q(t)$, also known as the "logistic" function $Q(t)$. It ranges from 0 in the past to Q_∞ in the future ($t \gg \tau$).

$$Q(t) = \frac{Q_\infty}{1+e^{\omega(\tau-t)}} \tag{1}$$

The appropriateness of this function may be best understood from the three forms derived from it. I will state those first, then later show the necessary steps of elementary calculus to see all forms follow from this one.

The derivative of the cumulative history $Q(t)$ is the current production history $P(t)$, also known as "Hubbert's pimple." The current production $P = dQ/dt$ is

$$P(t) = Q_\infty\ \omega\ \frac{1}{(e^{-(\omega/2)(\tau-t)} + e^{(\omega/2)(\tau-t)})^2} \tag{2}$$

A plot of equation (2) looks like a Gaussian, but it isn't. (A Gaussian decays much faster.) The equation is clearly symmetric about the point $t = \tau$. When τ and t are much different (away from the center) we can ignore one of the terms in the denominator and bring the other up to the numerator. What was increasingly large in the denominator becomes small in the numerator. It decays exponentially away from the center of the blob, growing as we approach it, decaying as we recede. Exponential growth is common in ecological systems which may also decay exponentially as resources are depleted or predator numbers grow exponentially.

Hubbert's favorite mathematics can also be expressed as a non-linear differential equation.

$$P = \frac{dQ}{dt} = \omega\ Q \left(1 - \frac{Q}{Q_\infty}\right) \tag{3}$$

This equation is non-linear in Q, but it reduces to familiar linear equations for growth and decay near the beginning at $Q \approx 0$ and near the end at $Q \approx Q_\infty$. This can be seen from this equation itself, or from the coming proof that equation (1) is the solution to equation (3).

Dividing equation (3) by Q we get the important form of the Hubbert equation sometimes called the *Hubbert Linearization*.

$$\frac{P}{Q} = \omega \left(1 - \frac{Q}{Q_\infty}\right) \tag{4}$$

RESULTS

The important thing about this equation is that it is linear in the two variables Q and P/Q. If you have historical measurements of P_i and Q_i, you can plot these points in the $(Q, P/Q)$-plane and hope for them to reasonably fit a straight line. Fitting the best line to the scattered points we can read the axis intercepts. At $Q = 0$ with equation (4) we can read off the value of the growth/decay parameter $\omega = (P/Q)_{\text{intercept}}$. For world oil, according to Deffeyes it is 5.3 percent/year. At the other intercept, $P/Q = 0$ we must have $Q = Q_\infty$. Again, according to Deffeyes, Q_∞ is two trillion barrels.

As a practical matter, all that remains is to figure out τ. The Hubbert curve is symmetrical and reaches its maximum when half the oil is gone. That happens when $Q = Q_\infty/2$. In the case of USA production which has passed its peak we can find the year that Q reached that value (about 1973). There is some debate about what year world production peaks, but general agreement is that it is about now (2008). Under Hubbert assumptions the decline curve is a mirror of the rise curve. That means we start down gently over the next decade, but about 25 years from now we hit the inflection point and see a 5 percent/year decline every year thereafter.

In real life there is no reason for the decay rate to match the growth rate. The decay could be faster because of horizontal drilling. The decay could be slower because we tax to conserve or successfully invest in technologies. As liquid oil depletes, society is switching to mining tar sands which have their own Hubbert pimple (see references).

VERIFICATION THE FOUR FORMS ARE EQUIVALENT

If you buy the idea that your data scatter in $(Q_i, P_i/Q_i)$-space is a straight line, then you have bought equation (4). If you buy any one of equations (1), (2), (3), or (4), then you have bought them all because they are mathematically equivalent. Starting from the definition (1) using the rule from calculus that $d(1/v)/dt = -(dv/dt)/v^2$ yields equation (2),

$$\frac{dQ}{dt} = P(t) = Q_\infty \, \omega \, \frac{e^{\omega(\tau-t)}}{(1+e^{\omega(\tau-t)})^2} \tag{5}$$

$$P(t) = Q_\infty \, \omega \, \frac{1}{(e^{-(\omega/2)(\tau-t)} + e^{(\omega/2)(\tau-t)})^2} \tag{6}$$

which is equation (2).

Equation (1) allows us to eliminate the denominator in equation (5) getting equation (4)

$$P/Q = (Q/Q_\infty) \, \omega \, e^{\omega(\tau-t)} \tag{7}$$

$$P/Q = (Q/Q_\infty) \, \omega \, ((1+e^{\omega(\tau-t)}) - 1) \tag{8}$$

$$P/Q = (Q/Q_\infty) \, \omega \, (Q_\infty/Q - 1) \tag{9}$$

$$P/Q = \omega \, (1 - Q/Q_\infty) \tag{10}$$

which is equation (4). Multiplying both sides by Q gives equation (3). Thus all the mathematical forms are equivalent.

REFERENCE

1. http://www.hubbertpeak.com/hubbert/1956/1956.pdf contains Hubbert(1956, M.King Hubbert, Nuclear energy and fossil fuels, Publication 95, Shell Oil Company.
2. Kenneth S. Deffeyes, 2006, Beyond Oil: The view from Hubbert's Peak: Hill and Wang ISBN 0-8090-2957-X.
3. http://sep.stanford.edu/sep/jon/tarsand/ by Jon Claerbout, Tar sands: reprieve or apocalypse?

POSTFACE

One day I learned that Firefox had a much better way of zooming web pages, zooming the pictures too. Knowing that equations are pictures I went to Wikipedia, and looked up "Fourier Analysis". I was delighted. A table of equations looked beautiful and could be zoomed up to a size suitable for public lectures! It was as if html had finally incorporated math. In reality the math had been done via LaTeX and inserted as photos. Wanting to have on-line lectures drawn exactly from my books I learned to contribute to Wikipedia including equations.

At the same time I was reading Deffeyes book “Beyond Oil” (a play on the slogan “Beyond Petroleum”). I wanted to play with Hubbert’s curve fitting of worldwide oil production. Francis Muir gave me the algebraic tips I needed. I prepared my contribution in my “sandbox” and then moved it to the main encyclopedia. One of their volunteer managers soon found it and didn’t like it. Rather than quote his opinions, I’ll just sum up by saying apparantly derivations do not belong in an encyclopedia. This paper might still be there in my “sandbox”, though the rules tell that sandboxes are subject to being wiped clean. If you’d like to check, you can find my sandbox with a search for “User:jclaer”.

My coworkers installed some Wikipedia style math on our wiki site. There are many places to choose the software required. After we got it running I made the unpleasant discovery that wiki math is far from standard. The input source language varies from one place to another. Thus wiki math as it is today is not a stable place to invest energy preparing lectures.

So, I gave up. Instead I did what I usually do, wrote the document in LaTeX and converted it to PDF. It’s not as seamlessly web viewable as html, but I’m much happier with it – and I am able to include it in this report!