UNKNOWN SHOT WAVEFORM

Consider a one-dimensional seismogram $d(t)$ is unknown reflectivity $c(t)$ convolved with unknown source waveform $s(t)$. The number of data points ND≈NC is less than the number of unknowns NC+NS. Clearly we need a "smart" regularization. Let us see how this problem can be set up so reflectivity $c(t)$ comes out with sparse spikes so the integral of $c(t)$ is blocky.

This is a nonlinear problem so we need to represent everything as a “known” part plus a perturbation part which we will find and add into the known part. This is most easily expressed in the Fourier domain.

$$0 \quad \approx \quad (S+\Delta S)(C+\Delta C) \; - \; D \tag{1}$$

Linearize by dropping $\Delta S \Delta C$.

$$0 \quad \approx \quad S \; \Delta C \; + \; C \; \Delta S \; + \; (CS - D) \tag{2}$$

Let us change to the time domain with a matrix notation. Put the unknowns ΔC and ΔS in vectors $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{s}}$. Put the knowns C and S in convolution matrices $\mathbf{C}$ and $\mathbf{S}$. Express $CS - D$ as a column vector $\bar{\mathbf{d}}$ The data fitting regression is now

$$\mathbf{0} \quad \approx \quad \mathbf{S}\tilde{\mathbf{c}} \; + \; \mathbf{C}\tilde{\mathbf{s}} \; + \; \bar{\mathbf{d}} \tag{3}$$

This regression is expressed more explicitly below.

$$\mathbf{0} \quad \approx \quad \begin{bmatrix} s_0 & . & . & . & . & . & . & c_0 & . & . \\ s_1 & s_0 & . & . & . & . & . & c_1 & c_0 & . \\ s_2 & s_1 & s_0 & . & . & . & . & c_2 & c_1 & c_0 \\ . & s_2 & s_1 & s_0 & . & . & . & c_3 & c_2 & c_1 \\ . & . & s_2 & s_1 & s_0 & . & . & c_4 & c_3 & c_2 \\ . & . & . & s_2 & s_1 & s_0 & . & c_5 & c_4 & c_3 \\ . & . & . & . & s_2 & s_1 & s_0 & c_6 & c_5 & c_4 \\ . & . & . & . & . & s_2 & s_1 & . & c_6 & c_5 \\ . & . & . & . & . & . & s_2 & . & . & c_6 \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \tilde{c}_4 \\ \tilde{c}_5 \\ \tilde{c}_6 \\ \hline \tilde{s}_0 \\ \tilde{s}_1 \\ \tilde{s}_2 \end{bmatrix} + \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \tag{4}$$

Before you can solve this regression you need to define $\bar{\mathbf{d}}$ whose components are d_t. What are the terms defining d_0? What are the terms defining d_1? These terms come from $CS - D$. Expressing the CS product in the time domain we have $d_0 = c_0 s_0 - d_0$ and $d_1 = c_0 s_1 + c_1 s_0 - d_1$, etc.

The model styling regression is simply $0 \approx C + \Delta C$, which in familiar matrix form is

$$\mathbf{0} \quad \approx \quad \mathbf{I}\tilde{\mathbf{c}} + \bar{\mathbf{c}} \tag{5}$$

It is this regression, along with a composite norm and its associated threshold that makes $c(t)$ come out sparse. Now we have the danger that $\mathbf{c} \to 0$ while $\mathbf{s} \to \infty$ so we need one more regression

$$\mathbf{0} \quad \approx \quad \mathbf{I}\tilde{\mathbf{s}} + \bar{\mathbf{s}} \tag{6}$$

We can use ordinary least squares on the data fitting regression and the shot waveform regression. Thus

$$\mathbf{0} \quad \approx \quad \begin{bmatrix} \mathbf{r}_d \\ \mathbf{r}_s \end{bmatrix} \quad = \quad \begin{bmatrix} \bar{\mathbf{S}} & \bar{\mathbf{C}} \\ \mathbf{0} & \bar{\mathbf{I}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{s}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{d}} \\ \bar{\mathbf{s}} \end{bmatrix} \tag{7}$$

$$\mathbf{0} \quad \approx \quad \mathbf{r} \quad = \quad \mathbf{F}\mathbf{m} + \mathbf{d} \tag{8}$$

The model styling regression is where we seek spiky behavior.

$$\mathbf{0} \quad \approx \quad \mathbf{r}_c \; = \; \mathbf{I}\tilde{\mathbf{c}} + \bar{\mathbf{c}} \tag{9}$$

The big picture is that we minimize the sum

$$\min_{\tilde{\mathbf{c}},\tilde{\mathbf{s}}} \quad N_d(\mathbf{r}) + \epsilon N_m(\mathbf{r}_c) \tag{10}$$

Inside the big picture we have updating steps

$$\Delta\mathbf{m} \quad = \quad \mathbf{F}'\mathbf{r} \tag{11}$$

$$\mathbf{g}_d = \Delta\mathbf{r} \quad = \quad \mathbf{F}\Delta\mathbf{m} \tag{12}$$

We also have a gradient for changing $\tilde{\mathbf{c}}$, namely $\mathbf{g}_c = -\bar{\mathbf{c}}$ One update step is to choose a line search for α

$$\min_{\tilde{\mathbf{c}},\tilde{\mathbf{s}}} \quad N_d(\mathbf{r}+\alpha\mathbf{g}_d) + \epsilon N_m(\bar{\mathbf{c}}+\alpha\mathbf{g}_c) \tag{13}$$

That was steepest descent. The extension to conjugate direction is obvious (I think).

As with all nonlinear problems there is the danger of bizarre behavior and multiple minima. To avoid frustration, while learning you should spend about half of your effort directed toward finding a good starting solution. This normally amounts to defining and solving one or two linear problems. In this application we would get our starting solution for $s(t)$ and $c(t)$ from conventional deconvolution analysis.