PLATE WAVE NORMAL MODES

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impromptu seminar

ABSTRACT

Shallow marine data shown here exhibits normal-mode phase and group velocities. Such modes are readily modeled using $k_z = 2\pi/4H(x, y)$.

INTRODUCTION

Geography, physics, and mathematics find themselves comfortably mixed in a water layer. Easy math quickly brings us the basic relations of layer thickness, phase velocity, group velocity, and frequency cutoff. The mathematics of a soil layer are more complicated, but the results are broadly similar. Researchers often present results in (ω, k_x)-space leaving listeners to puzzle what the (t, x)-space looks like. It is easy to measure layer depth if you have near vertical incidence reflections, but for data at wide offset the math here enables you to measure layer thickness H from either (t, x) space data or (f, k) space data.

We have here some shallow-water marine data. Basic scalar plate theory should fit the marine data quite well. We should study this marine data more carefully to see how good a fit we actually have. Surprisingly, the laterally moving wavefield yields a reasonable first estimate of the water depth. This math also gives a first-order understanding of land data by defining a map of effective weathered-layer thickness. More elaborate theories are available, but they tend to require the presumption of a laterally-constant effective-layer thickness, whereas for much land data, variable thickness might be about all of what we will be able to extract.

Figure 1 shows a marine shot gather before and after linear moveout (LMO) with water velocity. To recognize that the LMO is done at water velocity, look carefully at the moved-out direct arrival. It is weak and barely visible only in the first few traces. It lies horizontally. Also, there are a huge number of multiples that would asymptote to horizontal if the data included further offsets. Observe a clustering of energy at an apparent velocity a little slower than water velocity. This velocity is called the group velocity. In the middle of the group packet observe the phase packets moving upwards with offset.

Next look at Figure 2. I believe there are two clusters of energy at two different group velocities. I'll claim these are the "fundamental mode" and the "first higher mode". Let us examine some theory to see how this data relates to the water depth

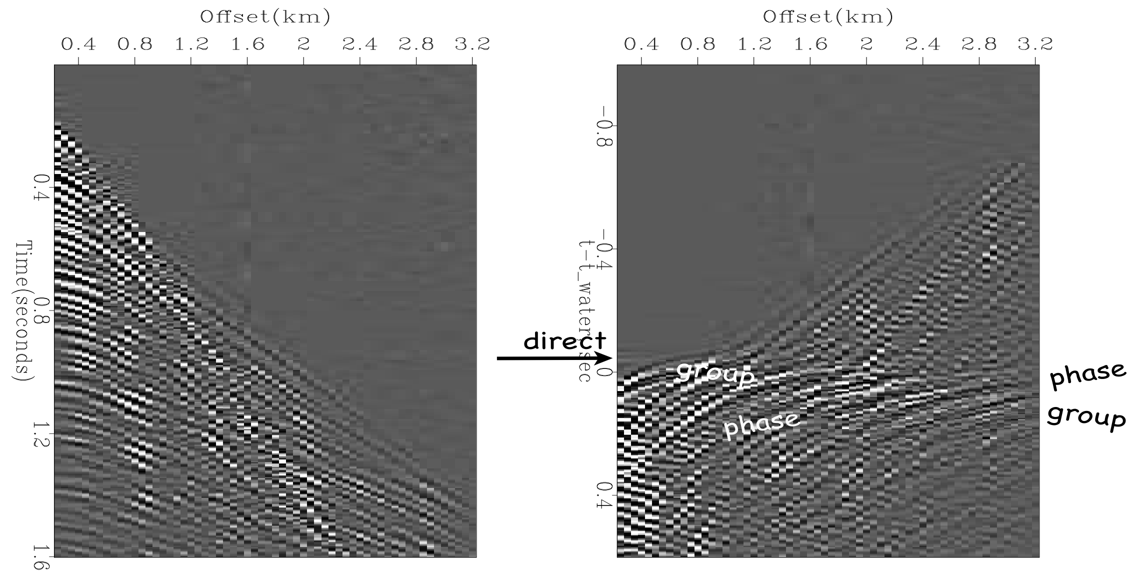

Figure 1: Shallow water marine data (left) and moved out linearly at water velocity (right) shows the fundamental mode with group velocity a bit slower than water velocity and phase velocity a bit higher. Data at vostok:/book/bei/Data/wz.34.H

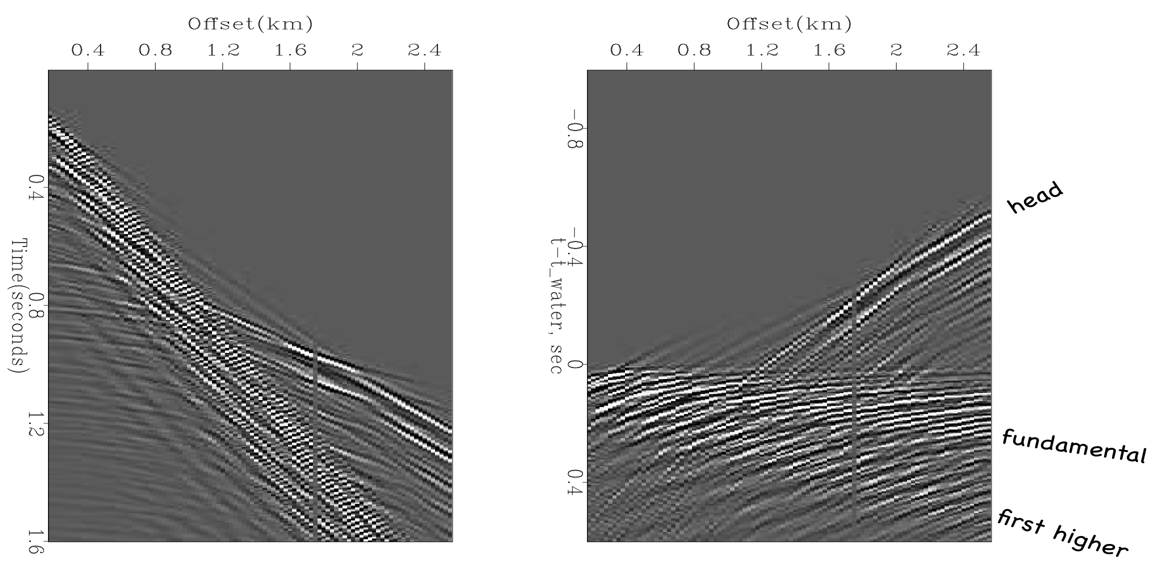

Figure 2: Shallow water marine, LMOed at water velocity, shows fundamental and one higher mode. Data at vostok:/book/bei/Data/wz.32.H

and see if my modal interpretation is quantitatively plausible and if we can deduce the water depth for each of the two figures.

The normal modes we see in figures 1 and 2 are much stronger than they look. Both shot arrays and receiver arrays are k_x filters that work against them.

Consider the water layer to be a plate with a free surface on top and a rigid surface on the bottom. Mathematically we will have a zero value boundary on one surface and a zero slope boundary on the other. The longest vertical wavelength λ_z that will fit a quarter wave in the plate is four times the plate thickness H. Thus

$$\lambda_z = 4H$$

and

$$k_z = \frac{2\pi}{\lambda_z} = \frac{2\pi}{4H} \tag{1}$$

The dispersion relation of the scalar wave equation is

$$\begin{aligned}
\frac{\omega^2}{v^2} &= k_x^2 + k_z^2 && (2)\\
\frac{\omega^2}{v^2} &= k_x^2 + \left(\frac{2\pi}{4H}\right)^2 && (3)\\
1 &= \left(\frac{vk_x}{\omega}\right)^2 + \left(\frac{2\pi v}{4H\omega}\right)^2 && (4)\\
1 &= \sin^2\theta + \left(\frac{v}{4Hf}\right)^2 && (5)
\end{aligned}$$

where we have used $\omega = 2\pi f$. If we imagine the last term should be a cosine, then we have a problem when $4Hf < v$. Thus there is a low frequency cutoff.

If the plate had identical boundary conditions on top and bottom it could contain a ray going exactly horizontally. Since the two boundary conditions differ, there must be a slightly upgoing ray and another slightly downgoing. At very short wavelengths these two rays would depart very slightly from the horizontal. At lower and lower frequencies the ray angle gets steeper and steeper until it is going straight up and down where we encounter the low frequency cutoff when the wavelength gets so long it fails to fit vertically in the plate.

The equation for a monochromatic plane wave is $\exp(-i\omega t + ik_x x)$. To see a wavefront, set the phase equal a constant, const $= -i\omega t + ik_x x$, and take the derivative of that constant by x getting $0 = -\omega dt/dx + k_x$ or $dt/dx = k_x/\omega$.

Using the definition of group velocity $v_g = \partial\omega/\partial\vec{k}$ (see Google) and taking the implicit derivative of (3) we have

$$\frac{2\omega}{v^2}\frac{\partial\omega}{\partial k_x} = 2k_x \tag{6}$$

$$\frac{\omega}{k_x}\frac{\partial\omega}{\partial k_x} = v^2 \tag{7}$$

$$\frac{dx}{dt}\frac{\partial\omega}{\partial k_x} = v^2 \tag{8}$$

which shows the product of group and phase velocities is the water velocity squared. Thus one must be faster than v while the other is slower. We saw this in both data sets.

With $dt/dx = k_x/\omega$ equation (4) becomes

$$\frac{1}{v^2} = \left(\frac{dt}{dx}\right)^2 + \left(\frac{1}{4Hf}\right)^2 \tag{9}$$

The expression $dt/dx = k_x/\omega$ assumes that we are dealing with a single plane wave. Real data always has a mess of plane waves. To deal with this reality you need to average Equation (9) over time and space to get the best fitting plane wave. This is a least squares problem with a classic solution. Looking at the field data you solved this problem with your eyeballs. Such averaging can also be done by computer. See for example my current book http://sep.stanford.edu/sep/prof/gee/book-sep.pdf , page 35, section 2.1.4 "The plane-wave destructor."

Exercises

I have not yet done these exercises. You will be the first. Hooray for you! Sjoerd's data showed a nice sharp frequency cutoff. I tried (unsuccessfully) to convince him to use it to measure the effective layer thickness H.

1. Do we need to have a frequency cutoff to measure depth H?
2. Using your eyeballs and a ruler on the enlarged data copies below what do you measure for dt/dx? Give also a standard error $\pm$ or a range.
3. What measurements do you need to make from the data enlargements to determine H?
4. Using your eyeballs and a ruler what do you estimate for frequency f and its standard error or range?
5. Using equation (9) what do you get for $H(x)$ including its standard error or range?

6. How do the equations change for the first higher mode? Measure the depth H from the higher mode.

7. Goal: Find numerator and denominator that can be smoothed locally whose ratio estimates $H(x)$.

8. Cook up a question for the group.

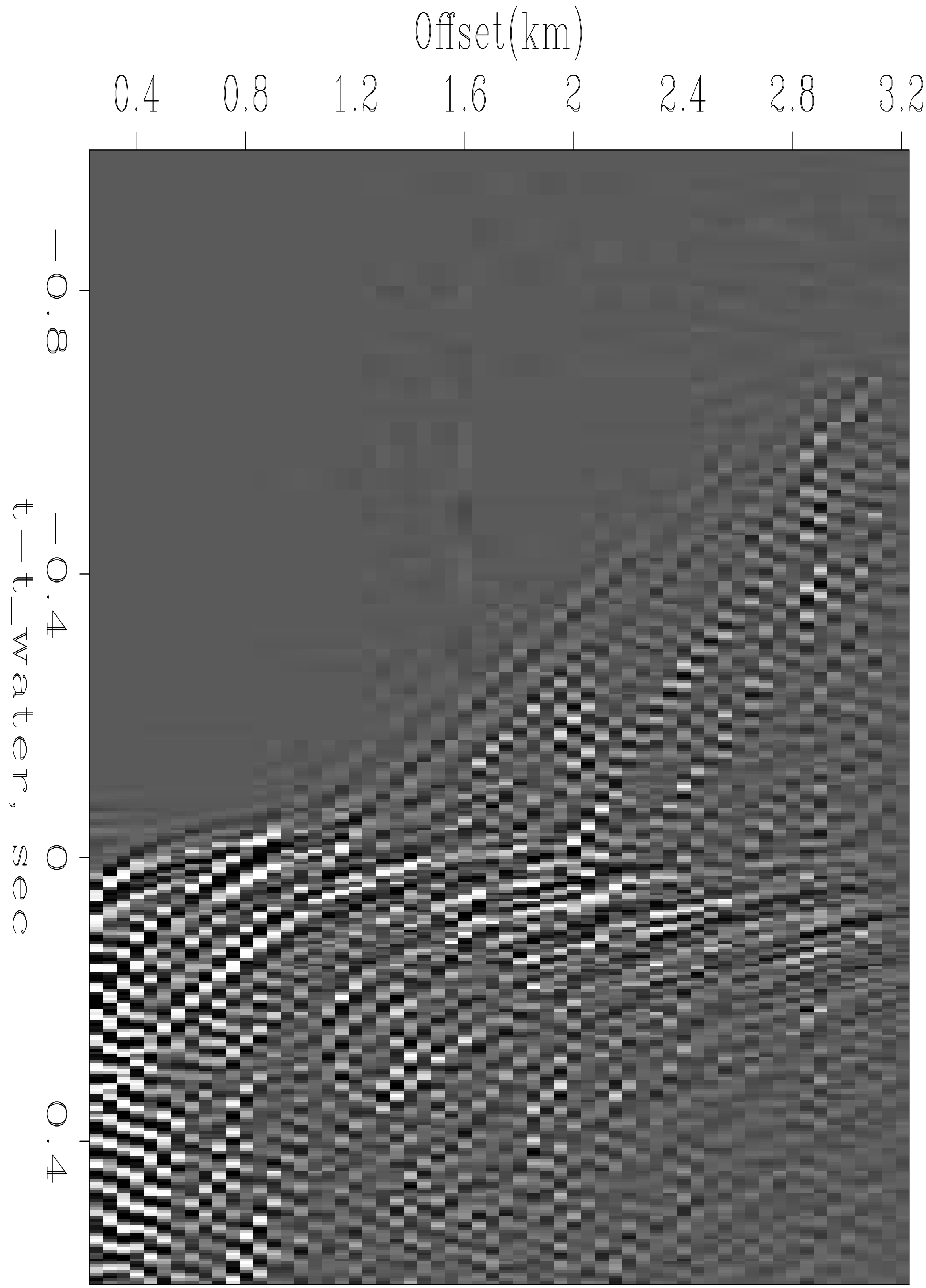

Figure 3: Enlargement of Fig 3. Ignore the PDF compression artifacts.

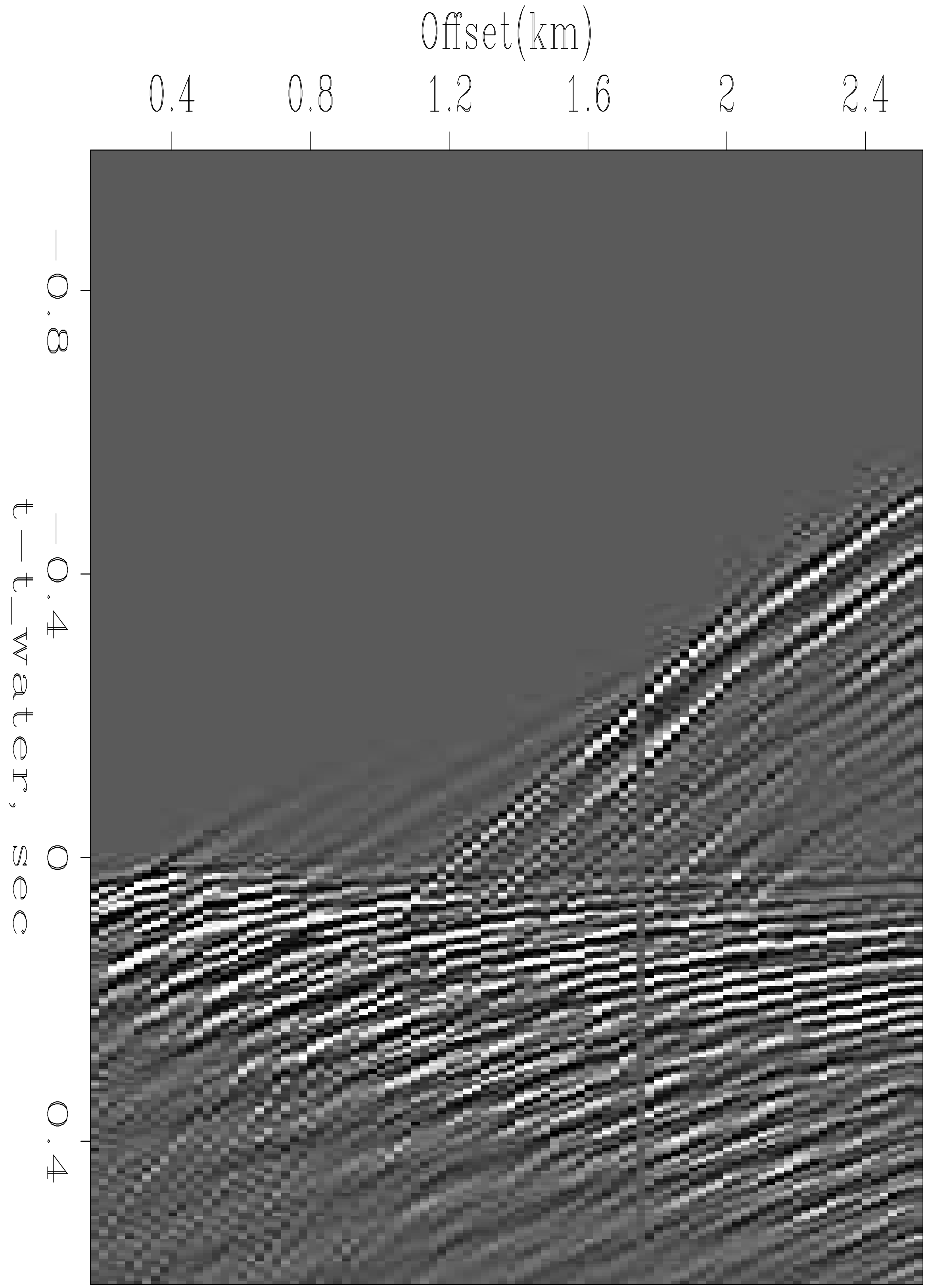

Figure 4: Enlargement of Fig 4. Ignore the PDF compression artifacts.