ON THE

Theory of Numeric Equations

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I. — Descartes' Rule of Signs

1. Descartes' Rule of Signs consists of the following two propositions:

If $F(x)$ designates a polynomial ordered by powers of x, the number of positive roots of the equation $F(x) = 0$ is at most equal to the number of variations of the [signs of the coefficients of the] polynomial $F(x)$.

If the number of positive roots is less than the number of variations, the difference is an even number.

To establish the first proposition, I will show that, if it is true when the polynomial has $(m-1)$ variations, then it is equally true when the polynomial has m variations. The proposition will thereby be immediately established in full generality since it is evident for the case where all the terms of the polynomial have the same sign.

Thus let

$$F(x) = Ax^p + \ldots + Mx^r + Nx^s + \ldots + Rx^u$$

be a polynomial ordered by increasing or decreasing powers of x and presenting m variations of sign. The equation

$$F(x)x^{-\alpha} = 0 \ ,$$

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where α denotes an arbitrary real number, has the same positive roots as the equation

$$F(x) = 0 \tag{1}$$

and the function on the left hand side remains finite and continuous when x grows indefinitely beginning from a positive number ε as small as one might wish. One can thus apply Rolle's Theorem between the limits 0 and ∞, and one sees that the number of roots of equation (1) is at most one greater than the number of roots of the equation $x^{-(\alpha+1)}[xF'(x) - \alpha F(x)] = 0$ or, equivalently,

$$xF'(x) - \alpha F(x) = 0 \ . \tag{2}$$

The coefficients of that equation are respectively

$$A(p-\alpha),\ \ldots,\ M(r-\alpha),\ N(s-\alpha),\ \ldots, R(u-\alpha)\,.$$

The polynomial $F(x)$ having m variations, let us suppose that M and N are of opposite sign and choose the arbitrary number α so that it falls between the numbers r and s[1]; one sees that, in the preceding series, the numerical values of the coefficients multiplying $A, \ldots, M$ and those multiplying the quantities $N, \ldots, R$ have opposite signs.

The left hand side of equation (2) has specifically the sign variations of the series

$$A,\ \ldots,\ M,\ -N,\ \ldots,\ -R\,,$$

that is to say $(m-1)$ sign variations; it follows that that equation has at most $(m-1)$ positive roots and that equation (1) has as most m positive roots. Proposition I is thus completely established.

To demonstrate proposition II, it is sufficient, as one knows, to remark that the number of positive roots of equation (1) and the number of sign variations of the polynomial $F(x)$ always have the same parity.

2. The preceding proof made no assumption that the values of the exponents $p,\ \ldots,\ r,\ s,\ \ldots,\ u$ are integers; they can be fractions or even irrational.*

*In Laguerre's work, the term *polynomial* was employed for any finite sum of real powers of x. The phrase *integral polynomial* was reserved for what we now call polynomials. I drop this distinction when it does not cause confusion.

[1] One could simply take α equal to r or s; but, in some applications of the preceding arguments, it is useful to provide for, between certain limits, control of the value of α.

Thus the equation

$$x^3 - x^2 + x^{\frac{1}{3}} + x^{\frac{1}{7}} - 1 = 0\,,$$

presenting three variations, has at most three positive roots; it is clear, moreover, that it cannot have any negative roots.

One can equally well suppose that $F(x)$ is a power series ordered by increasing powers of x. If it is convergent for all positive values of x smaller than a given number a but does not converge for $x = a$, it follows from the preceding proof that *the number of positive values of x for which $F(x)$ is convergent and takes the value zero is at most equal to the number of variations in sign of the series.*

Moreover, *if the number of values of x which have that property is less than the number of variations of the series, then the difference is an even number.*[*]

In fact, the number of variations of the terms of the series being supposed finite (this is necessary to be able to apply the preceding theorem meaningfully), $F(x)$ is equal to a polynomial $\Phi(x)$ followed by an indefinite number of terms all having the same sign as the last term of $\Phi(x)$. For $x = 0$, the series has the same sign as that of the first term of $\Phi(x)$. When x approaches the value a, $\Phi(x)$ approaches a finite value; the remaining terms, which are infinite in number, all have the sign of the last term of $\Phi(x)$, and their absolute value grows indefinitely[*], since the series is divergent for $x = a$.

Thus, when x approaches arbitrarily close to a, the series of $F(x)$[‡] grows indefinitely in absolute value and exhibits the sign of of the last term of $\Phi(x)$; the number of sign variations of the series and the number of roots under consideration are thus of the same parity; from whence immediately follows the last proposition above.

All the above considerations likewise apply to the case where $F(x)$ is a series ordered by decreasing powers of x and also in the event that $F(x)$ is a series beginning with increasing powers of x followed by decreasing powers of the variable.

[*]Pólya [*Über einige Verallgemeinerungen der Descartesschen Zeichenregel*, Sonderabdruck aus Archiv der Mathematik und Physik. III. Reihe. XXIII. Heft 1, 1914] has shown that the divergence at $x = a$ is necessary here using the example $F(x) = 1 - \frac{\pi^2}{3}x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots$ which has only one positive root in $(0, 1)$.

[*]That is, the absolute value of the partial sums of the series, to be precise.

[‡]The text has the misprint $\Phi(x)$ here.

3. Let

$$f(x) = A_0x^m + A_1x^{m-1} + A_2x^{m-2} + \ldots + A_{m-1}x + A_m \tag{3}$$

be a polynomial of degree m; I will consider the sequence of polynomials

$$\begin{aligned}
f_m(x) &= A_0\,, \\
f_{m-1}(x) &= A_0x + A_1\,, \\
f_{m-2}(x) &= A_0x^2 + A_1x + A_2\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
f_1(x) &= A_0x^{m-1} + A_1x^{m-2} + \ldots + A_{m-1}\,, \\
f(x) &= A_0x^m + A_1x^{m-1} + \ldots + A_{m-1}x + A_m\,.
\end{aligned}$$

where the last is precisely the given polynomial.

The values taken by these polynomials, for a given value of the variable equal to a, are easily calculated by recurrence; one has, certainly, the well known relation

$$f_i(a) = af_{i+1}(a) + A_{m-i}\,,$$

and the quantities $f_m(a)$, $f_{m-1}(a)$, $\ldots$, $f_1(a)$, $f(a)$ arise when one obtains the result of substituting a in $f(x)$.*

Thus posed, one can state the following proposition:

If a is a positive number, the number of variations of the terms of the series

$$f_m(a),\ f_{m-1}(a)^\dagger,\ f_{m-2}(a),\ \ldots,\ f_1(a),\ f(a)$$

is at least equal to the number of roots of the equation $f(x) = 0$ *which are greater than a, and, if there are more, the difference of these two numbers is an even number.*

For proof, I consider the identity

$$\frac{f(x)}{x-a} = f_m(a)x^{m-1} + f_{m-1}(a)x^{m-2} + \ldots + f_1(a) + \frac{f(a)}{x-a}$$

*This *nested factorization* $f(x) = A_m + x(A_{m-1} + x(A_{m-2} + x(\ldots)))$ is often used to evaluate polynomials in computer applications.

†The original text has the misprint $f_{m+1}(a)$ here.

for values of x greater than a; the right hand side may be expanded in a convergent power series in decreasing powers of x and one has

$$\begin{aligned}\frac{f(x)}{x-a} &= f_m(a)x^{m-1} + f_{m-1}(a)x^{m-2} + \dots \\ &\quad + f_1(a) + \frac{f(a)}{x} + \frac{af(a)}{x^2} + \frac{a^2 f(a)}{x^3} + \dots .\end{aligned}$$

The number of values of x for which the series converges and takes the value zero is precisely the number of roots of the equation $f(x) = 0$ which are greater than a*; this number, by virtue of the fundamental proposition that I demonstrated earlier, is at most equal to the number of variations of the right hand side, which reduces evidently to the number of variations of the terms of the sequence

$$f_m(a),\ f_{m-1}(a),\ f_{m-2}(a),\ \dots,\ f_1(a),\ f(a)\,,$$

from which results the stated theorem.

As an application, I will consider the equation

$$f(x) = x^4 - 3x^3 + x^2 - 8x - 10 = 0\,.$$

It has no negative roots; in calculating successively the result of the substitution in the left hand sides of the numbers 1, 2, and 3, one forms the following table:

x	$f_5(x)$	$f_3(x)$	$f_2(x)$	$f_1(x)$	$f(x)$
$+1$	$+1$	-2	$-\ 1$	$-\ 9$	$-\ 19$
$+2$	$+1$	$+1$	$+\ 3$	$-\ 2$	$-\ 14$
$+3$	$+1$	$+6$	$+19$	$+49$	$+137$

All the numbers relative to $+3$ being positive, one thereby concludes that there are no roots of the equation which are greater than $+3$; furthermore, as the numbers relative to $+2$ exhibit only a single variation, one is certain that there is a root between $+2$ and $+3$ and only one. Moreover, as the number relative to $+1$ exhibits no more than one variation, one concludes that there is only one root greater than $+1$: precisely the one we have already separated; finally if one considers the transformation by $\frac{1}{x}$,

$$10x^5 + 8x^4 - x^3 + 3x^2 - 1 = 0\,,$$

*Laguerre implicitly assumes $f(a) \neq 0$.

the substitution of $+1$ yields the sequence of numbers $+10$, $+18$, $+17$, $+20$, $+19$, which hasn't any variation. The equation therefore has no root smaller than $+1$ and, consequently, has only one positive root, which lies between $+2$ and $+3$.

4. The preceding proposition can be restated in another fashion.

With a being positive, it is clear that the quantities

$$A_0a^m,\ A_0a^m + A_1a^{m-1},\ A_0a^m + A_1A^{m-1} + A_2a^{m-2},\ \ldots$$

have, respectively, the same signs as the quantities $f_m(a)$, $f_{m-1}(a)$, $f_{m-2}(a)$, $\ldots$; we may thus say that the number of roots of the equation $f(x) = 0$ is at most equal to the number of variations of the terms of the sequence

$$A_0a^m,\ A_0a^m + A_1a^{m-1},\ \ldots,\ A_0a^m + A_1a^{m-1} + \ldots + A_{m-1}a + A_m\,.$$

In general, if $P + Q + R + S + \ldots$ is some sequence of terms, I will term the number of *alternations* of that sequence as the number of variations of the sequence

$$P,\ P+Q,\ P+Q+R,\ P+Q+R+S,\ \ldots\,.$$

Given this definition, the preceding theorem may be restated in the following fashion:

Given the polynomial

$$F(x) = Ax^\alpha + Bx^\beta + Cx^\gamma + \ldots + Lx^\lambda\,,$$

where the right hand side is ordered according to decreasing powers of x. Then the number of roots of the equation $F(x) = 0$ which are greater than a positive number a is at most equal to the number of alternations of the series

$$Aa^\alpha + Ba^\beta + Ca^\gamma + \ldots + La^\lambda\,,$$

and if these two numbers differ, their difference is an even number.

The proof that I have just given for this theorem clearly assumed that the numbers α, β, γ, $\ldots$ are positive integers, but it is easy to see that this restriction is unnecessary.

In the first place, if any of them are negative, upon multiplying $F(x)$ by any appropriate power of x (which does not change the number of positive roots of the equation), one can make all the exponents positive.

In the second place, if any of the numbers α, β, γ, ... are rational fractions, one can make them integers by changing x to x^ω, ω being the smallest common multiple of the denominators of the numbers α, β, γ, ... The proposition thus holds even with the exponents are negative or rational, and, by a standard argument, one can deduce that it continues to hold when the exponents are irrational.

Nothing even prevents one from supposing that the number of terms of the function $F(x)$ are infinite, so long as the series formed by these terms is convergent at $x = a$.

5.* One may investigate the bound on the number of positive roots of an equation $f(x) = 0$ which are less than a positive number a by considering the expression $\dfrac{f(x)}{a-x}$ which, for all the values of x between zero and a, may be expanded in a series of increasing powers of the variable. The argument follows exactly that which I have employed above and, without stopping to go into the details of the proof, I will immediately assert the following fundamental proposition:

Given the polynomial

$$F(x) = Ax^\alpha + Bx^\beta + Cx^\gamma + \ldots + Lx^\lambda\,,$$

where the right hand side is ordered by increasing powers of x and where the exponents are arbitrary real numbers, positive or negative, rational or irrational, then the number of positive roots of the equation $F(x) = 0$ which are less then a given positive number a is at most equal to the number of alternations of the series

$$Aa^\alpha + Ba^\beta + Ca^\gamma + \ldots + La^\lambda\,,$$

and, if these two numbers differ, their difference is an even number[1].

*The original text has the incorrect section number 6 here.

[1] The case where $F(x)$ is ordered by decreasing powers of x leads likewise to the following proposition:

The number of positive roots of the equation $F(x) = 0$ which are greater than unity is

This proposition continues to hold when the number of terms of $F(x)$ is infinite, provided that the series composed of its terms is convergent for $x = a$; the number of its variations will evidently remain finite if the series tends, for $x = a$, towards a limit different from zero.

I will mention, as a special case and because of its importance in applications, the following corollary:

The number of roots of the equation $F(x) = 0$ *which lie between* 0 *and* $+1$ *is at most equal to the number of alternations of the series*

$$A + B + C + \ldots + L\,,$$

and, if these two numbers differ, their difference is an even number.

6. Let $f(x)$ be a polynomial and write

$$F(x) = f(a+x) = f(a) + xf'(a) + \frac{x^2}{1\cdot 2}f''(a) + \ldots\,.$$

If h is a positive number, it follows from the preceding discussion that the number of roots of the equation $F(x) = 0$ that lie between 0 and h, or, in other words, the number of roots of the equation $f(x) = 0$ which lie between a and $a + h$, is at most equal to the number of alternations of the expression

$$f(a) + hf'(a) + \frac{h^2}{1\cdot 2}f''(a) + \ldots\,.$$

Similarly, posing

$$F(x) = f(a-x) = f(a) - xf'(a) + \frac{x^2}{1\cdot 2}f''(a) + \ldots\,,$$

one finds that, for a positive quantity h, the number of roots of the equation $f(x) = 0$ which lie between a and $a - h$ is at most equal to the number of alternations of the series

$$f(a) - hf'(a) + \frac{h^2}{1\cdot 2}f''(a) + \ldots\,.$$

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at most equal to the number of alternations of the series

$$A + B + C + \ldots + L\,,$$

and, if these two numbers differ, their difference is an even number.

One may thus state this proposition:

If $f(x)$ is a polynomial and a and h are two arbitrary numbers, positive or negative, then the number of roots of the equation $f(x)=0$ which lie between a and $a+h$ is at most equal to the number of alternations of the series

$$f(a)+hf'(a)+\frac{h^2}{1\cdot 2}f''(a)+\dots\,,$$

and, if these numbers differ, their difference is an even number.

Remark.—Let us consider the various quantities

$$f(a),\ f(a)+hf'(a),\ f(a)+hf'(a)+\frac{h^2}{1\cdot 2}f''(a),\ \dots\,,$$

of which the last is precisely $f(a+h)$, and let P and Q be respectively the smallest and largest of them; all the expressions

$$f(a)-P,\ f(a)-P+hf'(a),\ f(a)-P+hf'(a)+\frac{h^2}{1\cdot 2}f''(a),\ \dots\,,$$

will positive; it follows that, if one writes $f(x)-P=\varphi(x)$, that the series

$$\varphi(a)+h\varphi'(a)+\frac{h^2}{1\cdot 2}\varphi''(a)+\dots$$

will not have any alternations. The equation $f(x)-P=0$ thus hasn't any root between a and $a+h$; one would similarly prove the same for the equation $f(x)-Q=0$; whence follows this important conclusion:

When x ranges from $x=a$ to $x=a+h$, the value of the polynomial $f(x)$ always stays between the numbers P and Q.

7. The preceding theorem is only a special case of a more general proposition which is easy to establish directly and that one may state in the following fashion:

Let $f(x)$ be a polynomial of degree n, ω an arbitrary number, and a and b any two numbers not containing ω between them. If one denotes by V the number of alternations that arise in the series

$$(1)\qquad f(x)+(\omega-x)f'(x)+\frac{(\omega-x)^2}{1\cdot 2}f''(x)+\dots+\frac{(\omega-x)^n}{1\cdot 2\cdots n}f^n(x)\,,$$

when one substitutes a for x, and by V′ the number of alternations of that series when one substitutes b for x, then the number of roots of the equation $f(x)=0$ that lie between a and b is at most equal to the absolute value of the difference $V-V'$. If ω does lie between a and b, then the number of roots between a and b is at most equal to the sum $V+V'$; in the latter case, the difference of these two numbers, if it is nonzero, is an even number.

To establish this proposition, I will remark that the number of alternations of series (1), which I will abbreviate by*

$$\begin{aligned}
U_0 &= f(x)\,,\\
U_1 &= f(x)+(\omega-x)f'(x)\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
U_i &= f(x)+(\omega-x)f'(x)+\frac{(\omega-x)^2}{1\cdot 2}f''(x)+\ldots+\frac{(\omega-x)^i}{1\cdot 2\cdots i}f^i(x)\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
U_n &= f(\omega)
\end{aligned}$$

is, for a given value of x, the number of variations of the terms of the sequence U_0, U_1, …, U_{i-1}, U_i, U_{i+1}, …, U_n, of which the last is the constant $f(\omega)$. Supposing, to fix our ideas, that $a<b<\omega$, let us examine how the number of variations could change as x increases continuously from $x=a$ to $x=b$.

Should an intermediate function U_i vanish for a value α of x between a and b, the number of variations of the sequence can only change if U_{i-1} and U_{i+1} have opposite signs†.

But one evidently has

$$U_{i+1}(\alpha)=\frac{(\omega-\alpha)^{i+1}}{1\cdot 2\cdots(i+1)}f^{i+1}(\alpha)\,,$$

a quantity which has the same sign as $f^{i+1}(\alpha)$. Moreover, an easy calculation gives

$$U_i'(x)=\frac{(\omega-x)^i}{1\cdot 2\cdots i}f^{i+1}(x)\,,$$

from which one sees that $U_i'(\alpha)$ and $U_{i+1}(\alpha)$ have the same sign. Thus if

*The original text contains the misprint U instead of U_0 here.

†Laguerre does not consider the case where U_i and U_{i+1} vanish together.

$U_{i-1}(\alpha)$ and $U_{i+1}(\alpha)$ are positive, $U_i'(\alpha)$ is also positive and $U_i(x)$, being increasing for $x = \alpha$, passes from negative to positive, causing the loss of two variations in the sequence under consideration. If, on the other hand, $U_{i-1}(\alpha)$ and $U_{i+1}(\alpha)$ are both negative, $U_i(x)$ passes from positive to negative, again causing the loss of two variations. It is thus only possible to lose variations, and only an even number, if one of the intermediate functions vanishes when x varies continuously from $x = a$ to $x = b$.

When the function $U_0 = f(x)$ vanishes, one sees that there will always be one variation lost; the proposition is thus demonstrated in the case where a and b are both less than ω, and a entirely similar argument will easily establish the other cases.

Remark I. — If the arbitrary number ω is sufficiently large and positive, the functions U_0, U_1, U_2, ... have respectively the same signs as the functions $f(x)$, $f'(x)$, $f''(x)$, ..., and one then recovers Budan's theorem.

Remark II. — When the number ω is an upper bound for the roots of the equation, the preceding proposition gives the exact number of roots whenever all the roots are real and the numbers a and b are less than ω. The same conclusion holds when ω is a lower bound for the roots of the equation having only real roots and a and b are greater than ω.

8. The method that I have just employed for obtaining the number of roots of the equation $f(x) = 0$ which are greater than a positive number a is based upon the observation that the equation $\dfrac{f(x)}{x-a} = 0$ has the same roots and that the development of $\dfrac{f(x)}{x-a}$ in decreasing powers of x converges for all values of x greater than a.

It is clear that I could equally well have used the expansion of the expression $\dfrac{f(x)}{(x-a)^p}$, where p is an arbitrary integer, and it is easily proved in the same fashion that one would then obtain, in general, a sharper bound. Denoting by $\Phi(x)$ an ordered series of integer powers (increasing or decreasing) of x, on may easily demonstrate that, α being an arbitrary positive number, the expression $\Phi(x)(x-\alpha)$ (which is generally a series but may fortuitously reduce to a polynomial) exhibits at least as many variations as the series $\Phi(x)$; the proof is entirely similar to that of Segner's lemma on which that geometer's proof of Descartes' Rule of Signs was based.

Conversely it follows that, $F(x)$ denoting a polynomial or an ordered series of integer powers (increasing or decreasing) of x and α denoting an arbitrary positive number, the expansion of the expression $\dfrac{F(x)}{x-\alpha}$ exhibits at most as many variations as the expansion of $F(x)$; one can add that, if the numbers of these variations differ, their difference is an even number.

The same situation evidently holds if one considers the more general expression $\dfrac{F(x)}{\varphi(x)}$ where $\varphi(x)$ is an arbitrary polynomial constructed from factors of the form $x-\alpha$, α being real and positive.

Having made this important point, I now consider the expression $\dfrac{f(x)}{(x-a)^p}$, where $f(x)$ is a polynomial, a a positive number, and p an arbitrary integer.

Let n be the number of roots of $f(x)=0$ which are greater than a, and V the number of variations exhibited in the expansion of the previous expression in decreasing powers of x; it follows from the abovementioned propositions that n is at most equal to V (their difference, if nonzero, furthermore being an even number); the number V can only be reduced as the integer p increases: it is not possible to reduce it below a certain limit, since it must be always greater than or equal to n.*

The essential point in this method, in order to infer the number n with the best possible approximation, would be to precisely determine the limit of the number V when p increases indefinitely; but this investigation appears to present great difficulties.

In preference, I will use the following proposition:

If one writes the fraction $\dfrac{f(x)}{(x-a)^p}$ *in the form:*

$$Ax^{\alpha}+Bx^{\beta}+\ldots+Lx^{\lambda}+x^{\lambda}\left[\frac{\mathcal{A}}{x-a}+\frac{\mathcal{B}}{(x-a)^2}+\ldots+\frac{\mathcal{L}}{(x-a)^p}\right]$$

where the exponents α, β, ..., λ are in decreasing order (λ may be negative), a form, moreover, which can be realized in an infinite number of ways, then the number of roots of the equation $f(x)=0$ which are greater than a positive number a is at most equal to the number of variations of the terms of the

*The text omits *or equal to*, relying on the proximity of *limite* and *supérieur* (upper bound) for the intended meaning.

sequence

$$A,\ B,\ \dots,\ L;\ \mathcal{A},\ \mathcal{B},\ \dots,\ \mathcal{L}\,,$$

and, if these two numbers differ, their difference is an even number.

Let n be the number of roots of the proposed equation which are greater than a and V be the number of variations exhibited in the expansion of $\dfrac{f(x)}{(x-a)^p}$ in decreasing powers of x; one has, as I have shown,

$$n \le V\,.$$

Let us now denote by V_0 the number of variations of the sequence

$$A,\ B,\ \dots,\ L,\ \mathcal{A}$$

and by V_1 the number of variations that appear in the series expansion of the expression

$$\text{(1)} \qquad \frac{\mathcal{A}}{x-a} + \frac{\mathcal{B}}{(x-a)^2} + \dots + \frac{\mathcal{L}}{(x-a)^p}\,;$$

then one will evidently have

$$V = V_0 + V_1\,.$$

It follows from the preceding that V_1 is at most equal to the number of variations that appear in multiplying the expression in (1) by $(x-a)$, that is to say equal to the number of variations of the expansion of

$$\mathcal{A} + \frac{\mathcal{B}}{x-a} + \frac{\mathcal{C}}{(x-a)^2} + \dots + \frac{\mathcal{L}}{(x-a)^{p-1}}\,;$$

if we denote by $V(\mathcal{A},\mathcal{B})$ the number of variations between the two quantities $\mathcal{A}$ and $\mathcal{B}$ (a number which is precisely zero or one) and by V_2 the number of variations exhibited in the expansion of the expression

$$\frac{\mathcal{B}}{x-a} + \frac{\mathcal{C}}{(x-a)^2} + \dots + \frac{\mathcal{L}}{(x-a)^{p-1}}\,,$$

one will thus have

$$V_1 \le V(\mathcal{A},\mathcal{B}) + V_2$$

and, similarly,

$$V_2 \leq V(\mathcal{B}, \mathcal{C}) + V_3 \,,$$

where V_3 represents the number of variations exhibited in the expansion of the expression

$$\frac{\mathcal{C}}{x-a} + \frac{\mathcal{D}}{(x-a)^2} + \ldots + \frac{\mathcal{L}}{(x-a)^{p-2}} \,;$$

from which one readily deduces that

$$V_1 \leq V(\mathcal{A}, \mathcal{B}) + V(\mathcal{B}, \mathcal{C}) + \ldots + V(\mathcal{K}, \mathcal{L}) \,,$$

and the stated proposition follows immediately.

9. Applying the preceding theorem is simplest in the case a is equal to unity, a case that leads easily to the general case by a change of variable, making use of an algorithm which has already been employed by Horner and by Budan.

This algorithm consists of forming successively, by means of recursion, the different coefficients of the expansions of $\frac{f(x)}{x-1}$, $\frac{f(x)}{(x-1)^2}$, $\frac{f(x)}{(x-1)^3}$, $\ldots$ in decreasing powers of x.

To illustrate this, letting

$$f(x) = \alpha_0 x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 \,,$$

one would first write (Table A) the coefficients of this equation (the coefficients of missing terms being replaced by zeros), and append to them an infinite series of zeros.

Below that, in a first horizontal line, one would write a series of numbers a_0, a_1, a_2, $\ldots$, where the first is α_0, and each successive element being the sum of the previous term with the term of the series immediately above this one in the same vertical column, so that $a_1 = a_0 + \alpha_1$, $a_2 = a_1 + \alpha_2$, $\ldots$; one sees that all the terms a_5, a_6, a_7, $\ldots$ following a_4 are all equal to each other. The numbers thus obtained are, as it is easy to see, the coefficients of the expansion of $\frac{f(x)}{x-1}$ in decreasing powers of x.

Below that, in a second horizontal line, one would write a series of numbers b_0, b_1, b_2, $\ldots$ computed from the numbers a_0, a_1, a_2, $\ldots$ in the same way we earlier used α_0, α_1, α_2, $\ldots$ so that

$$b_0 = a_0, \quad b_1 = b_0 + a_1, \quad b_2 = b_1 + a_2, \quad \ldots \,,$$

these new numbers being the coefficients of the expansion of $\dfrac{f(x)}{(x-1)^2}$ in decreasing powers of x.

Continuing in the same fashion, one would form a series of horizontal lines

$$\begin{array}{cccccc}
c_0 & c_1 & c_2 & c_3 & c_4 & \dots\,, \\
d_0 & d_1 & d_2 & d_3 & d_4 & \dots\,, \\
e_0 & e_1 & e_2 & e_3 & e_4 & \dots\,, \\
.\,. & .\,. & .\,. & .\,. & .\,. & \dots\,,
\end{array}$$

where the various terms give the coefficients of the expansions $\dfrac{f(x)}{(x-1)^3}$, $\dfrac{f(x)}{(x-1)^4}$, $\dfrac{f(x)}{(x-1)^5}$, $\dots$, in decreasing powers of x.

If, in particular, one considers the numbers a_5, b_4, c_3, d_2, c_1, f_0, it is easy to see that, up to positive numeric factors, they are equal to $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$, $f^{\text{IV}}(1)$, $f^{\text{V}}(1)$; it is for the goal of forming these numbers in a straightforward and rapid fashion that Budan made use of the above tableau and it follows from his theorem that the number of variations exhibited by these terms gives an upper limit to the number of roots of the equation which are greater than 1. But one can make use of this tableau in an even more advantageous manner.

Table A.

$$\begin{array}{cccccccccccccccccc}
\alpha_0 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & 0 & & 0 & & 0 & \dots \\
\hline
a_0 & & a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 & & a_8 & \dots \\
 & & & & & & & & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \dots \\
b_0 & & b_1 & & b_2 & & b_3 & & b_4 & & b_5 & & b_6 & & b_7 & & b_8 & \dots \\
 & & & & & & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & & & \dots \\
c_0 & \cdots & c_1 & \cdots & c_2 & \cdots & c_3 & & c_4 & & c_5 & & c_6 & & c_7 & & c_8 & \dots \\
 & & & & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & & & & & \dots \\
d_0 & \cdots & d_1 & \cdots & d_2 & \cdots & d_3 & \cdots & d_4 & & d_5 & & d_6 & & d_7 & & d_8 & \dots \\
 & & & \cdot^{\cdot^{\cdot}} & & & & & & \cdot^{\cdot^{\cdot}} & & & & & & & & \dots \\
e_0 & & e_1 & & e_2 & & e_3 & & e_4 & & e_5 & & e_6 & & e_7 & & e_8 & \dots \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \cdot^{\cdot^{\cdot}} & & & & & & & & & & \dots \\
f_0 & \cdots & f_1 & \cdots & f_2 & \cdots & f_3 & & f_4 & & f_5 & & f_6 & & f_7 & & f_8 & \dots \\
g_0 & & g_1 & & g_2 & & g_3 & & g_4 & & g_5 & & g_6 & & g_7 & & g_8 & \dots
\end{array}$$

Reviewing the manner in which this table is constructed, one sees quite easily that one has the following identities:

$$
\begin{aligned}
\frac{f(x)}{(x-1)^3} &= c_0x^2 + c_1x + c_2 + \frac{c_3}{x-1} + \frac{b_4}{(x-1)^2} + \frac{a_5}{(x-1)^3}\,, \\
\frac{f(x)}{(x-1)^4} &= d_0x + d_1 + \frac{d_2}{x} + \frac{d_3}{x^2} + \frac{d_4}{x^2(x-1)} \\
&\qquad + \frac{c_5}{x^2(x-1)^2} + \frac{b_6}{x^2(x-1)^3} + \frac{a_7}{x^2(x-1)^4}\,;
\end{aligned}
$$

from which it follows, by virtue of the theorem demonstrated above, that the number of roots of the equation $f(x) = 0$ which are greater than one is at most equal to the number of variations that appears in each of the two sequences

$$c_0,\ c_1,\ c_2,\ c_3,\ b_4,\ a_5$$

and

$$d_0,\ d_1,\ d_2,\ d_3,\ d_4,\ c_5,\ b_6,\ a_7\,.$$

More generally, if one labels as the *principal diagonal* the diagonal which contains the numbers a_5, b_4, c_3, d_2, e_1, f_0 and which are (up to positive numeric factors) the values of $f(x)$ and its derivatives for $x = 1$, one may state the following proposition:

Having formed Table A, if one follows an arbitrary row horizontally until one has reached or passed the entry corresponding to the principal diagonal and then follows the table obliquely upwards parallel to that diagonal until one reaches the first row, then the number of roots of the equation $f(x) = 0$ which are greater than one is at most equal to the number of variations exhibited by the successive terms of the table one has encountered during this traverse; and if these numbers differ, their difference is an even number.

Returning to Table A, one thus sees that the number of positive roots greater than unity is at most equal to to the number of variations displayed by the terms of the series

$$f_0,\ f_1,\ f_2,\ f_3,\ e_4,\ d_5,\ c_6,\ b_7,\ a_8\,.$$

10. Some examples are not amiss at this point to clarify the preceding.

Example I. — Consider the equation $x^3 - 4x + 6 = 0$; to obtain a limit for the number of roots greater than one, one forms the following table:

$$\begin{array}{ccccccccc}
1 & & 0 & & -4 & & 6 & & 0 \\
\hline
1 & & 1 & & -3 & & 3 & & 3 \\
 & & & & & \iddots & & \iddots & \\
1 & & 2 & & -1 & & 2 & & \\
 & & & \iddots & & \iddots & & & \\
1 & \cdots & 3 & \cdots & 2 & & & & \\
 & \iddots & & & & & & & \\
1 & & & & & & & &
\end{array}$$

The principal diagonal contains the three terms 1, 3, -1, 3, which exhibit two variations; application of Budan's theorem indicates the possibility of two roots.

But the series 1, 3, 2, 2, 3, formed by following the third row until the term $+2$ and moving up parallel to the diagonal, shows no variations; the equation thus has no root greater than unity.

To see if it has roots smaller than one, let us consider the transformation by $\dfrac{1}{x}$,

$$6x^3 - 4x^2 + 1 = 0\,;$$

one would form the following table

$$\begin{array}{cccc}
6 & -4 & 0 & 1 \\
\hline
6 & 2 & 2 & 3
\end{array}$$

which shows immediately that the proposed equation has no positive roots; application of Descartes' Rule of Signs to the transformation by $-x$ makes it clear moreover that it has a single negative root.

Example II. — Consider the equation

$$x^4 - 5x^3 + 12x^2 - 15x + 9 = 0\,,$$

which clearly has no negative root. To obtain a limit for the number of roots greater than one, we would form the following table:

1	-5	12	-15	9	0
1	-4	8	-7	2	2
1	-3	5	-2	0	
1	-2	3	1		
1	-1	2			
1 $\cdots$	0				

The terms of the principal diagonal 1, -1, 3, -2, 2 exhibit four variations here, for which Budan's theorem permits up to four roots; but the series 1, 0, 2, 1, 0, 2 does not show any variations, so one concludes that the equation does not have any root greater than unity.

To investigate the number of roots less than one, I consider the transformation by $\dfrac{1}{x}$,

$$9x^4 - 15x^3 + 12x^2 - 5x + 1 = 0\,,$$

which gives the following table:

9	-15	12	-5	1
9	-6	6	1	2
9 $\cdots$	3 $\cdots$	9 $\cdots$	10	

Since the series 9, 3, 9, 10, 2 has no variations, one sees that the equation yields no positive roots less than one; all roots are thus imaginary.

Example III. — For the equation $x^4 - 3x^3 + 9x - 9 = 0$, the transformation by $-x$,

$$x^4 + 3x^3 - 9x - 9 = 0\,,$$

shows immediately that it has only one negative root. To find a bound for the number of positive roots greater than one, I form the following table:

$$
\begin{array}{ccccccccc}
1 & -3 & 0 & 9 & -9 & & & & \\
\hline
1 & -2 & -2 & 7 & -2 & -2 & -2 & -2 & -2 \\
1 & -1 & -3 & 4 & 2 & 0 & -2 & -4 & \\
1 & 0 & -3 & 1 & 3 & 3 & 1 & & \\
1 & 1 & -2 & -1 & 2 & 5 & & & \\
1 & 2 & 0 & -1 & 1 & & & & \\
1 \cdots & 3 \cdots & 3 \cdots & 2 & & & & &
\end{array}
$$

The terms on the principal diagonal show three variations, but the series

$$1,\ 3,\ 3,\ 2,\ 1,\ 5,\ 1,\ -4,\ -2$$

only exhibiting one, one sees that the proposed equation has only a single root greater than one.

In regards the positive roots less than one, I would consider the transformation by $\dfrac{1}{x}$,

$$9x^4 - 9x^3 + 4x - 1 = 0\,,$$

which yields the following table:

$$
\begin{array}{cccc}
9 & -9 & 4 & -1 \\
\hline
9 & 0 & 4 & 3
\end{array},
$$

from which one concludes that the equation has no [positive] roots less than one.

11. Let $f(x)$ be a polynomial; designating by ω a positive quantity and by m an arbitrary integer, let us consider the expansion, in increasing powers of x, of the fraction

$$\frac{f(x)}{\left(1 - \frac{x}{\omega}\right)^m}\,.$$

Let V be the number of variations of that expansion; it follows from the above that the number V can only decrease as the number m grows; moreover it is at least equal to the number p of positive roots of the equation $f(x) = 0$ which are less than ω. With this established, let the numbers ω and m grow indefinitely in such a way that the ratio $\frac{m}{\omega}$ approaches a given positive limit z; since $\dfrac{1}{\left(1-\frac{x}{\omega}\right)^m}$ has e^{zx} as its limit, one can state the following proposition:

If z denotes a positive number, the number of variations V which appear in the expansion of $e^{zx}f(x)$ in increasing powers of x can only decrease as z grows and it is at least equal to the number p of positive roots of the equation $f(x) = 0$.

Let

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

and*

$$e^{zx}f(x) = A_0 + A_1x + A_2\frac{x^2}{1\cdot 2} + A_3\frac{x^3}{1\cdot 2\cdot 3} + \ldots ;$$

one easily finds that

$$A_0 = a_0, \;\; A_1 = a_0z + a_1, \;\; A_2 = a_0z^2 + 2a_1z + 2a_2, \;\; \ldots ,$$

and, in general,

$$A_i = a_0z^i + ia_1z^{i-1} + i(i-1)a_2z^{i-2} + i(i-1)(i-2)a_3z^{i-3} + \ldots .$$

From this one sees that, for z positive, A_i has the same sign as the expression

$$a_0z^n + ia_1z^{n-1} + i(i-1)a_2z^{n-2} + i(i-1)(i-2)a_3z^{n-3} + \ldots ;$$

if one then forms the polynomial

$$\begin{aligned} F(x) \;\; = \;\; & a_0z^n + a_1z^{n-1}x + a_2z^{n-2}x(x-1) + \ldots \\ & + a_nx(x-1)\ldots(x-n+1)\,, \end{aligned}$$

the number V is equal to the number of variations of the sequence

$$F(0), \;\; F(1), \;\; F(2), \;\; \ldots .$$

*The original text has the misprint $A_2\dfrac{x^2}{1\cdot\alpha}$ here.

Let us pose $z = \dfrac{1}{\omega}$ and, by changing x into $\dfrac{x}{\omega}$,

$$\text{(A)} \quad \left\{ \begin{aligned} \Phi(x) \;\; = \;\; & a_0 + a_1 x + a_2 x(x-\omega) + a_3 x(x-\omega)(x-2\omega) + \ldots \\ & + a_n x(x-\omega)\cdots(x-(n-1)\,\omega)\,. \end{aligned} \right.$$

V is also equal to the number of variations of the series

$$\Phi(0), \;\; \Phi(\omega), \;\; \Phi(2\omega), \;\; \ldots\,.$$

Denoting by p' the number of positive roots of the equation $\Phi(x) = 0$, one has moreover $V \leq p'$; whence, by virtue of the relation $V \geq p$,

$$p' \geq p\,.$$

Thus the equation $\Phi(x) = 0$ has at least as many positive roots as the equation $f(x) = 0$; in particular, if the equation $f(x) = 0$ has all its roots real and positive, the same is true for the equation

$$\Phi(x) = 0\,.$$

I will further remark that, V being in this case equal to p, the substitution in $\Phi(x)$ of the numbers 0, ω, $2\,\omega$, $\ldots$ must yield precisely p variations; from this it follows that, i denoting an arbitrary integer, the equation $\Phi(x) = 0$ has, at most, one root between $i\omega$ and $(i+1)\omega$.

Letting, for example,

$$f(x) = (1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \ldots + x^n\,;$$

one will have

$$\begin{aligned} \Phi(x) \;\; = \;\; & 1 + nx + \frac{n(n-1)}{1\cdot 2}x(x-\omega) + \ldots \\ & + x(x-\omega)\cdots(x-(n-1)\,\omega)\,. \end{aligned}$$

One sees that the equation $\Phi(x) = 0$ has all real roots and, further, that one may separate all of them by substituting in the polynomial $\Phi(x)$ the series of numbers

$$0, \;\; \omega, \;\; 2\,\omega, \;\; 3\,\omega, \;\; \ldots\,.$$

12. As I have demonstrated, the number V of variations of the terms of the expansion of $e^{zx}f(x)$ is at least equal to the number p of positive roots of the equation $f(x) = 0$; this number can only diminish when z takes on larger and larger values. One may ask if, for sufficiently large values of z (and, in consequence all values greater), V will be precisely equal to p.

Let us suppose, without loss of generality, that the equation $f(x) = 0$ does not have a zero root. From that which I have established above, it follows that, for z positive, V is at most equal to the number of positive roots of the equation $\Phi(x) = 0$, where $\Phi(x)$ represents the polynomial defined by equation (§11.A).

Let $\omega^k\Theta(\omega)$ be the discriminant of this polynomial; the number k will generally be zero, except in the case where the equation $f(\omega) = 0$ has a multiple root. Let us denote by ω_1 a positive number less than the smallest positive root of the equation $\Theta(x) = 0$, and gradually vary ω from 0 up to ω_1. The equation $\Phi(x) = 0$ never having a zero root, since a_0 is different from zero, any negative root cannot become positive; the roots which are imaginary for $\omega = 0$ would not become positive, because they would only be able to become equal in pairs, which is impossible since ω is smaller than ω_1. It could happen, if the equation $f(x) = 0$ has some equal roots, that certain positive multiple roots may become imaginary; in every case, letting p' be the number of positive roots of the equation $\Phi_1(x) = 0$, where $\Phi_1(x)$ is the polynomial $\Phi(x)$ after substituting ω_1 for ω, one has $p' \leq p$.

But one has $V \leq p'$ and, thus, $\leq p$; on the other hand, $V \geq p$; therefore $V = p$ and thus, the number ω_1 having been chosen as detailed above, one is assured that, if one takes $z = \dfrac{1}{\omega_1}$, for that value of z (and for all values greater) the number of variations exhibited in the expansion of $e^{zx}f(x)$ is exactly equal to the number of positive roots of the equation $f(x) = 0$.

This theorem continues to hold if that equation, contrary to my previous assumption, does have zero as a root.

This results provide a *method* entirely different from that of Lagrange and that of Sturm *for determining the exact number of positive roots of an equation.*

This method only requires the calculation of the discriminant $\omega^k\Theta(\omega)$ of the polynomial $\phi(x)$; but, this polynomial being a function of the variable ω, the calculation of this discriminant is nevertheless very laborious.

One has then to determine a lower bound ω_1 for the positive roots of the equation $\Theta(\omega) = 0$ and, this accomplished, the number of variations of the infinite sequence

$$\Phi_1(0), \;\; \Phi_1(\omega_1), \;\; \Phi_1(2\,\omega_1), \;\; \ldots$$

gives exactly the number of positive roots of the proposed equation.

Although this procedure is rarely practical, I nevertheless believe I must mention it in view of the small number of methods which permit one to determine the exact number of roots of an equation which lie between two given limits.

II. — On Equations of the Form

$$A_1 F(\alpha_1 x) + A_2 F(\alpha_2 x) + \ldots + A_n F(\alpha_n x) = 0\,.$$

13. Let

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

be an infinite series ordered by increasing powers of x, in which I assume all the coefficients are positive or zero, with the first distinct from zero.

Let us consider the equation

$$f(x) = A_1 F(\alpha_1 x) + A_2 F(\alpha_2 x) + \ldots + A_n F(\alpha_n x) = 0\,, \tag{1}$$

where the α_i's denote some positive quantities that I will suppose are arranged in decreasing order so that one has

$$\alpha_1 > \alpha_2 > \alpha_2 > \ldots > \alpha_{n-1} > \alpha_n\,.$$

Thus posed, if we expand the second member of equation (1) in a power series, we will have

$$\begin{aligned} f(x) \;\; = \;\; & a_0(A_1 + A_2 + \ldots + A_n) \\ & + a_1(A_1\alpha_1 + A_2\alpha_2 + \ldots + A_n\alpha_n)x \\ & + a_2(A_1\alpha_1^2 + A_2\alpha_2^2 + \ldots + A_n\alpha_n^2)x^2 \\ & + a_3(A_1\alpha_1^3 + A_2\alpha_2^3 + \ldots + A_n\alpha_n^3)x^3 \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ; \end{aligned}$$

and it follows from Descartes' Rule of Signs that the number p of positive roots of equation (1) [*that is to say the number of positive quantities at which $f(x)$ vanishes and for which the series expansion of the function is convergent*] is at most equal to the number of variations appearing in the terms of the infinite sequence

$$\text{(A)} \qquad \left\{\begin{array}{ccccccc} A_1 & + & A_2 & + & \ldots & + & A_n\,, \\ A_1\alpha_1 & + & A_2\alpha_2 & + & \ldots & + & A_n\alpha_n\,, \\ A_1\alpha_1^2 & + & A_2\alpha_2^2 & + & \ldots & + & A_n\alpha_n^2\,, \\ \multicolumn{7}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,.} \end{array}\right.$$

To determine an upper limit to the number of variations, let us truncate this series at index i,*

$$A_1\alpha_1^i + A_2\alpha_2^i + \ldots + A_n\alpha_n^i\,,$$

and let us denote by V' the number of variations that appear in these $i+1$ initial terms.

Denoting by V'' the number of variations of the infinite series

$$A_1\alpha_1^i + A_2\alpha_2^i + \ldots + A_n\alpha_n^i,\ A_1\alpha_1^{i+1} + A_2\alpha_2^{i+1} + \ldots + A_n\alpha_n^{i+1},\ \ldots\,,$$

one has evidently $V = V' + V''$.

But the second series is composed of the values taken by the function

$$\Phi(x) = A_1\alpha_1^i\alpha_1^x + A_2\alpha_2^i\alpha_2^x + \ldots + A_n\alpha_n^i\alpha_n^x\,,$$

when one successively substitutes $x = 0$, $x = 1$, $x = 2$, $\ldots$; the number of variations of the terms of that series is at most equal to the number of positive roots of the equation $\Phi(x) = 0$, or, equivalently, the number of roots greater than unity of the equation

$$\text{(2)} \qquad A_1\alpha_1^i z^{\log\alpha_1} + A_2\alpha_2^i z^{\log\alpha_2} + \ldots A_n\alpha_n^i z^{\log\alpha_n} = 0\,,$$

that one obtains by the substitution $e^x = z$.

As the positive numbers α_1, α_2, …, α_n are decreasing, so are the exponents $\log\alpha_1$, $\log\alpha_2$, …, $\log\alpha_n$; it follows, by virtue of a proposition demonstrated earlier (§5), that the number of roots of equation (2) which are greater than one is at most equal to the number of alternations of the series

$$A_1\alpha_1^i + A_2\alpha_2^i + \ldots + A_n\alpha_n^i\,.$$

*The original text has the misprint $A_n\alpha^i$.

Letting R be the number of these alternations, one will thus have

$$V'' \leq R \text{ and } V \leq V' + R\,;$$

from whence

$$p \leq V' + R\,.$$

14. Let us consider in particular the case where one truncates series (A) at its first term; it follows from the above that p is at most equal to the number of roots of the equation

$$A_1 z^{\log\alpha_1} + A_2 z^{\log\alpha_2} + \ldots A_n z^{\log\alpha_n} = 0$$

which are greater than one, and one may state this important proposition:

The number of positive roots of the equation

$$A_1 F(\alpha_1 x) + A_2 F(\alpha_2 x) + \ldots + A_n F(\alpha_n x) = 0\,,$$

where the quantities α_1, α_2, ..., α_n are positive numbers in decreasing order, is at most equal to the number of alternations of the series

$$A_1 + A_2 + A_3 + \ldots + A_n\,.$$

15. In a more general fashion one could consider the equation

$$A_1 F(\alpha_1 x) + A_2 F(\alpha_2 x) + \ldots + A_n F(\alpha_n x) = \Phi(x)\,,$$

where $\Phi(x)$ is a polynomial. But I believe it would be unproductive for me to expound further on this topic; what I have already covered is more than sufficient for seeing how one could treat this question.

III. — On the Equation $\int_a^b e^{-zx}\Phi(z)\,dz = 0$.

16. Let us apply the preceding results to the case where $F(x) = e^x$; the expansion of this function is, as one knows, convergent for all values of the variable and has only positive coefficients.

Consider the equation

$$A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + \ldots + A_n e^{\alpha_n x} = 0\,, \tag{1}$$

where the numbers α_1, α_2, …, α_n appear in decreasing order and are otherwise arbitrary, whether positive or negative.

This equation clearly has the same roots as the equation

$$A_1 e^{(k+\alpha_1)x} + A_2 e^{(k+\alpha_2)x} + \ldots + A_n e^{(k+\alpha_n)x} = 0\,,$$

where k denotes an arbitrary positive number that one can always choose so that the numbers $(k+\alpha_1)$, $(k+\alpha_2)$, …, $(k+\alpha_n)$ are all positive.

Applying the theorem of §14, one sees that equation (1) has at most as many positive roots as the series

$$A_1 + A_2 + \ldots + A_n \tag{A}$$

has alternations.

Let us now suppose that the numbers α_1, α_2, …, α_n are terms of an arithmetic progression in which the increment is very small and whose first and last terms are respectively $-a$ and $-b$ with $a < b$. The coefficients A_1, A_2, …* being completely arbitrary, one sees that the equation can, as the increment of the progression tends towards zero, be written in the form

$$\int_a^b e^{-zx}\Phi(z)\,dz = 0\,, \tag{2}$$

where $\Phi(z)$ denotes an entirely arbitrary function, continuous or discontinuous; it could, for example, vanish in as many intervals as one would like.

On the other hand, the number of alternations of the series (A) is at most equal to the number of roots of the equation $\int_a^x \Phi(x)dx = 0$ which lie between a and b† (it could be less in the event that this equation would have roots in this interval of even multiplicity); from whence we have the following proposition:

*The original text incorrectly writes A_0, A_1, ….

†Pólya (*Sur un théorème de Laguerre*, Comptes rendus, **156** (1913), 996-99) points out this argument is fallacious as it assumes that no new roots appear in the limit. He gives the counterexample $f_n(x) = (x-1)^2 + \frac{1}{n^2}$.

The number of roots of equation (2) *is at most equal to the number of roots of the equation* $\int_a^x \Phi(x)dx = 0$ *which lie between* a *and* b.

One can estimate the number of alternations of the series (A) in another way; dividing the interval between a and b into [sub]intervals such that, within each of them, the function $\Phi(x)$ does not identically vanish, is continuous and *has the same sign.*

One could, on writing*

$$\begin{aligned}&\int_a^b e^{-zx}\Phi(z)\,dz \\ &\qquad = \int_a^{a_1} \Phi_1(z)\,dz + \int_{a_1}^{a_2} \Phi_2(z)\,dz + \ldots + \int_{a_n}^{b} \Phi_n(z)\,dz\,,\end{aligned}$$

state the following proposition:

The number of positive roots of equation (2) is at most equal to the number of alternations of the series†

$$\int_a^{a_1} \Phi_1(x)\,dx + \int_{a_1}^{a_2} \Phi_2(x)\,dx + \ldots + \int_{a_n}^{b} \Phi_n(x)\,dx\,.$$

*The original text has the misprint $\int_a^{a_2} \Phi_2(z)dz$ here.

†Pólya, *op. cit.*, provides the following argument (with $a = 0$, $b = \infty$) that the number of roots of (2) is less then the number of intervals of constant sign:

The theorem is evident for $V = 0$; suppose that it is true for $V = n - 1$. The function $\Phi(z)$ having precisely $n + 1$ *intervals of constant sign*, let z_0 be the boundary between two such adjacent intervals. If the function $f(x)$ given by the left hand side of equation (2) has r roots greater than any x_0 for which the integral converges, then the function

$$f^*(x) = \frac{d}{dx} e^{z_0 x} f(x) = e^{z_0 x} \int_0^{\infty} (z_0 - z) e^{-zx} \Phi(z)\,dz\,,$$

will have *at least* $r - 1$; this is a simple consequence of Rolle's theorem.

On the other hand, the function $(z_0 - z)\Phi(z)$ has precisely $(n - 1) + 1$ intervals of constant sign; by our induction hypothesis for $V = n - 1$, $f^*(x)$ has *at most* $n - 1$ roots greater than x_0.

From this one concludes

$$r - 1 \le n - 1 \ \text{ or } \ r \le n = V\,. \qquad \text{Q.E.D.}$$

17. As an application of the preceding theorems, take $a = 0$ and $b = \infty$ and

$$\Phi(z) = \frac{a_0}{\Gamma(\alpha_0)} z^{\alpha_0 - 1} + \frac{a_1}{\Gamma(\alpha_1)} z^{\alpha_1 - 1} + \ldots + \frac{a_n}{\Gamma(\alpha_n)} z^{\alpha_n - 1},$$

where the α_i denote arbitrary positive numbers and Γ is the Eulerian function of the second type.*

The equation† $\int_0^\infty e^{-zx}\Phi(z)\,dz = 0$ becomes

$$\frac{a_0}{x^{\alpha_0}} + \frac{a_1}{x^{\alpha_1}} + \ldots + \frac{a_n}{x^{\alpha_n}} = 0,$$

and I observe that the number of its positive roots is precisely equal to the number of positive roots of the equation

$$a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \ldots + a_n x^{\alpha_n} = 0. \tag{1}$$

On the other hand, the equation $\int_0^x \Phi(x)\,dx = 0$ becomes

$$\frac{a_0 x^{\alpha_0}}{\Gamma(\alpha_0 + 1)} + \frac{a_1 x^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \ldots + \frac{a_n x^{\alpha_n}}{\Gamma(\alpha_n + 1)} = 0, \tag{2}$$

and it follows from the preceding proposition that the number of positive roots of equation (1) is at most equal to the number of positive roots of equation (2).

18. Let us consider the n^{th} degree equation

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0, \tag{1}$$

which I rewrite in the form

$$a_0 x^\omega + a_1 x^{1+\omega} + a_2 x^{2+\omega} + \ldots + a_n x^{n+\omega} = 0,$$

where ω denotes a nonnegative number.

It follows from the preceding that equation (1) has at most as many positive roots as the equation

$$\frac{a_0}{\Gamma(\omega+1)} + \frac{a_1 x}{\Gamma(\omega+2)} + \frac{a_2 x^2}{\Gamma(\omega+3)} + \ldots + \frac{a_n x^n}{\Gamma(\omega+n+1)} = 0,$$

*I.e., the standard generalization of the factorial function.

†dz is missing in the original text.

or, equivalently, the equation

$$(2)\ a_0 + \frac{a_1 x}{\omega+1} + \frac{a_2 x^2}{(\omega+1)(\omega+2)} + \ldots + \frac{a_n x^n}{(\omega+1)(\omega+2)\cdots(\omega+n)} = 0\,;$$

ω designating, as I have said, an arbitrary positive number or zero.

The same arguments hold with regards to negative roots, as it is easy to see on considering the substitution of $-x$. In particular, one can state the following important property:

If equation (1) *has all its roots real, then equation* (2) *equally has all its roots real.*

19. Consider the polynomial $a_0 + a_1 x + a_2 x^2 + a_3 x^3$; let us form the product

$$e^{zx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = U_0 + U_1 z + U_2 z^2 + \ldots\,,$$

where the U_i are functions of z. As it is easy to prove, one has in general $\frac{dU_i}{dz} = U_{i-1}$, so that all the functions with index lower than U_i are the successive derivatives of the latter.

One evidently has

$$U_i = \frac{a_0 x^i}{1\cdot 2\cdots i} + \frac{a_1 x^{i-1}}{1\cdot 2\cdots(i-1)} + \frac{a_2 x^{i-2}}{1\cdot 2\cdots(i-2)} + \frac{a_3 x^{i-3}}{1\cdot 2\cdots(i-3)}\,;$$

from whence one sees that the equation $U_i = 0$ has as many real roots distinct from zero as the equation

$$a_3 + \frac{a_2 x}{i-2} + \frac{a_1 x^2}{(i-2)(i-1)} + \frac{a_0 x^3}{(i-2)(i-1)i} = 0\,.$$

But this equation has, according the previous theorem, all of its roots real if $i-2$ is greater than zero, and if the equation $a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$ itself has all real roots. The equation $U_i = 0$ thus equally has all its roots real if i is > 2, and the same proposition holds for the equations $U_2 = 0$, $U_1 = 0$, since U_2 and U_1 are the first two derivatives of U_3.

This demonstration generalizes to a polynomial of arbitrary degree; from whence it is easy, indeed, to establish the following theorem directly:

Let $f(x)$ be an arbitrary polynomial decomposable into linear factors and

$$F(x) = e^{zx} f(x) = U_0 + U_1 x + U_2 x^2 + \dots\,;$$

where U_i are the coefficients of this expansion; then the equation in z

$$U_i = 0$$

has all its roots real.

20. $F(x)$ representing, as above, the function $e^{zx} f(x)$, let us write

$$F(x+h) = e^{zh} e^{zx} f(x+h) = V_0 + V_1 x + V_2 x^2 + \dots\,;$$

V_k denoting the coefficients of the expansion, let us write $V_k = \varphi(z)$, so that $V_{k-1} = \varphi'(z)$, $V_{k-2} = \varphi''(z)$, …; the equation $\varphi(z+t) = 0$ has, for any z, all real roots and may be expanded as

$$\varphi(z) + t\varphi'(z) + \frac{t^2}{1 \cdot 2}\varphi''(z) + \dots + \frac{t^k}{1 \cdot 2 \cdots k}\varphi^{(k)}(z) = 0$$

or

$$V_k + tV_{k-1} + \frac{t^2}{1 \cdot 2} V_{k-2} + \dots + \frac{t^k}{1 \cdot 2 \cdots k} V_0 = 0$$

or again

$$\frac{F^{(k)}(h)}{1 \cdot 2 \cdots k} + \frac{F^{(k-1)}(h)}{1 \cdot 2 \cdots (k-1)} + \frac{t^2}{1 \cdot 2}\,\frac{F^{(k-2)}(h)}{1 \cdot 2 \cdots (k-2)} + \dots + \frac{t^k}{1 \cdot 2 \cdots k} F(h) = 0\,,$$

or finally, on changing h to x, t to $\dfrac{1}{t}$ and clearing denominators,

$$F(x) + kF'(x)t + \frac{k(k-1)}{1 \cdot 2} F''(x)t^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} F'''(x)t^3 + \dots + F^{(k)}(x)t^k = 0\,;$$

and one sees that this equation in t has, for any x, all of its roots real.

If one thus writes the following system

$$\begin{aligned}
&F(x) + tF'x = 0\,,\\
&F(x) + 2\,tF'(x) + t^2F''(x) = 0\,,\\
&F(x) + 3\,tF'(x) + 3\,t^2F''(x) + t^3\,F'''(x) = 0\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\,,
\end{aligned}$$

one sees that, for any value of x, these equations have all real roots. The same holds for the equations

$$\begin{array}{lll} F'(x) + tF''(x) = 0\,, & F'(x) + 2\,tF''(x) + t^2F'''(x) = 0\,, & \ldots\,, \\ F''(x) + tF'''(x) = 0\,, & F''(x) + 2\,tF'''(x) + t^2F^{\mathrm{IV}}(x) = 0\,, & \ldots\,; \end{array}$$

the derivative of $F(x)$ being given by $e^{zx}[f(x) + zf'(x)]$ and the equation $f(x) + zf'(x) = 0$ having all its roots real, it is quite clear that $F'(x)$ is a function of the same type as $F(x)$, and it is the same for all of its derivatives.

21. The preceding propositions are likewise easily established when one supposes them proven for the case where $F(x)$ is a polynomial; it is sufficient to remark that $e^{zx}f(x)$ can be considered as the limit of the polynomial $\left(1 + \dfrac{zx}{n}\right)^n f(x)$, which is decomposable into real linear factors.

The same holds true with regards to the function $e^{-ux^2+zx}f(x)$, where I suppose u is positive; this function can be, in effect, considered as the limit of $\left(1 - \dfrac{ux^2}{n}\right)^n \left(1 + \dfrac{zx}{n}\right)^n f(x)$; but the preceding propositions are not applicable to a function of the form $e^{\varphi(x)}f(x)$, if the polynomial $\varphi(x)$ is of degree greater than two or, if quadratic, the coefficient of x^2 is positive.

These very simple remarks will find useful applications in the theory of transcendental functions.

22. Upon making a change of variable, the theorem established in §18 can be restated as follows:

If the equation

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0 \tag{1}$$

has all of its roots real, then so does the equation

$$a_0 + \frac{a_1x}{\alpha+\omega} + \frac{a_2x^2}{(\alpha+\omega)(2\alpha+\omega)} + \frac{a_3x^3}{(\alpha+\omega)(2\alpha+\omega)(3\alpha+\omega)} + \ldots = 0\,,$$

where α and ω denote arbitrary positive quantities, the latter may be zero. From the result that, given an equation having all its root real, one can derive an infinite number of others enjoying the same property; on applying the preceding theorem a second time one sees, for example, that the equation

$$a_0 + \frac{a_1 x}{(\alpha+\omega)(\alpha'+\omega')} + \frac{a_2 x^2}{(\alpha+\omega)(2\alpha+\omega)(\alpha'+\omega')(2\alpha'+\omega')}$$
$$+ \frac{a_3 x^3}{(\alpha+\omega)(2\alpha+\omega)(3\alpha+\omega)(\alpha'+\omega')(2\alpha'+\omega')(3\alpha'+\omega')} + \ldots = 0$$

has all its root real, α' and ω' being subject to the same conditions as α and ω.

Let, in general, $\Theta(x)$ be a polynomial of arbitrary degree decomposable into real factors of the first degree and not becoming negative for any positive value of the variable, so that $\Theta(x)$ takes the form

$$\Theta(x) = x^p (ax+b)^q \, (a'x+b')^{q'} (a''x+b'')^{q''} \ \ldots \,,$$

the numbers p, q, q', q'', … being whole numbers or being equal to zero, a, a', a'', …, and b, b', b'', … being positive, one will see easily from the above that the equation*

$$a_0 + \frac{a_1 x}{\Theta(1)} + \frac{a_2 x^2}{\Theta(1)\,\Theta(2)} + \frac{a_3 x^3}{\Theta(1)\,\Theta(2)\,\Theta(3)} + \ldots + \frac{a_n x^n}{\Theta(1)\,\Theta(2)\,\cdots\,\Theta(n)} = 0$$

has all its roots real.

But one can provide an even greater generalization to that proposition; equation (1) having, indeed, all of its roots real, the same goes for the equation

$$a_0 + \frac{a_1 x}{1+\omega} + \frac{a_2 x^2}{(1+\omega)(1+2\,\omega)} + \frac{a_3 x^3}{(1+\omega)(1+2\,\omega)(1+3\,\omega)} + \ldots = 0\,,$$

and also the equation

$$a_0 + \frac{a_1 x}{(1+\omega)^2} + \frac{a_2 x^2}{(1+\omega)^2(1+2\,\omega)^2} + \frac{a_3 x^3}{(1+\omega)^2(1+2\,\omega)^2(1+3\,\omega)^2} + \ldots = 0\,,$$

and in general the equation

$$a_0 + \frac{a_1 x}{(1+\omega)^k} + \frac{a_2 x^2}{(1+\omega)^k(1+2\,\omega)^k} + \frac{a_3 x^3}{(1+\omega)^k(1+2\,\omega)^k(1+3\,\omega)^k} + \ldots = 0\,,$$

*The original text has the misprint $\frac{a_1}{\Theta(1)}$.

where k denotes an arbitrary whole number.

Let us now make the arbitrary positive number $\frac{1}{\omega}$ and the whole number k grow indefinitely is such a way that $k\omega$ approaches the limit $\log\frac{1}{q}$, where q denotes any positive number less than or equal to unity.

The preceding equation will become

$$a_0 + a_1 qx + a_2 q^3 x^2 + a_3 q^6 x^3 + \ldots + a_n q^{\frac{n(n+1)}{2}} x^n = 0\,,$$

and will have all its root real; the same holds for the equation obtained by substituting ω^2 for q and $\frac{x}{\omega}$ for x,

$$a_0 + a_1 \omega x + a_2 \omega^4 x^2 + a_3 \omega^9 x^3 + \ldots + a_n \omega^{n^2} x^n = 0\,,$$

where ω is any real quantity with absolute value at most equal to one.

From these considerations that I have just presented immediately results the following proposition:

If equation (1) *has all of its roots real, the equation*

$$a_0 + \frac{a_1\omega x}{\Theta(1)} + \frac{a_2\omega^4 x^2}{\Theta(1)\Theta(2)} + \frac{a_3\omega^9 x^3}{\Theta(1)\Theta(2)\Theta(3)} + \ldots + \frac{a_n\omega^{n^2} x^n}{\Theta(1)\Theta(2)\cdots\Theta(n)} = 0$$

also has all of its roots real; $\Theta(x)$ denoting an arbitrary polynomial satisfying the conditions stated above and ω being an arbitrary real number having absolute value equal to or less than unity.

23. Consider, as an application of the preceding theorem, the equation

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \ldots + nx^{n-1} + x^n = 0\,;$$

ω and $\Theta(x)$ continuing to be defined as above, I will write

$$F(x) = 1 + \frac{n\omega x}{\Theta(1)} + \frac{n(n-1)}{1\cdot 2}\,\frac{\omega^4 x^2}{\Theta(1)\Theta(2)} + \ldots + \frac{\omega^{n^2} x^n}{\Theta(1)\Theta(2)\cdots\Theta(n)}$$

The polynomials of this form possess the following remarkable properties:

1. The equation $F(x) = 0$ has all roots real.

2. The various derivatives of $F(x)$ may be expressed in terms of polynomials of the same type.

One has, certainly,

$$\begin{aligned} F'(x) &= \frac{n\omega}{\Theta(1)}\left[1 + (n-1)\frac{\omega^3 x}{\Theta(2)} + \frac{(n-1)(n-2)}{1\cdot 2}\,\frac{\omega^8 x^2}{\Theta(2)\Theta(3)}\right. \\ &\qquad \left. + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}\,\frac{\omega^{15} x^3}{\Theta(2)\Theta(3)\Theta(4)} + \dots\right]. \end{aligned}$$

If then one defines $\Theta(x+1) = H(x)$ and

$$\begin{aligned} \Phi(x) &= 1 + \frac{(n-1)\omega x}{H(1)} + \frac{(n-1)(n-2)}{1\cdot 2}\,\frac{\omega^4 x^2}{H(1)H(2)} \\ &\qquad + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}\,\frac{\omega^9 x^3}{H(1)H(2)H(3)} + \dots, \end{aligned}$$

one obtains

$$F'(x) = \frac{n\omega}{\Theta(1)}\,\Phi(\omega^2 x)\,;$$

but $\Phi(x)$ is a polynomial of the same type as $F(x)$, since $H(x)$ is decomposable into linear real factors and is never negative for any positive value of x.

Having just established this for the first derivative, it clearly follows for all the subsequent derivatives.

3. If one writes, separating real and imaginary parts,

$$F(ix) = V(x) + iW(x)\,,$$

the equations $V(x) = 0$ and $W(x) = 0$ have all their roots real and, more generally, if a denotes an arbitrary real constant, the equation

$$V(x) + aW(x) = 0$$

has all its roots real.

To show this it suffices to remark that this equation can be written

$$\begin{aligned} 1 &+ \frac{n\omega x}{\Theta(1)}\,a - \frac{n(n-1)}{1\cdot 2}\,\frac{\omega^4 x^2}{\Theta(1)\Theta(2)} \\ &\qquad - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\,\frac{\omega^9 x^3}{\Theta(1)\Theta(2)\Theta(3)}\,a + \dots = 0\,, \end{aligned}$$

and that the equation

$$\left[1-\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4+\dots\right] \\ +a\left[nx-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\dots\right]=0$$

has all its roots real, whatever the real constant a; it is, in fact, the equation which determines $\tan\dfrac{\alpha}{n}$ when one is given $\tan\alpha$.

In particular, if one sets $\Theta(x)=x$ and $\omega=1$, one has

$$\begin{aligned} F(x) &= 1+\frac{nx}{1}+\frac{n(n-1)}{1\cdot 2}\,\frac{x^2}{1\cdot 2} \\ &\quad +\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\,\frac{x^3}{1\cdot 2\cdot 3}+\dots+\frac{x^n}{1\cdot 2\cdot 3\cdots n}\,, \end{aligned}$$

a polynomial which arises in several important questions of Analysis.[1]

Setting, instead, $\Theta(x)=1$, one has

$$F_n(x)=1+n\omega x+\frac{n(n-1)}{1\cdot 2}\omega^4x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\omega^9x^3+\dots+\omega^{n^2}x^n\,;$$

the polynomials thus defined satisfy the equation

$$F_n(x)=n\omega F_{n-1}(\omega^2x)\,.$$

24. One particular case of interest is the equation $\int_a^b e^{-zx}\Theta(z)\,dz=0^*$, where one supposes that $\Theta(z)$ is a polynomial whose form changes successively[†] when the variable z increases from a to b.

This equation may be manipulated[‡] into the form

$$e^{\alpha_0x}f_0(x)+e^{\alpha_1x}f_1(x)+\dots+e^{\alpha_nx}f_n(x)=0\,,$$

[*] x is missing in the original text.

[†] i.e., a piecewise-polynomial integrand.

[‡] integration by parts followed by clearing powers of x from the denominator.

[1] On this subject see a memoir of Chebyshev (*Mélanges mathématiques et astronomiques,* v. II, p. 182, St. Petersburg, 1859), my note *Sur l'intégrale* $\int_x^\infty \frac{e^{-x}\,dx}{x}$ (*Bulletin de la Société mathématique,* v. VII, p. 72) and a note of Halphen, *Sur une série pour développer les fonctions d'une variable* (*Comptes rendus des séances de l'Académie des Sciences,* 9 October 1882).

where the α_i are constants and the f_i are polynomials.

To examine the simplest case, let α_0, α_1, …, α_n be arbitrary real quantities arranged in increasing order and a_0, a_1, …, a_n be arbitrary real quantities; let us write, for compactness,

$$\begin{aligned} p_0 &= a_0\,, \\ p_1 &= a_0 + a_1\,, \\ p_2 &= a_0 + a_1 + a_2\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ p_n &= a_0 + a_1 + a_2 + \ldots + a_n\,, \end{aligned}$$

and let us consider the equation*

$$\int_{\alpha_0}^{\alpha_1} e^{-zx} p_0 dz + \int_{\alpha_1}^{\alpha_2} e^{-zx} p_1 dz + \int_{\alpha_2}^{\alpha_3} e^{-zx} p_2 dz + \ldots + \int_{\alpha_n}^{\infty} e^{-zx} p_n dz = 0\,.$$

On evaluting these integrals, it easy to see that it simply becomes

$$a_0 e^{-\alpha_0 x} + a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x} + \ldots + a_n e^{-\alpha_n x} = 0\,;$$

and the number p of its positive roots is the same as that of the roots of the equation

$$a_0 z^{\alpha_0} + a_1 z^{\alpha_1} + a_2 z^{\alpha_2} + \ldots + a_n z^{\alpha_n} = 0\,,$$

which lie between 0 and 1. This equation results in fact from the first when one substitutes $e^{-x} = z$.

One knows moreover (§16) that the number p is at most equal to the number of alternations of the series

$$\int_{\alpha_0}^{\alpha_1} p_0 dz + \int_{\alpha_1}^{\alpha_2} p_1 dz + \int_{\alpha_2}^{\alpha_3} p_2 dz + \ldots + \int_{\alpha_n}^{\infty} p_n dz\,,$$

from whence we have the following propositions:

The numbers α_0, α_1, α_2, …, α_n being arranged in increasing order, the number of roots of the equation

$$a_0 z^{\alpha_0} + a_1 z^{\alpha_1} + a_2 z^{\alpha_2} + \ldots + a_n z^{\alpha_n} = 0 \tag{1}$$

*The original text has the misprint $\int_{\alpha_2}^{\alpha_1} e^{-zx} p_2 dz$.

which lie between 0 *and* 1 *is at most equal to the number of alternations of the series*

$$(2)\qquad p_0(\alpha_1-\alpha_0)+p_1(\alpha_2-\alpha_1)+\ldots+p_{n-1}(\alpha_n-\alpha_{n-1})+p_n.\infty\,^1\,,$$

where p_0, p_1, p_2, …, p_n *have the meaning given above.*

Furthermore (which is the same theorem in another guise):

If the numbers α_0, α_1, α_2, …, α_n *are arranged in decreasing order, the number of roots of equation* (1) *which are greater than unity is at most equal to the number of alternations of the series* (2).

25. I have observed earlier (§14) that the number of positive roots of the equation

$$(1)\qquad A_0F(\alpha_0x)+A_1F(\alpha_1x)+A_2F(\alpha_2x)+\ldots+A_nF(\alpha_nx)=0$$

is at most equal to the number of roots of the equation

$$A_0z^{\log\alpha_0}+A_1z^{\log\alpha_1}+\ldots A_nz^{\log\alpha_n}=0$$

that are greater than one; provided the numbers $\log\alpha_0$, $\log\alpha_1$, …, $\log\alpha_n$, are in decreasing order.

Let us now write

$$\begin{aligned} p_0 &= A_0\,,\\ p_1 &= A_0+A_1\,,\\ p_2 &= A_0+A_1+A_2\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,,\\ p_n &= A_0+A_1+A_2+\ldots+A_n\,; \end{aligned}$$

it follows, from the preceding, that *the number of positive roots of equation* (1) *is at most equal to the number of alternations of the series*

$$p_0\log\frac{\alpha_1}{\alpha_0}+p_1\log\frac{\alpha_2}{\alpha_1}+\ldots+p_{n-1}\log\frac{\alpha_n}{\alpha_{n-1}}+p_n.\infty\,.$$

[1] According to the definition of the alternations of a series, it is clear that the number of alternations of the series $a+b+c+d.\infty$ is the number of variations of the terms of the sequence

$$a,\ a+b,\ a+b+c,\ d\,.$$

IV. — On Equations of the Form $\dfrac{a_0}{(x-\alpha_0)^\omega}+\dfrac{a_1}{(x-\alpha_1)^\omega}+\dfrac{a_2}{(x-\alpha_2)^\omega}+\ldots+\dfrac{a_n}{(x-\alpha_n)^\omega}=0.$

26. Consider the equation

$$\frac{a_0}{(x-\alpha_0)^\omega}+\frac{a_1}{(x-\alpha_1)^\omega}+\frac{a_2}{(x-\alpha_2)^\omega}+\ldots+\frac{a_n}{(x-\alpha_n)^\omega}=0\,, \tag{1}$$

where the numbers α_0, α_1, …, α_n are in decreasing order and ω is an arbitrary positive number.

Let us choose a positive number k large enough so that the quantities $k+\alpha_0$, $k+\alpha_1$, …, $k+\alpha_n$ are positive and make the transformation $y=x+k$,

$$\frac{a_0}{[y-(k+\alpha_0)]^\omega}+\frac{a_1}{[y-(k+\alpha_1)]^\omega}+\ldots+\frac{a_n}{[y-(k+\alpha_n)]^\omega}=0$$

or, for compactness,

$$\frac{a_0}{(y-\alpha_0')^\omega}+\frac{a_1}{(y-\alpha_1')^\omega}+\ldots+\frac{a_n}{(y-\alpha_n')^\omega}=0\,.$$

This equation may be written

$$\frac{a_0}{\left(1-\frac{\alpha_0'}{y}\right)^\omega}+\frac{a_1}{\left(1-\frac{\alpha_1'}{y}\right)^\omega}+\ldots+\frac{a_n}{\left(1-\frac{\alpha_n'}{y}\right)^\omega}=0\,. \tag{2}$$

Letting

$$\left(1-\frac{1}{y}\right)^{-\omega}=1+\frac{M_1}{y}+\frac{M_2}{y^2}+\frac{M_3}{y^3}+\ldots=F\left(\frac{1}{y}\right)\,;$$

the preceding equation may be written in the form

$$F\left(\frac{\alpha_0'}{y}\right)+F\left(\frac{\alpha_1'}{y}\right)+\ldots+F\left(\frac{\alpha_n'}{y}\right)=0\,,$$

where the left hand side converges for all values of y greater than α_0'.

If one now notices that all the coefficients M_i are positive, one will establish, as earlier (§13), that the number of roots of equation (2) which are greater than α_0' is at most equal to the number of roots of the equation

$$a_0 x^{\log \alpha_0'} + a_1 x^{\log \alpha_1'} + \ldots + a_n x^{\log \alpha_n'} = 0\,,$$

which are greater than unity; this number is thus at most equal to the number of alternations of the series

$$a_0 + a_1 + \ldots + a_n\,.$$

Furthermore the number of roots of equation (2) which are greater than α_0', that is $\alpha_0 + k$, is precisely equal to the number of roots of equation (1) which are greater than α_0. We can thus state the following proposition:

With the quantities α_0, α_1, ..., α_n, being arranged in decreasing order, the number of roots of the equation

$$\frac{a_0}{(x-\alpha_0)^\omega} + \frac{a_1}{(x-\alpha_1)^\omega} + \ldots + \frac{a_n}{(x-\alpha_n)^\omega} = 0$$

which are greater than α_0 is at most equal to the number of alternations of the series

$$a_0 + a_1 + \ldots + a_n\,.$$

The number of these alternations is either even or odd, according to whether the two quantities a_0 and $(a_0 + a_1 + \ldots + a_n)$ have the same or opposite signs; the same holds for the number of roots of the equation greater than α_0, as one sees on successively substituting in the left hand side of the equation $+\infty$ and the quantity $\alpha_0 + \varepsilon$, where ε denotes an infinitesimal quantity. This does not apply in the case where one has $a_0 + a_1 + \ldots + a_n = 0$; this case excepted, one can say that:

If the number of roots of the equation greater than α_0 and the number of alternations of the series formed by the coefficients differ, their difference is an even number.

27.* Putting aside, for the moment, the general case, I will focus in particular on the equation

$$\text{(1)} \qquad \frac{a_0}{x-\alpha_0} + \frac{a_1}{x-\alpha_1} + \frac{a_2}{x-\alpha_2} + \ldots + \frac{a_n}{x-\alpha_n} = 0\,.$$

*The original text incorrectly numbered this as §28. I have renumbered this and the remaining sections correctly in this version.

In the interval between α_i and α_{i+1}, insert two numbers ξ and ξ', in such a way that the numbers

$$\text{(A)} \qquad \alpha_0, \ \ldots, \ \alpha_i, \ \xi, \ \xi', \ \alpha_{i+1}, \ \ldots, \alpha_n$$

are arranged in either increasing or decreasing order, and let us make the substitution

$$x = \frac{X\xi - \xi'}{X - 1},$$

from which one recovers

$$X = \frac{x - \xi'}{x - \xi}.$$

To the series (A) correspond the following quantities:

$$\alpha'_0, \ \ldots, \ \alpha'_i, \ \infty, \ 0, \ \alpha'_{i+1}, \alpha'_n,$$

arranged in order of value[1].

One easily sees that all the quantities α'_k are positive: α'_i is thus the largest of them; equation (1) becomes, after the substitution indicated above,

$$\sum \frac{a_k}{\frac{X\xi - \xi'}{X-1} - \alpha_k} = 0$$

or equivalently

$$\sum \frac{a_k}{X(\xi - \alpha_k) - (\xi' - \alpha_k)} = \sum \frac{a_k}{\xi - \alpha_k} \ \frac{1}{X - \alpha'_k}.$$

[1]It is well to be precise about what I mean here. The quantities are said to be arranged in order of increasing or decreasing value if a variable quantity, which always varies in the same sense, successively takes the values of the terms of the series of these quantities, passing through infinity as necessary.

Thus the quantities

$$+4, \ -3, \ 0, \ +1$$

are arranged by increasing value, and the quantities

$$1, \ 0, \ -1, \ +5$$

by decreasing value.

Instead of arranging the quantities along a straight line, one could arrange them thus: along a *cycle* (a circle traversed in a specific direction).

But, α'_i being the largest of the numbers α'_k which are all positive, upon applying the proposition proven earlier, one sees that the number of roots of the equation

$$\sum \frac{a_k}{\xi - \alpha_k} \frac{1}{X - \alpha'_k} = 0 \tag{2}$$

which are greater than α'_k, i.e. the number of roots of equation (1) which are within the interval (ξ, α_i)[1], is at most equal to the number of alternations of the series

$$\frac{a_i}{\xi - \alpha_i} + \frac{a_{i-1}}{\xi - \alpha_{i-1}} + \frac{a_{i-2}}{\xi - \alpha_{i-2}} + \dots + \frac{a_{i+2}}{\xi - \alpha_{i+2}} + \frac{a_{i+1}}{\xi - \alpha_{i+1}}\,.$$

28. As an application, let is consider the equation

$$\frac{14}{x+2} - \frac{1}{x+1} + \frac{2}{x} - \frac{1}{x-1} + \frac{14}{x-2} = 0\,. \tag{1}$$

The quantities

$$-2,\ \ -1,\ \ 0,\ \ +1,\ \ +2$$

being arranged in increasing order, we will have to consider the five intervals

$$(-2,\, -1),\ (-1,\, 0),\ (0,\, +1),\ (+1,\, +2),\ (+2,\, -2)\,,$$

of which the last contains infinity.

Denoting by ξ an arbitrary real quantity, we deduce, from the above, the following consequences:

1. For ξ in the interval $(-2, -1)$, the number of roots of equation (1) which line between ξ and -2 is at most equal to the number of alternations of the series

$$\frac{14}{\xi+2} + \frac{14}{\xi-2} - \frac{1}{\xi-1} + \frac{2}{\xi} - \frac{1}{\xi+1}\,,$$

and the number of roots which line between ξ and -1, at most equal to the number of alternations of the series

$$-\frac{1}{\xi+1} + \frac{2}{\xi} - \frac{1}{\xi-1} + \frac{14}{\xi-2} + \frac{14}{\xi+2}\,.$$

[1] The quantities α, β, $\dots$, λ, μ, $\dots$, τ, ω being arranged in increasing order, I call the interval (λ, μ) that interval determined by these two numbers which does not contain any of the other numbers. This interval can include ∞ if λ and μ have opposite signs; thus given the sequence

$$+4,\ \ -3,\ \ 0,\ \ +1\,,$$

the interval $(+4, -3)$ includes the point at infinity.

In particular, let us substitute, in the first series, $\xi = -1 - \varepsilon$, where ε denotes an infinitesimal positive quantity; the series becomes

$$14 - \frac{14}{3} + \frac{1}{2} - 2 + \infty\,;$$

since it exhibits no alternations, one concludes that equation (1) has no roots in the interval $(-2, -1)$.

2. For ξ in the interval $(-1, 0)$, the number of roots of the equation which lie between ξ and -1 is at most equal to the number of alternations of the series

$$-\frac{1}{\xi+1} + \frac{14}{\xi+2} + \frac{14}{\xi-2} - \frac{1}{\xi-1} + \frac{2}{\xi}\,,$$

and the number of roots between ξ and 0 at most equal to the number of alternations of the series

$$\frac{2}{\xi} - \frac{1}{\xi-1} + \frac{14}{\xi-2} + \frac{14}{\xi+2} - \frac{1}{\xi+1}\,. \tag{2}$$

In particular, let us substitute in the first series $\xi = -\varepsilon$;* the series becomes

$$-1 + 7 - 7 + 1 - \infty\,;$$

since it exhibits two alternations, one sees that the interval $(-1, 0)$ contains either two roots or none.

3. For ξ in the interval $(0, +1)$, the number of roots of the equation which lie between 0 and ξ is at most equal to the number of alternations of the series

$$\frac{2}{\xi} - \frac{1}{\xi+1} + \frac{14}{\xi+2} + \frac{14}{\xi-2} - \frac{1}{\xi-1}\,, \tag{3}$$

and those roots between ξ and $+1$, at most equal to the number of alternations of the series

$$-\frac{1}{\xi-1} + \frac{14}{\xi-2} + \frac{14}{\xi+2} - \frac{1}{\xi+1} + \frac{2}{\xi}\,.$$

Substituting, for example, $\xi = 1 - \varepsilon$, the first series becomes

$$2 - \frac{1}{2} + \frac{14}{3} - 14 + \infty\,;$$

*The original text has the misprint x for ξ.

which exhibits two alternations: thus the equation has in the interval $(0, +1)$ either 0 or 2 roots.

One arrives at the same conclusion on substituting $+\varepsilon$ in the second series; it becomes, in fact,

$$+1 - 7 + 7 - 1 + \infty\,,$$

and this series also exhibits two alternations.

4. For ξ in the interval $(+1, +2)$, the number of roots which lie between ξ and $+1$ is at most equal to the number of alternations of the series

$$-\frac{1}{\xi-1} + \frac{2}{\xi} - \frac{1}{\xi+1} + \frac{14}{\xi+2} + \frac{14}{\xi-2}\,,$$

and the number of roots between ξ and $+2$, at most equal to the number of alternations of the series

$$\frac{14}{\xi-2} + \frac{14}{\xi+2} - \frac{1}{\xi+1} + \frac{2}{\xi} - \frac{1}{\xi-1}\,.$$

Substituting $\xi = 1 + \varepsilon$, for example, in the second series, produces

$$-14 + \frac{14}{3} - \frac{1}{2} + 2 - \infty\,;$$

as it shows no alternation, the equation hasn't any roots in the interval under consideration.

5. Finally, let us consider the interval $(+2, -2)$ which contains the point at ∞; for ξ in this interval, the number of roots which lie between ξ and 2 is at most equal to the number of alternations of the series

$$\frac{14}{\xi-2} - \frac{1}{\xi-1} + \frac{2}{\xi} - \frac{1}{\xi+1} + \frac{14}{\xi+2}\,,$$

and the number of roots between ξ and -2, at most equal to the number of alternations of the series

$$\frac{14}{\xi+2} - \frac{1}{\xi+1} + \frac{2}{\xi} - \frac{1}{\xi-1} + \frac{14}{\xi-2}\,.$$

Substituting in the second series, for example, $\xi = 2 + \varepsilon$, it becomes

$$\frac{7}{2} - \frac{1}{3} + 1 - 1 + \infty\,;$$

and, as it exhibits no alternation, the proposed equation has no roots in the interval $(+2, -2)$.

29. The preceding equation can thus only have roots in the intervals $(-1, 0)$ and $(0, +1)$.

Substituting $\xi = -\frac{3}{4}$ in the series (2), it becomes

$$-\frac{8}{3} + \frac{4}{7} - \frac{14 \cdot 4}{11} + \frac{14 \cdot 4}{5} - 4\,,$$

and exhibits one alternation; the equation thus has one and only one root between 0 and $-\frac{3}{4}$ and, likewise, exactly one root between $-\frac{3}{4}$ and -1.

Secondly, substituting $\xi = \frac{3}{4}$ in the series (3), it becomes

$$\frac{8}{3} - \frac{4}{7} + \frac{14 \cdot 4}{11} - \frac{14 \cdot 4}{3} + 4\,;$$

as it does not exhibit any alternation, it follows that the interval $\left(0,\, \frac{3}{4}\right)$ contains only a single root and it is likewise the same for the interval $\left(\frac{3}{4},\, 1\right)$.

One sees therefore that the proposed equation has all of its roots real; the first lies between -1 and $-\frac{3}{4}$, the second between $-\frac{3}{4}$ and 0, the third between zero and $+\frac{3}{4}$, and the fourth between $+\frac{3}{4}$ and $+1$.

These conclusions may be, in fact, easily verified, the equation in the form of a polynomial being

$$(2x^2 - 1)(7x^2 - 4) = 0\,[1]\,.$$

[1]This memoir formed the first chapters of a Treatise that Laguerre proposed to publish on the *theory of the resolution of numerical equations.* The majority of the propositions which it contains had already been presented in other papers by the author, but in a less complete fashion. This is why, in order to avoid repetition and to present the results in the form in which Laguerre indicated definite preference, we have thought it appropriate to place this memoir at the beginning of the contributions concerning the theory of equations, contrary to the rule which we have adopted to organize in chronological order the works concerning a given subject. E.Rouché