A comparison of two models for the elastic properties of fractured rock

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ABSTRACT

Models of the elastic properties of fractured rock, those proposed by Hudson and by Schoenberg are compared and shown to be equivalent. If the corrections to the stiffness matrix are transformed to the group domain suggested by Schoenberg and Muir they are greatly simplified. The angular variation of P-wave velocity in fractured rock is controlled by the relative magnitudes of two parameters in the Schoenberg/Muir group representation of the fracture system.

INTRODUCTION

There has been a recent increase in interest in the elastic properties of fractured rocks because Crampin (1978) and others have suggested that the presence of azimuthal anisotropy may be an indicator of aligned cracks. Schoenberg and Douma (1988) showed that Hudson's (1981) model of "penny shaped cracks" and Schoenberg's (1980) model of limited slip interfaces can predict the same values for the elastic properties of cracked rocks. I show that if the corrections to the stiffness matrix are transformed to the group domain suggested by Schoenberg and Muir (1989) they are greatly simplified. In particular if the cracks are have transverse isotropic symmetry they can be described by two scalar parameters in the group domain. The nature of the angular dependence of quasi-P wave velocity is controlled by the relative magnitudes of these two parameters. I compare the predicted angular variation for rocks with dry cracks, wet incompressable cracks and wet compressable cracks. The predictions for dry cracks have angular variations that agree with the measurements made by Nur and Simmons (1968).

THE GROUP DOMAIN FOR ELASTIC PROPERTIES

Schoenberg and Muir rearrange the 6×6 stiffness matrix into the following 3×3 matrices,

$$\mathbf{C}_{TT} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix}, \quad \mathbf{C}_{NN} = \begin{bmatrix} c_{33} & c_{34} & c_{35} \\ c_{34} & c_{44} & c_{45} \\ c_{35} & c_{45} & c_{55} \end{bmatrix}, \quad \mathbf{C}_{NT} = \begin{bmatrix} c_{13} & c_{14} & c_{15} \\ c_{23} & c_{24} & c_{25} \\ c_{36} & c_{46} & c_{56} \end{bmatrix}.$$

where $\mathbf{C}_{TT}$ relates transverse stresses and strains, $\mathbf{C}_{NN}$ relates normal stresses and strains and $\mathbf{C}_{NT}$ relates transverse stresses to normal strains and normal stresses to transverse strains. In this instance normal and transverse are defined relative to the 1-2 plane.

In a layered medium we know that the normal stress components and the transverse strain components are the same in all layers. Using this information it was shown in by Schoenberg and Muir that the static equivalent medium is obtained by mapping these three matrices plus thickness and density to a new domain, the group domain, and adding the group elements for the various layers. The equivalent medium is obtained by mapping back to the original domain.

The mapping to the group domain is given by

$$\begin{bmatrix} H \\ H\,\rho \\ H\,\mathbf{C}_{NN}^{-1} \\ H\,\mathbf{C}_{NT}\,\mathbf{C}_{NN}^{-1} \\ H\,[\,\mathbf{C}_{TT} - \mathbf{C}_{NT}\mathbf{C}_{NN}^{-1}\mathbf{C}_{NT}^{T}\,] \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{g}(1) \\ \mathbf{g}(2) \\ \mathbf{g}(3) \\ \mathbf{g}(4) \\ \mathbf{g}(5) \end{bmatrix},$$

where H is the layer thickness and ρ is the layer density.

Simple symmetries

It is instructive to look at the group domain representation of some simple symmetries.

For an isotropic layer with thickness z density ρ and Lamé constants λ and μ the group representation is,

$$\begin{aligned}
\mathbf{g}(1) &= z \\
\mathbf{g}(2) &= z\rho \\
\mathbf{g}(3) &= z\begin{bmatrix} 1/(\lambda+2\mu) & 0 & 0 \\ 0 & 1/\mu & 0 \\ 0 & 0 & 1/\mu \end{bmatrix} \\
\mathbf{g}(4) &= z\lambda\begin{bmatrix} 1/(\lambda+2\mu) & 0 & 0 \\ 1/(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\mathbf{g}(5) = z2\mu \begin{bmatrix} 2(\lambda+\mu)/(\lambda+2\mu) & \lambda/(\lambda+2\mu) & 0 \\ \lambda/(\lambda+2\mu) & 2(\lambda+\mu)/(\lambda+2\mu) & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

for a transverse isotropic layer with a vertical axis of symmetry the group representation is,

$$\mathbf{g}(1) = z$$

$$\mathbf{g}(2) = z\rho$$

$$\mathbf{g}(3) = z \begin{bmatrix} 1/c_{33} & 0 & 0 \\ 0 & 1/c_{44} & 0 \\ 0 & 0 & 1/c_{44} \end{bmatrix}$$

$$\mathbf{g}(4) = z \begin{bmatrix} c_{13}/c_{33} & 0 & 0 \\ c_{13}/c_{33} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{g}(5) = z \begin{bmatrix} c_{12}-c_{13}^2/c_{33}+2c_{66} & c_{12}-c_{13}^2/c_{33} & 0 \\ c_{12}-c_{13}^2/c_{33} & c_{12}-c_{13}^2/c_{33}+2c_{66} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} .$$

In particular notice that adding a diagonal matrix of the form

$$\begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y \end{bmatrix}$$

to the third group element of an isotropic layer will produce a transverse isotropic layer.

The limited slip interface model of fractures

In the Schoenberg/Muir formalism fractures are treated as very thin very compliant layers (Schoenberg and Douma, 1988). The fracture filling material is assumed to be softer than the background material. The elastic moduli $c_{ij,f}$ of the fracture are much smaller than a typical non-zero modulus of the background material e.g. c_{33}.

In deriving the effect of the fractures it is assumed that the moduli are of the order of the volume fraction of the fractures, i.e. the thickness of fractures in a unit interval. If this volume ratio is h_f we require $c_{ij,f}/c_{33} = O(h_f)$. This assumption means that in the limit as $h_f \to 0$ we can write $c_{ij,f}$ as $h_f \tilde{c}_{ij}$. As h_f becomes small the fracture layers become soft layers that act as linear slip interfaces. Across each slip interface, the traction components are continuous, as with any layer in the static case. However the components of displacement need not be continuous, indicating that the strains in the fractures can become infinite (i.e. slip can occur) as the elastic moduli approach zero.

We can write the submatrices of the soft layer as,

$$C_{TT} = h_f \tilde{C}_{TT}, \quad C_{NN} = h_f \tilde{C}_{NN}, \quad C_{NT} = h_f \tilde{C}_{NT}.$$

If the total thickness of the fractured medium is H and the thickness of fractures is $H_f = h_f H$ we can write the group mapping of the fracture layers.

$$\begin{bmatrix} \mathbf{g}(1) \\ \mathbf{g}(2) \\ \mathbf{g}(3) \\ \mathbf{g}(4) \\ \mathbf{g}(5) \end{bmatrix} = \begin{bmatrix} h_f H \\ h_f H \, \rho \\ H \, \tilde{\mathbf{C}}_{NN}^{-1} \\ h_f H \, \tilde{\mathbf{C}}_{NT} \, \tilde{\mathbf{C}}_{NN}^{-1} \\ h_f H \, [\, \tilde{\mathbf{C}}_{TT} \; - \; \tilde{\mathbf{C}}_{NT} \tilde{\mathbf{C}}_{NN}^{-1} \tilde{\mathbf{C}}_{NT}^{T} \,] \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ H \, \tilde{\mathbf{C}}_{NN}^{-1} \\ 0 \\ 0 \end{bmatrix}$$

Therefore a particular fracture system is characterized by the symmetric 3×3 matrix $\tilde{\mathbf{C}}_{NN}^{-1}$ with six independent constants. We can follow Schoenberg and Douma (1988) and define the excess compliance due to the fracture system $\mathbf{Z} = \tilde{\mathbf{C}}_{NN}^{-1}$.

If we denote the background medium by the subscript b we can write the group form of the fractured medium as,

$$\begin{bmatrix} H \\ H \, \rho_b \\ H \, (\mathbf{C}_{NN,b}^{-1} + \mathbf{Z}) \\ H \, \mathbf{C}_{NT,b} \, \mathbf{C}_{NN,b}^{-1} \\ H \, [\, \mathbf{C}_{TT,b} \; - \; \mathbf{C}_{NT,b} \mathbf{C}_{NN,b}^{-1} \mathbf{C}_{NT,b}^{T} \,] \end{bmatrix}$$

The fracture system compliance matrix can have many symmetries. One of the simplest forms is the transversely isotropic fracture system,

$$\mathbf{Z} = \begin{bmatrix} Z_N & 0 & 0 \\ 0 & Z_T & 0 \\ 0 & 0 & Z_T \end{bmatrix} .$$

The tangential fracture displacements and tangential fracture stresses are collinear and the normal and tangential compliances are uncoupled. Schoenberg and Douma (1988), show that for a transverse isotropic fracture system in an isotropic medium the variation of phase velocity with angle from the vertical, θ, is approximately

$$\begin{aligned} v_{sh}^2(\theta) &\approx \frac{\mu_b}{\rho_b}(1 - E_T \cos^2 \theta) \ , \\ v_{qs}^2(\theta) &\approx \frac{\mu_b}{\rho_b}(1 - E_T \cos^2 2\theta - \gamma_b E_N \sin^2 2\theta) \ , \\ v_{qp}^2(\theta) &\approx \frac{\lambda_b + 2\mu_b}{\rho_b}(1 - \gamma_b E_T \sin^2 2\theta - E_N (1 - 2\gamma_b \sin^2 \theta)^2) \ , \end{aligned}$$

where $E_N = c_{33,b} Z_N$, $E_T = c_{44,b} Z_T$ and $\gamma_b = \mu_b / (\lambda_b + 2\mu_b)$. The variables E_T and E_N may be interpreted as the dimensionless excess compliances that specify the fracture system compliances as a proportion of the background medium compliances.

$$Z_N = \frac{1}{c_{33}} E_N \ , \ Z_T = \frac{1}{c_{44}} E_T .$$

In this approximation terms of order E^2 were ignored, this means that the fracture system compliances should be a small fraction of the background compliances.

The pure shear wave velocity, v_{sh}, is always 2θ dependent and the quasi shear wave velocity, v_{qs}, is always 4θ dependent. However the quasi-P wave velocity, v_{qp}, has an angular dependence that depends on the relative magnitudes of the E_T and E_N terms. The excess tangential compliance gives a 4θ dependence and the excess normal compliance gives a 2θ dependence. Note that because the term involving E_T is the same at 0 and 90 degrees any difference in quasi-P wave velocity between waves traveling horizontally and vertically is controlled only by the excess normal compliance parameter, E_N.

This prediction is interesting in the light of the experimental results obtained by Nur and Simmons who observed a 2θ variation of both the quasi shear and quasi-P wave velocities of rocks with dry fractures. This means that if the assumptions made are correct the normal compliance must be greater than the tangential compliance. The assumptions made in deriving this result are, (1) small fracture compliance, (2) isotropy of the background medium and (3) transverse isotropy of the fracture system.

HUDSON'S MODEL OF FRACTURING

Hudson's (1981) model of the elastic properties of fractured rocks is based on Keller's (1964) equations for the mean field in a solid with a random distribution of inclusions. In this paper I will only consider Hudson's expressions to the first order in crack density, $\epsilon = Na^3/v$. Where N is the number of penny shaped cracks of radius a in volume v in an isotropic solid with Lamé constants λ and μ.

This is equivalent to only considering single scattering and ignoring crack/crack interaction. For this approximation to be valid the crack density must be low, $\epsilon << 1$. If this condition is satisfied the elastic constants of the cracked medium c_{ijkl} are given by a simple additive correction to the elastic constants of the uncracked medium ${c^0}_{ijkl}$:

$$c_{ijkl} = {c^0}_{ijkkl} + {c^1}_{ijkl}.$$

For cracks with normals aligned in the x_3 direction

$$c^1 = -\epsilon/\mu \begin{bmatrix} \lambda^2 U_{33} & \lambda^2 U_{33} & \lambda(\lambda+2\mu)U_{33} & 0 & 0 & 0 \\ \lambda^2 U_{33} & \lambda^2 U_{33} & \lambda(\lambda+2\mu)U_{33} & 0 & 0 & 0 \\ \lambda(\lambda+2\mu)U_{33} & \lambda(\lambda+2\mu)U_{33} & (\lambda+2\mu)^2 U_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu^2 U_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2 U_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the quantity U_{kl} n the equation above is the integrand over the crack face of μ/a times the k component of displacement discontinuity across the crack due to tractions in the l direction. This value depends on the boundary conditions at the crack face.

If these relations are transformed into the group domain we obtain expressions for the changes in the group elements that are equivalent to Hudson's equations for cracked rocks.

These changes are,

$$\begin{aligned}
\Delta \mathbf{g}(1) &= 0 \\
\Delta \mathbf{g}(2) &= 0 \\
\Delta \mathbf{g}(3) &= \begin{bmatrix} \frac{-U_{33}\epsilon z}{U_{33}\epsilon(\lambda+2\mu)-\mu} & 0 & 0 \\ 0 & \frac{-U_{11}\epsilon z}{U_{11}\epsilon\mu-\mu} & 0 \\ 0 & 0 & \frac{-U_{11}\epsilon z}{U_{11}\epsilon\mu-\mu} \end{bmatrix} \\
\Delta \mathbf{g}(4) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Delta \mathbf{g}(5) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

This is the same as the result obtained for fracture systems with transverse isotropic symmetry in the limited slip interface model. We now have a formula for calculating the fracture system excess compliances from the the parameters in Hudson's model of penny shaped cracks.

$$Z_N = \frac{-U_{33}\epsilon z}{U_{33}\epsilon(\lambda + 2\mu) - \mu}, \quad Z_T = \frac{-U_{11}\epsilon z}{U_{11}\epsilon\mu - \mu}$$

As discussed previously changing the group form of a isotropic layer in this way produces a transversely isotropic layer.

Long wavelength approximation

In the long wavelength limit Hudson(1981) derives expressions for U_{11} and U_{33} for penny shaped cracks with aspect ratio $\alpha << 1$ and a weak filling medium with isotropic constants λ' and μ'.

These are,

$$U_{11} = \frac{16}{3}\frac{\lambda + 2\mu}{3\lambda + 4\mu}\frac{1}{1 + M},$$

$$U_{33} = \frac{4}{3}\frac{\lambda + 2\mu}{\lambda + \mu}\frac{1}{1 + K},$$

where

$$K = \frac{\kappa' + 4/3\mu'}{\pi\alpha\mu}\frac{\lambda + 2\mu}{\lambda + \mu},$$

$$M = \frac{4\mu'}{\pi\alpha\mu}\frac{\lambda + 2\mu}{3\lambda + 4\mu},$$

and $\kappa' = \lambda' + 2/3\mu'$ is the bulk modulus of the filling material.

In deriving these formulae terms of order α, $\alpha^2\mu/\mu'$ and $\alpha^2\mu/\kappa'$ have been neglected relative to $\alpha\mu/\mu'$ and $\alpha\mu/\kappa'$. In order for U_{11} and U_{33} to be non-zero as the aspect ratio goes to zero we require that μ' and κ' be of order α. This is the same requirement as we had in deriving the limited slip interface model.

Incompressible fluid filled cracks

If the cracks are filled with fluid that has a high bulk modulus or the crack is so narrow that there is no displacement discontinuity normal to the crack we have, $U_{33} = 0$. (I.e. κ' does not go to zero as $\alpha \to 0$.) This give a fracture correction of

$$\Delta \mathbf{g}(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-3(3\lambda+4\mu))} & 0 \\ 0 & 0 & \frac{16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-3(3\lambda+4\mu))} \end{bmatrix}.$$

If $U_{33} = 0$ then E_N will be zero so the quasi-P wave velocity variation will only depend on E_T and will vary as a function of 4θ.

Dry cracks

For the case of dry cracks μ' and λ' are zero so the corrections become.

$$\Delta \mathbf{g}(3) = \begin{bmatrix} \frac{-4\epsilon z(\lambda+2\mu)}{4\epsilon(\lambda+2\mu)^2-3\mu(\lambda+\mu)} & 0 & 0 \\ 0 & \frac{-16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-9\lambda-12\mu)} & 0 \\ 0 & 0 & \frac{-16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-9\lambda-12\mu)} \end{bmatrix},$$

To determine the angular variation of quasi-P wave velocity we must look at the relative magnitudes of E_N and E_T.

$$E_N = (\lambda + 2\mu)\frac{-4\epsilon z(\lambda + 2\mu)}{4\epsilon(\lambda + 2\mu)^2 - 3\mu(\lambda + \mu)},$$

$$E_T = \mu\frac{-16\epsilon z(\lambda + 2\mu)}{\mu(16\epsilon(\lambda + 2\mu) - 9\lambda - 12\mu)}.$$

To see a 2θ variation we require $E_N >> E_T$. Dividing by $4\epsilon z$ and ignoring terms involving ϵ (as $\epsilon << 1$) we have,

$$(\lambda + 2\mu)\frac{-(\lambda + 2\mu)}{-3\mu(\lambda + \mu)} >> \mu\frac{-4(\lambda + 2\mu)}{\mu(-9\lambda - 12\mu)}.$$

This reduces to,

$$(\lambda + 2\mu)(9\lambda + 12\mu) >> 12\mu(\lambda + \mu)$$

For all reasonable values of λ and μ this is true, so we expect dry cracks to always produce a 2θ dependence in the quasi-P wave velocity. This is what Nur and Simmons observed in their measurements of wave speeds in granite under uniaxial stress.

Wet cracks

If the cracks have a fluid filling with bulk modulus $\kappa' = \lambda'$ the corrections become

$$\Delta \mathrm{g}(3) = \begin{bmatrix} \frac{-4\pi\epsilon z(\lambda+2\mu)}{(\lambda+2\mu)(4\pi\epsilon(\lambda+2\mu)-3\lambda'/\alpha)-3\pi\mu(\lambda+\mu)} & 0 & 0 \\ 0 & \frac{-16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-9\lambda-12\mu)} & 0 \\ 0 & 0 & \frac{-16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-9\lambda-12\mu)} \end{bmatrix}.$$

As the value of λ'/α becomes large $E_N \rightarrow 0$, so this will reduce to the formula for incompressible wet cracks given previously. To observe a 2θ dependence of quasi-P wave velocity we require $E_N >> E_T$. In this case we have,

$$\begin{aligned} E_N &= (\lambda+2\mu)\frac{-4\pi\epsilon z(\lambda+2\mu)}{(\lambda+2\mu)(4\pi\epsilon(\lambda+2\mu)-3\lambda'/\alpha)-3\pi\mu(\lambda+\mu)}, \\ E_T &= \mu\frac{-16\epsilon z(\lambda+2\mu)}{\mu(16\epsilon(\lambda+2\mu)-9\lambda-12\mu)} \end{aligned}$$

If we ignore terms in ϵ we have to compare the terms,

$$(\lambda+2\mu)\frac{-\pi(\lambda+2\mu)}{(\lambda+2\mu)(-3\lambda'/\alpha)-3\pi\mu(\lambda+\mu)}$$

and

$$\mu\frac{-4(\lambda+2\mu)}{\mu(-9\lambda-12\mu)}$$

Figure 1 shows these two values plotted for a range of aspect ratios for water filled cracks in a rock matrix. The water has a P-wave velocity of 1500m/s and density of 1g/cc and the rock matrix has a P-wave velocity of 4000m/s, S-wave velocity of 2500m/s and a density of 4g/cc. For aspect ratios less than .01 we see that $E_T > E_N$ so the velocity will vary as a function of 4θ. For larger aspect ratios the velocity will vary as a function of 2θ.

Cracks filled with weak solids

For the case of a filling material with coefficients λ' and μ' the corrections become,

$$\begin{aligned} Z_N &= \frac{-4\pi\epsilon z(\lambda+2\mu)}{(\lambda+2\mu)(4\pi\epsilon(\lambda+2\mu)-3(\lambda'+2\mu')/\alpha)-3\pi\mu(\lambda+\mu)} \\ Z_T &= \frac{-16\pi\epsilon z(\lambda+2\mu)}{(\lambda+2\mu)(16\pi\epsilon\mu-12\mu'/\alpha)-3\pi\mu(3\lambda+4\mu)} \end{aligned}$$

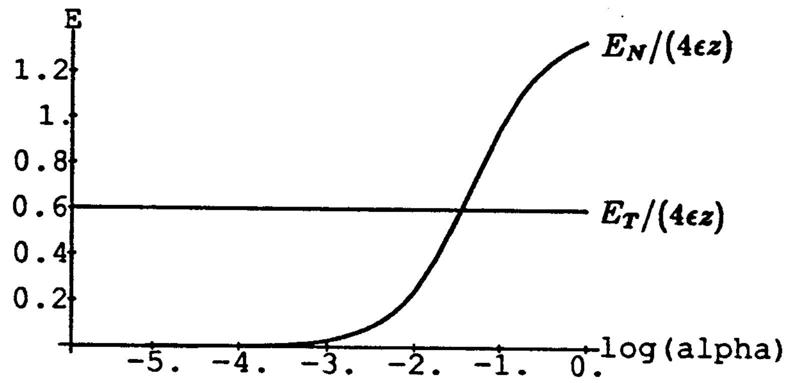

FIG. 1. $E_N/(4\epsilon z)$ and $E_T/(4\epsilon z)$ plotted against log_{10} of the aspect ratio

This result can be used to predict the angular variation of quasi-P wave velocity for any weak filling material by comparing the magnitudes of E_N and E_T. If the elastic constants of the filling go to zero the formula becomes that for dry cracks.

Limitations of Hudson's method

Schoenberg and Douma(1988) discuss the region of validity of Hudson's method by comparing the results with the model of Nishizawa(1982) which does not have the approximation of small aspect ratio. They come to the conclusion that Hudson's model is valid for aspect ratio's up to 0.3 for small values of ϵ. By using only the single scattering form we are restricted to crack densities that are small. Schoenberg and Douma suggest a limit of $\epsilon < .05$ for Hudson's model.

In this paper I have used the single scattering form of Hudson's model. Hudson derived a second order correction that honors the effects of crack-crack interactions. However the second order correction does not map to a simple representation in the group domain.

CONCLUSIONS

The Schoenberg/Muir model of limited slip interfaces and Hudson's first order model for penny shaped cracks produce identical changes in the elastic constants of a rock. Schoenberg and Douma have shown how to predict the angular dependence of quasi-P wave velocity from the parameters of the cracking. In this paper I have shown that the mapping of the Hudson model in the group domain is particularly simple. In future it may be preferable to work in the group domain so that the effects of layering and fracturing can be calculated in a uniform manner.

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