Chapter 2

Ray-theoretic vs. wave-theoretic tomography: equations

2.1 Ray-theoretic tomography

Seismic tomography in its ray-theoretic form is traveltime inversion. In this application the illuminating energy is acoustic and the tomographic integrals are line integrals measured as traveltimes along raypaths. The four steps of ray-theoretic tomography correspond to: the picking of traveltimes for each source-geophone experiment; the calculation of expected traveltimes by tracing rays through an assumed background velocity field; the development of a linear theory relating the resulting traveltime delays to possible velocity perturbations; and the production of an updated velocity field by projecting the traveltime delays back through the medium—along raypaths.

2.1.1 Step one: traveltime picks

Because ray-theoretic tomography relies on the high frequency approximation of ray theory, its traveltime picks must represent ray arrivals. Where traveltimes are picked from bandlimited wavelets, this requires either that the wavelet peaks are undistorted (that phase delay is linear with frequency) or that Fermat-path first breaks can be determined. In practical applications these requirements are often unmet. Where the velocity field varies quickly on the scale of the source wavelength, wavelets are distorted by geometrical frequency dispersion; where events overlap and signal level is low, first breaks are difficult to pick. The implications of these assumptions will be returned to in the next chapter.

2.1.2 Steps two and three: traveltime delays and linear theory

Given the preceding assumptions, the source-geophone traveltime integrals of ray-theoretic tomography are:

$$t(\mathbf{g}|\mathbf{s}) = \int w(\mathbf{r}) L\left[\mathbf{r}|\mathbf{s},\mathbf{g},w(\mathbf{r})\right] d\mathbf{r}. \tag{2.1}$$

Here the source-geophone pair is indicated by $\mathbf{s},\mathbf{g}$; w is inverse velocity or slowness; and L is the raypath from $\mathbf{s}$ to $\mathbf{g}$ through w. (I have transformed the usual line integral into a space integral by making L a function of $\mathbf{r}$ and nonzero only along the raypath.)

Following equation 2.1, the expected source-geophone traveltime integrals, calculated by ray-tracing though the first-guess background slowness field, are:

$$t_0(\mathbf{g}|\mathbf{s}) = \int w_0(\mathbf{r}) L_0\left[\mathbf{r}|\mathbf{s},\mathbf{g},w_0(\mathbf{r})\right] d\mathbf{r}. \tag{2.2}$$

Here w_0 is the background slowness field and L_0 the raypath through that field. Because L is a function of w, the acoustic tomography problem is nonlinear: instead of a linear relation between Δt and Δw, simple subtraction of t_0 from t creates:

$$\Delta t(\mathbf{g}|\mathbf{s}) = \int \left\{w(\mathbf{r}) L\left[\mathbf{r}|\mathbf{s},\mathbf{g},w(\mathbf{r})\right] - w_0(\mathbf{r}) L_0\left[\mathbf{r}|\mathbf{s},\mathbf{g},w_0(\mathbf{r})\right]\right\} d\mathbf{r}. \tag{2.3}$$

To create a linear relation, ray theory invokes Fermat's principle and approximates L by L_0, producing

$$\Delta t(\mathbf{g}|\mathbf{s}) = \int \Delta w(\mathbf{r}) L_0(\mathbf{r}|\mathbf{s},\mathbf{g}) d\mathbf{r}. \tag{2.4}$$

2.1.3 Step four: backprojection

The fourth, backprojection step of ray-theoretic tomography involves the solution of the system of linear equations resulting from consideration of a number of source-geophone pairs:

$$L_0 \Delta w = \Delta t. \tag{2.5}$$

For seismic experiments sampled evenly in space, this step may be performed analytically: in either the time-space domain as a generalized inverse Radon transform (Beylkin, 1982; Fawcett and Clayton, 1984), or in the time-wavenumber domain using the projection slice theorem (Mersereau and Oppenheim, 1974). However, given the irregular spatial coverage of the typical seismic experiment, most traveltime inversion is performed numerically in time-space, using an algebraic approach and iterative least squares (Dines and Lytle, 1979; Stork, 1988). Ray-theoretic tomography in this form is most often referred to as ray-trace tomography. Because ray-trace tomography retains the strongest physical connection with raypaths, it is the algorithm of choice in this thesis. (See Kak (1985) for a detailed survey of tomography in general and Worthington (1984) for a brief summary of geophysical tomography in particular.)

The nonlinear part of the acoustic tomography problem is attacked iteratively, with steps two through four being repeated for successively updated background velocity fields.

2.2 Wave-theoretic tomography

Seismic tomography in its wave-theoretic form is full waveform inversion. In this application the traveltimes measured as line integrals along raypaths in ray-trace tomography are replaced by scattered wavefields—measured as surface (2-d) or volume (3-d) integrals over wave-propagation paths. The four steps of wave-theoretic tomography correspond to: the recording of full wavefields as seismic traces for each source-geophone experiment; the calculation of expected wavefields by forward modeling though an assumed background velocity field; the development of a linear theory relating the resulting scattered wavefields to possible velocity perturbations; and the production of an updated velocity field by propagating the scattered wavefields back through the medium.

2.2.1 Step one: wavefield measurements

Because wave-theoretic tomography is full waveform inversion, it makes no assumptions about the characterization of an event by a single traveltime pick. However, it does assume both that the source is sufficiently well known for calculation of a background wavefield and that the recorded seismic traces are complete and free of noise. The implications of this latter assumption will be returned to in the next chapter.

2.2.2 Steps two and three: wavefield perturbations and linear theory

The wave-theoretic equivalent of equation 2.4 may be generated in two ways: corresponding to linearization of the scalar wave equation with either the first-order Born or the first-order Rytov approximation.

Born

While ray-trace tomography creates a linear relation between velocity and traveltime perturbations, the first-order Born approximation creates a linear relation between velocity and wavefield-amplitude perturbations: $\Delta\Psi(\omega) = \Psi(\omega) - \Psi_0(\omega)$. These are complex amplitudes ($\Psi = Ae^{i\phi}$), measured in the temporal-frequency domain for source-geophone pairs. $\Psi(\omega)$ is the full monochromatic wavefield; $\Psi_0(\omega)$ is the background monochromatic wavefield.

The Born approximation begins with the wave equation written as:

$$\Delta\Psi(\mathbf{g}|\mathbf{s}) = \int O(\mathbf{r})G_0[\mathbf{g}-\mathbf{r}, v_0(\mathbf{r})]\{\Psi_0[\mathbf{r}|\mathbf{s}, v_0(\mathbf{r})] + \Delta\Psi\,[\mathbf{r}|\mathbf{s}, O(\mathbf{r})]\}d\mathbf{r} \tag{2.6}$$

(Slaney et al., 1984). O is the object function, or the perturbed velocity field expressed as:

$$\begin{aligned} O(\mathbf{r}) &= k_0^2(\mathbf{r})\left[1 - v_0^2(\mathbf{r})/v^2(\mathbf{r})\right] \\ &\approx 2k_0^2(\mathbf{r})\Delta v(\mathbf{r})/v(\mathbf{r}). \end{aligned} \tag{2.7}$$

G_0 is the Green's function or impulse response for the background medium: for constant velocity and three dimensions,

$$G_0(\mathbf{r}) = \frac{e^{ik_0|\mathbf{r}|}}{4\pi|\mathbf{r}|}; \tag{2.8}$$

for constant velocity and two dimensions,

$$G_0(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(k_0|\mathbf{r}|). \tag{2.9}$$

k_0 is the background wavenumber ω/v_0, and $H_0^{(1)}$ is a zero-order Hankel function of the first kind.

Equation 2.6 has a simple physical interpretation based on the principle of Huygens construction. It says the anomalous wavefield at a specific geophone is generated by superposition—that each point in the medium acts as a scatterer, emitting an impulse response with a magnitude equal to the product of the full wavefield and the object function at that point. In spite of this interpretation, the basic form of the equation remains complex, bearing little resemblance to the simple tomographic integrals of ray theory in the previous section. Equation 2.6 can be made to look much more tomographic by regrouping terms and introducing the concept of a *wavepath* $\mathcal{L}$:

$$\Delta\Psi(\mathbf{g}|\mathbf{s}) = \int \frac{\Delta v(\mathbf{r})}{v(\mathbf{r})}\mathcal{L}\left[\mathbf{r}|\mathbf{s},\mathbf{g},v(\mathbf{r})\right]d\mathbf{r}$$
$$\mathcal{L}\left[\mathbf{r}|\mathbf{s},\mathbf{g},v(\mathbf{r})\right] = 2k_0^2(\mathbf{r})G_0(\mathbf{g}-\mathbf{r})\left\{\Psi_0(\mathbf{r}|\mathbf{s}) + \Delta\Psi\left[\mathbf{r}|\mathbf{s},O(\mathbf{r})\right]\right\}. \qquad (2.10)$$

With this rewriting, the equation says the scattered complex amplitudes of wave-theoretic tomography correspond to integrals through the perturbed velocity field over monochromatic wavepaths $\mathcal{L}$—just as the traveltime delays of ray-theoretic tomography correspond to integrals through the perturbed velocity field over raypaths L.

As in the ray-trace application, $\mathcal{L}$ (specifically $\Delta\Psi$) is a function of Δv, and the problem is nonlinear. Under the Born approximation, the equation is linearized by assuming the wavepath $\mathcal{L}$ to be independent of the velocity perturbation ($\Delta\Psi << \Psi_0$), yielding the monochromatic analog of equation 2.4:

$$\Delta\Psi(\mathbf{g}|\mathbf{s}) = \int \frac{\Delta v(\mathbf{r})}{v(\mathbf{r})}\mathcal{L}_0(\mathbf{r}|\mathbf{s},\mathbf{g})d\mathbf{r}. \qquad (2.11)$$

For a point source at $\mathbf{s}$, Ψ_0 is the Green's function for the background medium, and

$$\mathcal{L}_0(\mathbf{r}|\mathbf{s},\mathbf{g}) = 2k_0^2(\mathbf{r})G_0(\mathbf{g}-\mathbf{r})G_0(\mathbf{s}-\mathbf{r}). \qquad (2.12)$$

Rytov

The first-order Rytov approximation creates a linear relation between velocity and wavefield-phase perturbations: $\Delta\Phi(\omega) = \ln[\Psi(\omega)] - \ln[\Psi_0(\omega)]$. As with the Born approximation, these complex phases are measured in the temporal-frequency domain for source-geophone pairs. The Rytov approximation begins with the wave equation written as:

$$\Delta\Phi(\mathbf{g}|\mathbf{s}) = \int \frac{G_0(\mathbf{g}-\mathbf{r})\Psi_0(\mathbf{r}|\mathbf{s})}{\Psi_0(\mathbf{g}|\mathbf{s})} \left\{ \left[\nabla\left(\Delta\Phi(\mathbf{r}|\mathbf{s}, O(\mathbf{r}))\right)\right]^2 + O(\mathbf{r}) \right\} d\mathbf{r} \tag{2.13}$$

(Slaney et al., 1984). This equation is more difficult to interpret physically than the Born equivalent. However, under the Rytov approximation $[\nabla(\Delta\Phi)]^2 << O$, and the equation becomes

$$\Delta\Phi(\mathbf{g}|\mathbf{s}) = \int O(\mathbf{r}) \frac{G_0(\mathbf{g}-\mathbf{r})\Psi_0(\mathbf{r}|\mathbf{s})}{\Psi_0(\mathbf{g}|\mathbf{s})} d\mathbf{r}. \tag{2.14}$$

This is just the Born equation with $\Delta\Psi$ replaced by $\Psi_0\Delta\Phi$. (Indeed, the Rytov formula reduces to the Born formula in the weak-scattering limit, where $\Psi_0\Delta\Phi = \Delta\Psi$ (Devaney, 1981).) Following the Born development, equation 2.14 can be rewritten as:

$$\Delta\Phi(\mathbf{g}|\mathbf{s}) = \int \frac{\Delta v(\mathbf{r})}{v(\mathbf{r})} \mathcal{L}_0(\mathbf{r}|\mathbf{s},\mathbf{g}) d\mathbf{r}$$

$$\mathcal{L}_0(\mathbf{r}|\mathbf{s},\mathbf{g}) = 2k_0^2(\mathbf{r}) \frac{G_0(\mathbf{g}-\mathbf{r})\Psi_0(\mathbf{r}|\mathbf{s})}{\Psi_0(\mathbf{g}|\mathbf{s})} \tag{2.15}$$

—and the scattered complex phases again viewed as integrals through the perturbed velocity field over wavepaths $\mathcal{L}_0$. For a point source at $\mathbf{s}$, Ψ_0 is the Green's function for the background medium and

$$\mathcal{L}_0(\mathbf{r}|\mathbf{s},\mathbf{g}) = 2k_0^2(\mathbf{r}) \frac{G_0(\mathbf{g}-\mathbf{r})G_0(\mathbf{s}-\mathbf{r})}{G_0(\mathbf{s}-\mathbf{g})}. \tag{2.16}$$

2.2.3 Step three: backprojection

Whereas ray-theoretic tomography forms a system of equations through consideration of a number of sources and geophones, wave-theoretic tomography forms a system of equations—

$$\mathcal{L}_0 \Delta v/v = \Delta\Psi \tag{2.17}$$

or,

$$\mathcal{L}_0 \Delta v/v = \Delta\Phi \tag{2.18}$$

—through consideration of a number of sources, geophones and frequencies. The extra dimension of information in wave-theoretic as compared to ray-theoretic tomography will become apparent in subsequent chapters.

The backprojection step of wave-theoretic tomography is usually formulated in the frequency-wavenumber domain, under the title of diffraction tomography. The method solves the problem analytically, for plane-wave scattering and independent monochromatic sources (Mueller et al., 1979; Mueller, 1980; Devaney, 1981, 1982, 1984; Slaney, 1984; Wu and Toksöz, 1987). With its emphasis on *wavepaths*, this thesis solves the problem numerically in the frequency-space domain—under the title of *wave-equation tomography.* This reformulation not only facilitates the comparison of ray and wave propagation paths, but also makes wave-theoretic tomography more flexible in dealing with irregularly sampled surveys and inhomogeneous background media. As with ray-theoretic tomography, the nonlinear part of the problem is attacked iteratively with steps two through four being repeated for successively updated background velocity fields.

Full waveform acoustic and elastic inversion have also been formulated in the time-space domain under the title of nonlinear inversion (Tarantola, 1984; Tarantola, 1987; Mora, 1987). In contrast to tomography, these implementations do not take complete linear steps between nonlinear iterations. The background wavefield is remodeled and the data perturbations recalculated after each gradient step in the linear optimization problem.