Proof of an algebraic theorem.

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The subject of this article is the Cartesian theorem, often attributed to Harriot, on the relationship between the number of positive and negative roots of an algebraic equation and the number of variations and permanences in sign of its coefficients. Missed by the various authors' assorted proofs of this theorem is the clarity, conciseness and generality which can justly be expected for so elementary a topic, and a new treatment hence appears to be justified.

Let X be a polynomial function of x of order m, ordered by descending powers of x. We assume, without loss of generality, the highest power term to be x^m, and the constant term is not missing; missing terms being removed so that no remaining power has a coefficient of 0.

When not all coefficients are positive, there will be one or more sign variations present. Let $-Nx^n$ be the first negative term, the first subsequent positive term $+Px^p$, the first subsequent negative term $-Qx^q$, etc. Consequently m, n, p, q, etc. are decreasing whole numbers; N, P, Q, etc. are positive, and X may be written:

$$X = x^m + \ldots - Nx^n - \ldots + Px^p + \ldots - Qx^q - etc.$$

Let X be multiplied by the factor $x - \alpha$, where α is taken to be positive. One easily sees that in the product, x^{n+1} has a negative, x^{p+1} a positive, x^{q+1} a negative coefficient etc., so that the product takes the form

$$X(x - \alpha) = x^{m+1} \ldots - N'x^{n+1} \ldots + P'x^{p+1} \ldots - Q'x^{q+1} \ldots ,$$

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where N', P', Q', etc. are positive. The signs of the rest of the terms are not determined: generally it is clear that from the highest term through the power x^{n+1} there is at least one sign change, through x^{p+1} two, through x^{q+1} three, etc. If the last sign change in X is $\pm Ux^u$, denoting the coefficient of x^{u+1} in $X(x-\alpha)$ by $\pm U'$, then U' will be positive, and through the term $\pm U'x^{u+1}$ will be found as many sign changes as in X. The last term in $X(x-\alpha)$ turns out to have sign $\mp$; thus there must be at least one additional sign change. We thus conclude that $X(x-\alpha)$ has at least one more sign change than X.

Let X now be the product of factors of the the negative and imaginary roots for an equation $y=0$, then letting α, β, γ, etc. be the positive roots of the equation, it becomes

$$y = X(x-\alpha)(x-\beta)(x-\gamma)\ldots .$$

One finds according to the above argument that in $X(x-\alpha)$ there is at least one more sign change, in $X(x-\alpha)(x-\beta)$ at least two more, in $X(x-\alpha)(x-\beta)(x-\gamma)$ at least three more, etc., than in X; in consequence even if X has no sign changes, y has at least as many sign variations as positive roots. One notes that when the equation has neither negative nor imaginary roots, one has to set $X=1$ and this conclusion remains valid.

Let y' be constructed by reversing the signs of the coefficients of the powers x^{m-1}, x^{m-3}, x^{m-5}, etc. in y; the set of roots of the equation $y'=0$ are the roots of $y=0$ with the opposite sign. It follows that there are at least as many sign changes in y' as there are negative roots of the equation $y=0$.

We thus have the following theorem:

The equation $y=0$ cannot have more positive roots than sign variations in y, and cannot have more negative roots than sign variations in y'.

This form of the theorem appears to be the most appropriate. It combines the greatest simplicity with the most extensive generality, and all aspects of these propositions, formerly separated into special cases of validity, now result directly.

Should one want to know the limit of the number of negative roots directly from the signs of the coefficients of y, it will be necessary to distinguish between the direct sign changes and permanences (terms with exponents of x differing by unity) from those where missing powers intervene. Evidently each direct sign change and each one separated by an even number of missing powers in y' turns into a similar sign sequence in y, while a sign change separated by an odd number of missing powers in y' remains a similar sign change in y. The second part of the theorem therefore yields the statement:

The number of negative roots of the equation $y = 0$ cannot be greater than the number of direct and separated by an even number of missing terms sign permanences added to the number of sign changes separated by an odd number of missing terms in y.

When there are no missing powers in y, then the number of negative roots does not exceed the number of sign permanences.

Denoting by A the number of direct sign changes and by B the number of direct sign permanences in y, then, when no terms are missing, $A + B = m$, thus equal to the number of all roots. Insofar as these signs merely show that the number of positive roots cannot be greater than A and the number of negative ones greater than B, it remains unknown if and how many imaginary roots there are. If one knows by other means that the equation has no imaginary roots, then necessarily A must equal the number of positive, and B the number of negative, roots.

The situation is different when there are missing terms. For clearly perceiving the connection to the imaginary roots, we denote by a the number with an even, and by c the number with an odd, number of missing terms separating sign changes; by b and d respectively the number of sign permanences separated by an even and odd number of missing terms in y. One easily sees that $m - A - B - a - b - c - d$ will be equal to the total number of missing terms, which we will denote by e. Now according to our theorems the number of positive roots is bounded by $A + a + c$, the number of negative by $B + b + c$, thus the number of real roots is bounded by

$$A + B + a + b + 2c = m + c - d - e\,.$$

Thus the number of imaginary roots is at least $e - c + d$.

If one thus counts the total number of missing powers and then for each odd length gap decrease it by one if it is gap between sign changes and increase it by one if it separates a sign permanence, one thereby obtains a number which the number of imaginary roots must at least equal.