Wavefield separation in anisotropic media (and why it's not what you expect)

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ABSTRACT

The divergence and curl operators are commonly used to separate P and S waves in two-dimensional isotropic media (Clayton, 1981). In the Fourier domain a wavefield is viewed as a sum of plane wave components. In this domain the divergence and curl operators separate P and S waves by their orthogonal particle motion directions on each plane wave.

In generalizing this simple scheme to three dimensions, a complication arises even for the isotropic case. It is not enough merely to identify three wavetypes at each (k_x, k_y, k_z); three *global* modes must be identified for the Fourier-domain *as a whole*. Ideally each mode should be continuous and well behaved everywhere. Unfortunately, in three dimensions a mode cannot exhibit quasi-S motion everywhere and still be continuous.

A more confusing problem, mode-mode coupling, occurs with certain anisotropic media. For isotropic or transversely isotropic media, three modes can always be picked out (usually called quasi-P, quasi-SV, and SH). One might expect that slight perturbations away from transverse isotropy to orthorhombic symmetry should always leave these three modes identifiable. Instead, the three original modes can exchange fragments and pull apart from each other. This creates new continuous distinct modes that appear to be patched together out of the fragments of the original quasi-P, quasi-SV, and SH modes. This phenomena can create new arrivals in the impulse response of the orthorhombic medium with no counterpart in the unperturbed transversely isotropic medium (Crampin, 1981).

INTRODUCTION

An elastic migration or inversion program must be able to handle both P and S waves. Normally the two modes are separated from each other and each is treated independently. In the standard isotropic two-dimensional case the wave mode separation operators usually used are just $\nabla\cdot$ and $\nabla\times$.

An *anisotropic* elastic migration or inversion program must separate quasi-P and quasi-S waves. Fortunately, constructing wave separation operators proves to be straightforward in the (k_x, k_y) domain for any media with known elastic constants. The same method also easily generalizes to handle the three-dimensional case, in which there are three distinct modes (normally referred to as "P", "SV", and "SH" for isotropic media).

Unfortunately, a problem arises to bedevil the would-be wavefield separator: to be well separated the three modes must be well defined and everywhere continuous. I am forced to conclude that the very idea of wavefield separation for three-dimensional anisotropic media is a philosophically problematical. It is impossible to define a quasi-S mode that is everywhere continuous, even for the isotropic case. In the anisotropic case, things are even worse: even slight perturbations away from isotropy can cause the formerly identifiable SV and SH modes to exchange parts and separate, leaving behind "pure modes" that look like SV-like and SH-like fragments strangely patched together.

This crossing-over of the original SV and SH modes produces concavities in the slowness surfaces that are associated with cusps in the associated group velocity surfaces (Dellinger and Muir, 1985, SEP–**44**). This cusp is a new kind of event that has no counterpart in isotropic or transversely isotropic media; it only appears in media with orthorhombic (or worse) anisotropy. Interestingly, it also goes to zero amplitude on planes of symmetry. Slices of orthorhombic wavefields in the xy, yz, or xz planes will not show it. I show where to look for this event in this paper. The next step, not yet completed, is to demonstrate a clear example in the output of a standard elastic modeling program.

MODE SEPARATION

The isotropic, two-dimensional case

The divergence operator, $\nabla\cdot$, has the useful property that when applied to a two-dimensional elastic isotropic wavefield, it passes only P waves. Similarly, the curl operator, $\nabla\times$, passes only S waves. This wave separation property is commonly exploited by elastic inversion and migration programs to allow separate treatment of P and S waves. Since the divergence and curl operators operate in fundamentally the same way, we will only discuss the divergence operator here.

The divergence of a two-dimensional elastic wavefield $\vec{U}$ is

$$\frac{\partial U_x(x,y)}{\partial x} + \frac{\partial U_y(x,y)}{\partial y}. \qquad (1)$$

In the Fourier domain this is

$$i\ k_x\ \hat{U}_x(k_x,k_y) + i\ k_y\ \hat{U}_y(k_x,k_y), \tag{2}$$

where $\hat{\vec{U}}$ is the Fourier-domain representation of $\vec{U}$.

A more illuminating way to write equation (2) is

$$i\ k \quad (\vec{l}\cdot\hat{\vec{U}}), \tag{3}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wavenumber at (k_x, k_y) and $\vec{l}$ is the unit vector $(k_x/k, k_y/k)$ pointing in the direction of (k_x, k_y). Remember that each point (k_x, k_y) in the Fourier domain represents a plane wave traveling in the direction $\vec{l}$ with spatial frequency k. In general, both kinds of waves will be present at every point. For a plane wave in an isotropic medium, the particle motion is very simple: P waves have particle motion along the direction of travel ($\parallel \vec{l}$) and S waves have particle motion perpendicular to the direction of travel ($\perp \vec{l}$). Thus, the $\vec{l}\cdot\hat{\vec{U}}$ term passes P waves through unchanged (because the P particle motion points in the same direction as $\vec{l}$) but zeros S waves (because the S particle motion is perpendicular to $\vec{l}$).

The divergence operator is not quite a "perfect" wavetype separation operator, however, because of the additional presence of the k term. This term acts as a derivative operator, amplifying higher spatial frequencies. An operator that rejects S waves and passes P waves unchanged can be constructed by inverse Fourier transforming the operator $i\,\vec{l}\cdot$. Figure 1 shows a graphical Fourier-domain representation of this vector operator and the related operator that passes S waves.

The anisotropic, two-dimensional case

This method generalizes to the anisotropic case in the obvious way. The only change is that the particle motion directions for the wave modes are no longer simply parallel or perpendicular to $\vec{l}$, but may be complicated functions of k_x and k_y. In the Fourier domain our operator will now have the form $i\ \vec{v}(k_x, k_y)\cdot$, where $\vec{v}$ is a unit vector giving the particle motion direction of a plane wave of the desired mode traveling in the direction (k_x, k_y).

How do we find the required particle motion directions? The Christoffel equation is just the elastic wave equation Fourier transformed over space and time. It has the form

$$\mathbf{M}\,\mathbf{C}\,\mathbf{M}^T\ \vec{v} = -\rho\omega^2\ \vec{v}, \tag{4}$$

where ρ is density, $\mathbf{M}$ is a matrix involving only k and $\vec{l}$, $\mathbf{C}$ is the stiffness matrix that defines the elastic properties of the medium, and $\vec{v}$ is the particle motion direction (Auld, 1973). For a given $\mathbf{C}$, k, and $\vec{l}$, equation (4) always has two orthonormal solutions $\vec{v}_1$ and $\vec{v}_2$ (three in the three-dimensional case). Each of these solutions corresponds to one wave mode.

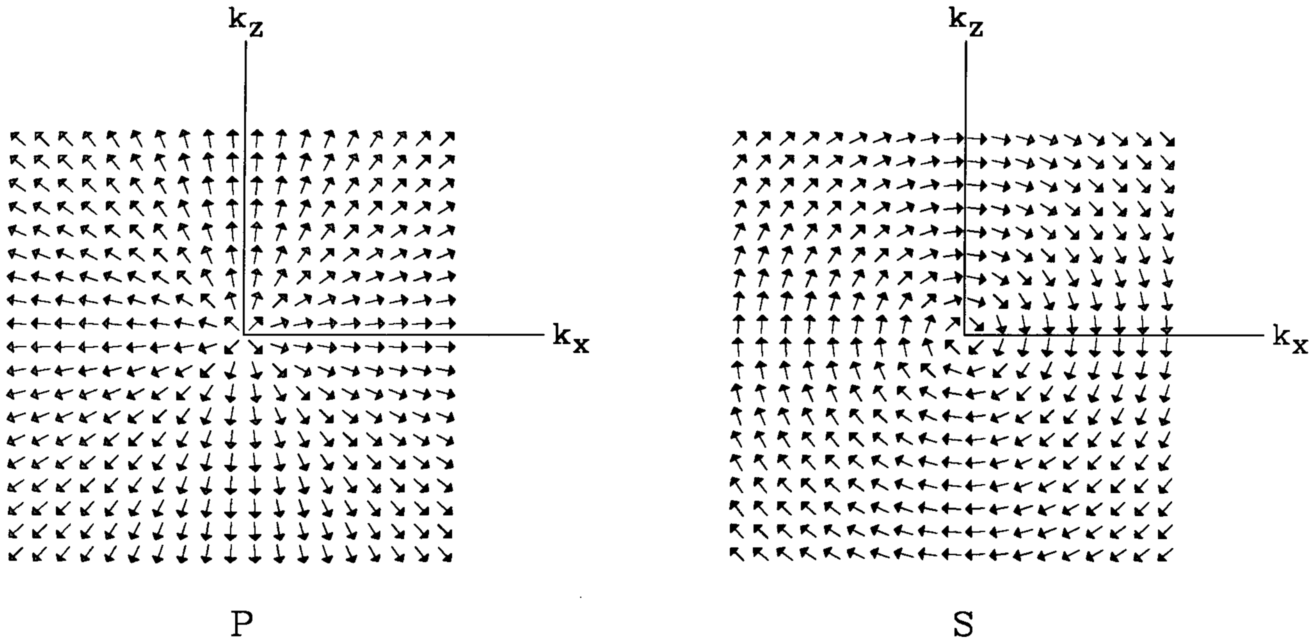

FIG. 1. Fourier-domain plots of the operator for passing only P waves (left) and S waves (right) for isotropic media.

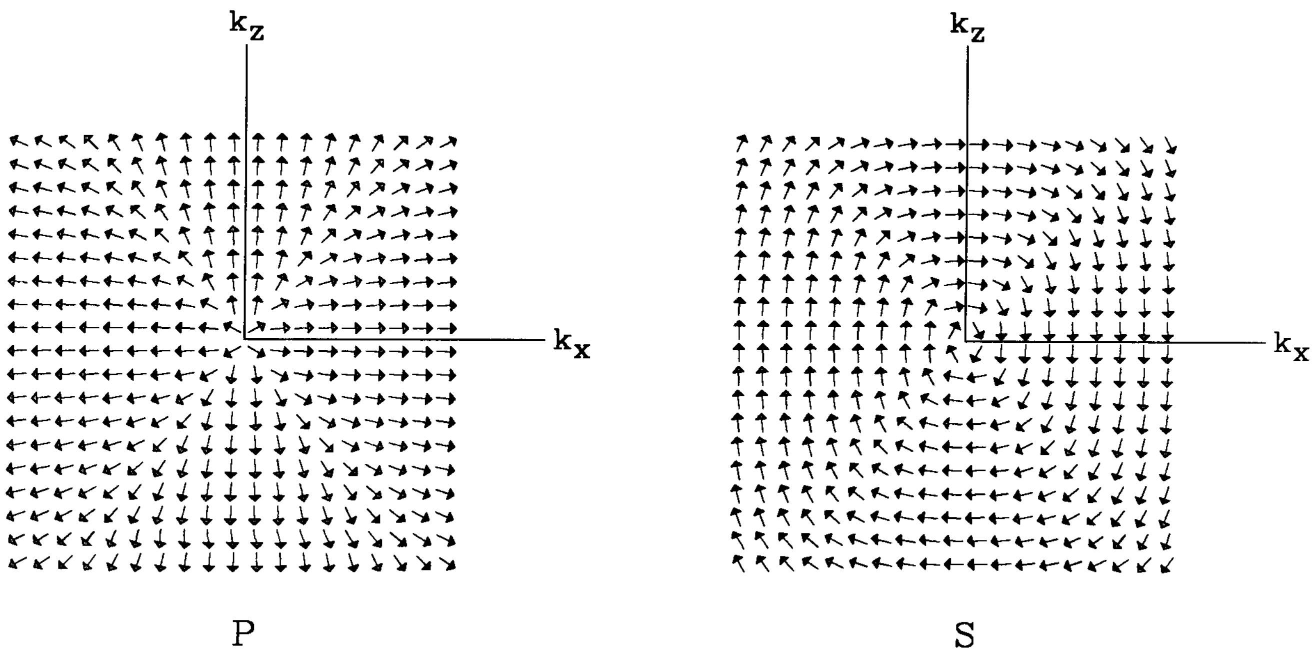

FIG. 2. Fourier-domain plots of the operator for passing only P waves (left) and S waves (right) for the anisotropic media used in Figure 3.

Now we have everything we need to extend our isotropic wave separation method to handle waves in anisotropic media.

(1.) Decide which mode ("quasi-P or quasi-S") this operator will pass.

(2.) For all (k_x, k_y):

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(A.) Substitute k_x and k_y into equation (4) and find the two solutions $\vec{v}_1$ and $\vec{v}_2$.

(B.) Decide whether solution 1 or 2 is the desired mode chosen in step (1). Call this solution $\vec{v}$.

(C.) Both $\vec{v}$ and $-\vec{v}$ are equally valid normalized solutions. Choose the one that is consistent with particle motion directions already determined for adjacent (k_x, k_y) points.

(D.) Store the result for this (k_x, k_y) in $\vec{L}(k_x, k_y)$.

}

(3.) Inverse Fourier transform the operator $\vec{L}\cdot$ to get the (x, y)-domain wave separation operator for this mode in this media.

An example

To demonstrate the method, I used John Etgen's anisotropic two-dimensional finite-difference elastic modeling program to generate a wavefield. The model is split into two parts. The left half is isotropic, and the right half is anisotropic. Figure 3 shows what happens when wavefield separation operators designed for each half are applied to the whole. The operators only work in the half of the medium they are tailored to.

Extending to three dimensions

This method easily extends to three dimensions. Instead of (k_x, k_y), there is now (k_x, k_y, k_z). The Christoffel equation has three solutions, corresponding to three different wave modes. For isotropic media, these three modes are normally described as "P", "SV", and "SH". For transversely isotropic media, the three modes are usually similar enough to the isotropic ones for them to be unambiguously referred to as "quasi-P", "quasi-SV", and "SH".[1]

[1] In transverse isotropy, it is perfectly valid to have $C_{44} > C_{11}$ or C_{33}, which means that pure SV waves can propagate faster than pure P waves in certain directions! When this happens, the names "quasi-P" and "quasi-SV" cease to have much meaning, as pure P and pure SV waves occur connected together on the same mode (Dellinger and Muir, 1985, SEP–**42**). This is known to happen in some pure crystals, and Helbig and Shoenberg (1987) show that it can also happen as a result of fine-layering together two "normal" media. I'm going use the usual sloppy "quasi-" terminology here anyway, as it is more widely understood.

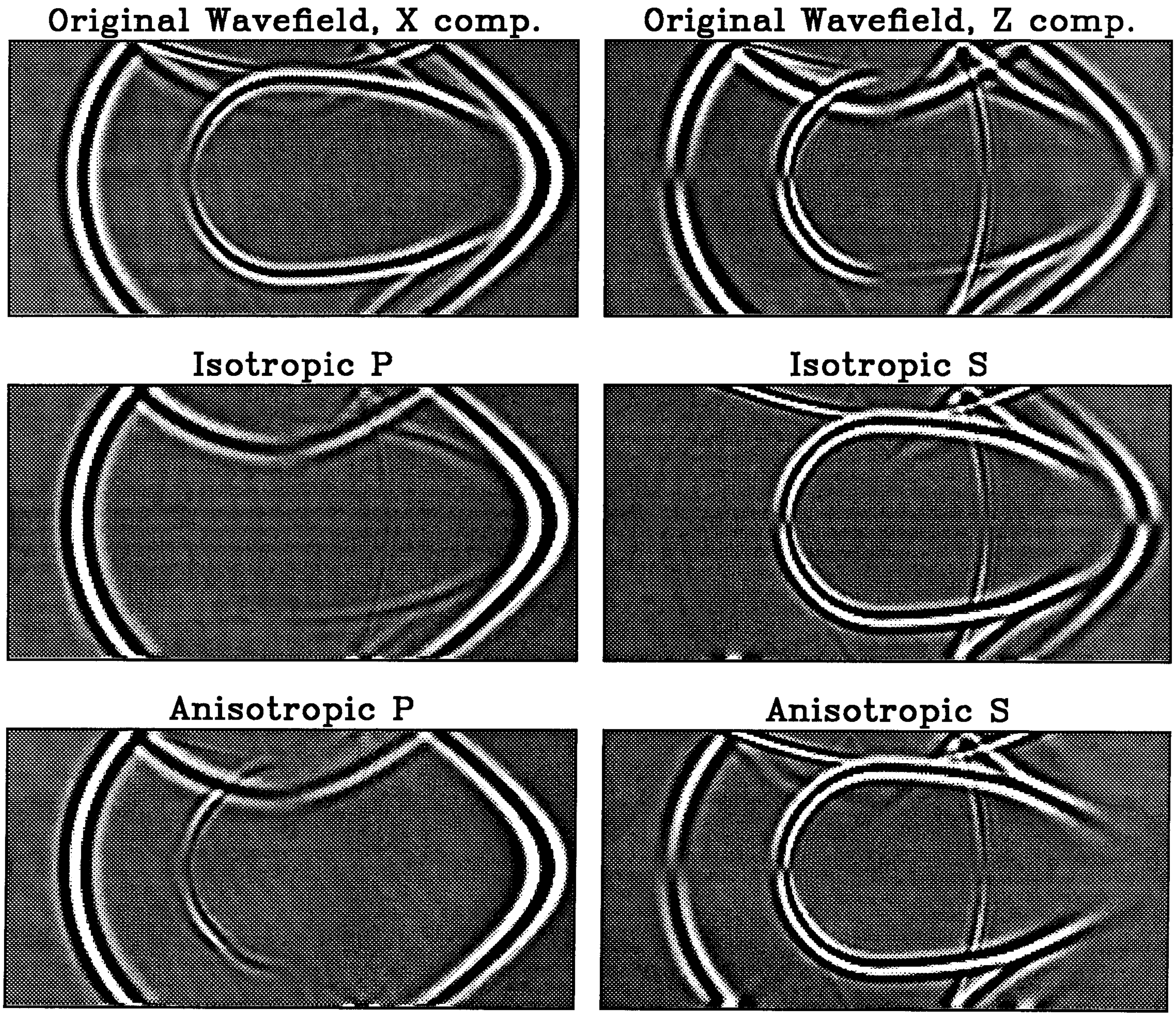


FIG. 3. Top: Snapshot of the X (left) and Z (right) components of a finite-difference wavefield generated by a horizontal force at the center of the model. The left half of the model is isotropic, and the right half is anisotropic. The density and vertical S and P velocities were chosen to be the same in both halves. Middle: The result of applying wavefield separation operators designed to work in the left (isotropic) half of the model. Bottom: The same for the right (anisotropic) half. All plots are plotted with the same parameters. Gain has been applied to make weak events (and artifacts) visible.

PITFALLS IN THREE-DIMENSIONAL MODE SEPARATION

A wave separation operator can perform correctly at each individual point in the Fourier domain, and yet fail to work properly as a whole: it must also be continuous everywhere. Local discontinuities in the Fourier domain create serious global artifacts in the space domain.

Our two-dimensional wave operators are discontinuous at zero spatial frequency (look at the center of the plots in Figure 1 and 2), but this is not a problem because wavefields never have any energy there.

Unfortunately, wave separation in three dimensions turns out to be more complicated than two. Things can go wrong in steps (2B) and (2C) of our algorithm even for seemingly "simple" cases. In part (2C), we may find that no matter what we do we always reach a point where neither sign is continuous in all directions with what we've already built up; in part (2B) we may find we always reach a point where none of the three modes are continuous in all directions with what we've already built up. Finally, we may find that even though we can construct a mode that is continuous everywhere except at a few points, what we get is nothing like the quasi-P, quasi-SH, and quasi-SV modes we intuitively expect. This can happen even for only marginally anisotropic media that otherwise seem very well behaved.

Problems in (2C): the "furry ball" theorem

Imagine a spherical ball covered with fur of a uniform length. The ball represents a wavefront; the fur on the ball represents particle motion direction. A ball with the fur sticking straight out everywhere would represent a pure P mode.

What about a pure S mode? Pure S-type motion only requires that each hair be flat against the ball's surface. We want more than that; we want a pure S mode with the particle motion continuous everywhere. Imagine what happens if you try to make such a mode by combing the fur on the ball flat and regular everywhere. As you work out from your starting point you are forced into making a whorl on each end. This whorl is a discontinuity: a pure S mode cannot be continuous everywhere. This theorem is a familiar one in radio antenna design; there it says that any antenna must have at least two nulls.

This theorem also applies to general anisotropic wave modes as well. The trick is to consider only the pure S component of the particle motion at each point. This allows a loophole of sorts in the theorem: instead of making a whorl, the S component of the particle motion can go to zero in a smooth fashion. Since the particle motion itself is always unit magnitude, however, such a point represents a pure P mode direction. So even for quasi-S waves we either have whorls (discontinuities) or a few hairs sticking up (mixed quasi-S and quasi-P waves together on one mode).

Problems in (2B): shear wave singularities

Steps (1) and (2B) contain the implicit assumption that there really are three distinct global modes to choose between, and that each mode may be considered in isolation from the others. We are fooled into thinking this is true because we so often look at the transverse isotropic case, where there really are clear global quasi-P, quasi-SV, and SH modes, and each of these defines a continuous surface by itself. In the orthorhombic case we usually only look at two-dimensional slices through the symmetry planes. Reassuringly, each such slice shows a quasi-P, quasi-SV, and quasi-SH wave, seemingly no more complicated than the transverse isotropic case.

Unfortunately, if you consider all three slices *together*, things no longer seem so simple. (See Figure 4.) The two shear surfaces interlock with one another to create a single two-sheeted continuous surface (Crampin, 1981). We can consider this single two-sheeted surface as two independent surfaces that happen to touch, but we create an artificial discontinuity by doing so. This is the first hint we have of the awful complexity that awaits us in true three-dimensional anisotropy.

More problems in (2B): mode-mode coupling

In Figure 4, we see that the two shear surfaces interconnect in a strange way. Figure 5 shows slices along four different longitude lines through a slowness surface for another orthorhombic medium.

The longitude 0° (upper left) and 90° (lower right) plots, which are symmetry planes, show intersection points like those we saw before in Figure 4. The upper intersection point on the longitude 0° plot performs the same SV-SH swapping role that the single intersection point in the previous example did.

Looking at these symmetry planes, there appear to be two modes, a quasi-SV and a quasi-SH one, just like in transverse isotropy. These two modes intersect but otherwise seem completely independent.

What happens off the symmetry planes? The longitude 30° and 60° plots, which are not symmetry planes, show that the two S surfaces break apart in a most peculiar way. There really *are* two modes, but they consist of patches of quasi-SV and quasi-SH abruptly joined together! This is quite a shock if you are used to only looking at the transverse isotropic case.

In Figure 4 we were led to wonder how the shear modes could be separated at all; at least now we see how to construct three global modes that are as continuous as possible. Unfortunately, the modes forced on us are extremely difficult to interpret. Sources designed to excite "SV" and "SH" waves will invariably excite pieces of both modes.

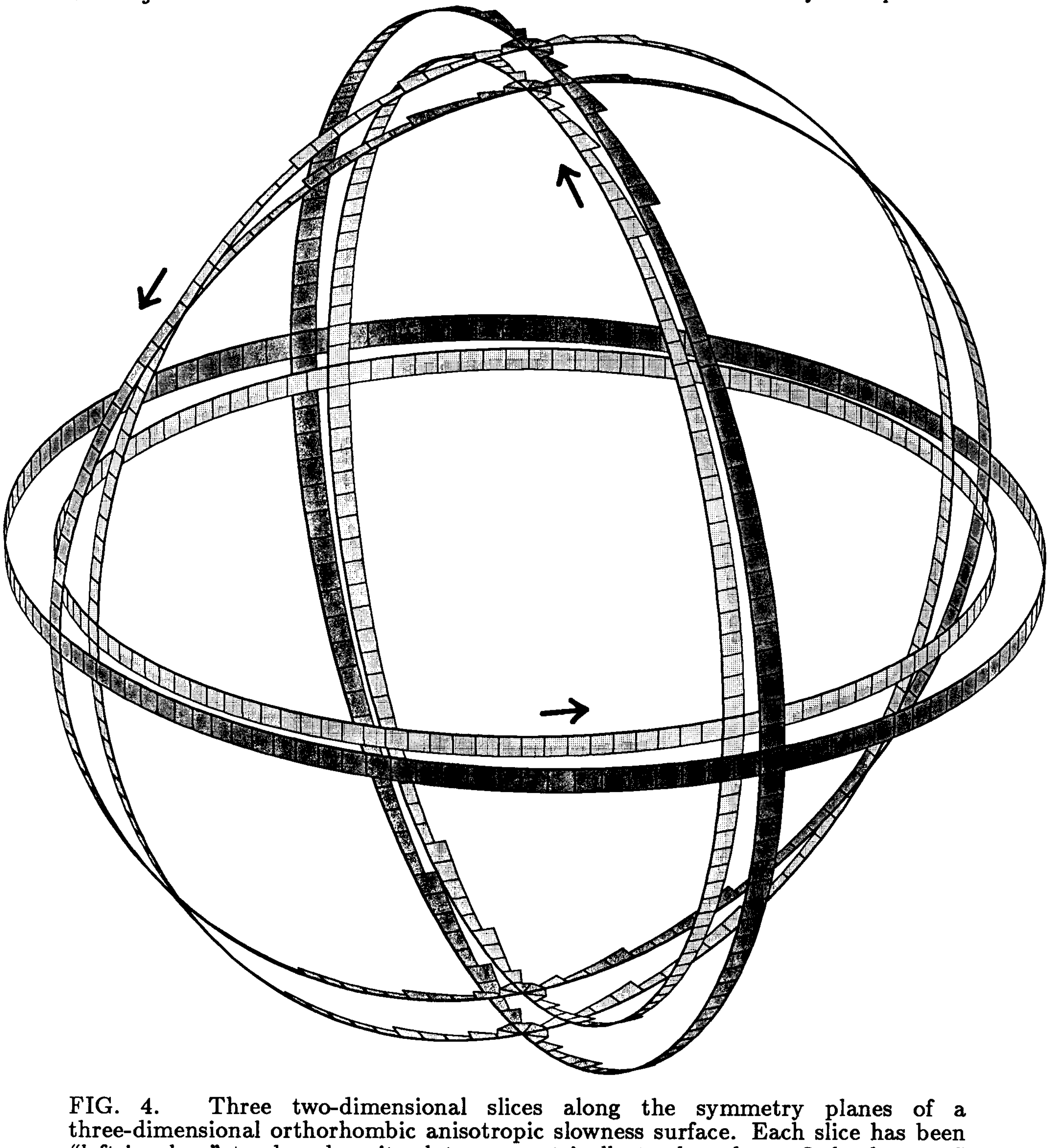


FIG. 4. Three two-dimensional slices along the symmetry planes of a three-dimensional orthorhombic anisotropic slowness surface. Each slice has been "left in place" to show how it relates geometrically to the others. Only the two S surfaces are shown. Each slice consists of an elliptical slowness surface with pure SH-type particle motion and a circular slowness surface with nearly pure SV-type particle motion. Things seem quite simple, but note what happens when you try to trace one surface around along the path marked by arrows! Why *must* this mobius-like effect occur? At the right-angle turn at the top, the same particle motion direction is called SV on one slice and SH on the perpendicular one. The crossing point near the leftmost arrow is necessary to swap the SV and SH back again so that they can connect to the correct surface on the horizontal slice.

"CONNECTION" WAVES

In a previous SEP report (Dellinger and Muir, 1985, SEP–**44**) I showed that a concavity in the slowness surface for a medium is always associated with a triplication in the impulse response. Figure 6 shows the impulse responses that go with each of the slowness surface slices in Figure 5. We see that indeed the concavities created when the two slowness surfaces break apart are associated with triplications.

The three-dimensional impulse response for a transversely isotropic medium perturbed to make it orthorhombic must look very much like it still consists of quasi-P, quasi-SV, and SH waves. The new triplications must be weak events connecting together the pieces of the now not quite continuous quasi-SV and SH surfaces. These "*connection*" events must have zero amplitude on the symmetry planes, since there the concavity on the slowness surface consists of only a single point. Since they are associated with small regions in the slowness surface domain near intersections of the quasi-SV and SH surfaces, in the impulse response domain they must be nearly planar events that are tangent to both the quasi-SV and SH surfaces.

The final two figures show various views of the impulse response associated with an octant cut out of the complete three-dimensional slowness surface for the same medium shown in Figures 5 and 6.

CONCLUSIONS

The two-dimensional divergence and curl operators separate out P and S waves in a two-dimensional isotropic medium. In three dimensions, the divergence operator still returns a scalar P potential as before. The curl operator, however, returns a 3-vector that contains information about both S modes.

We have seen that any attempt to define two distinct global S modes is bound to have problems. Perhaps the curl operator has the right idea: the two shear modes must always be considered together as a single complex object.

REFERENCES

Auld, B. A., 1973, Acoustic fields and waves in solids, vol. 1: John Wiley and Sons.

Clayton, R., 1981, Wavefield inversion methods: SEP–**27**, 81–93.

Crampin, S., 1981, A review of wave motion in anisotropic and cracked elastic-media: Wave Motion, **3**, 343–391.

Dellinger, J., and Muir, F., 1985, Axisymmetric anisotropy I: Kinematics: SEP–**42**, 1–23.

Dellinger, J., and Muir, F., 1985, Two domains of anisotropy: SEP–**44**, 59–62.

Helbig, L., and Schoenberg, M., 1987, Anomalous polarization of elastic waves in transversely isotropic media: J. Acoust. Soc. Am., **81**, no. 5, 1235–1245.

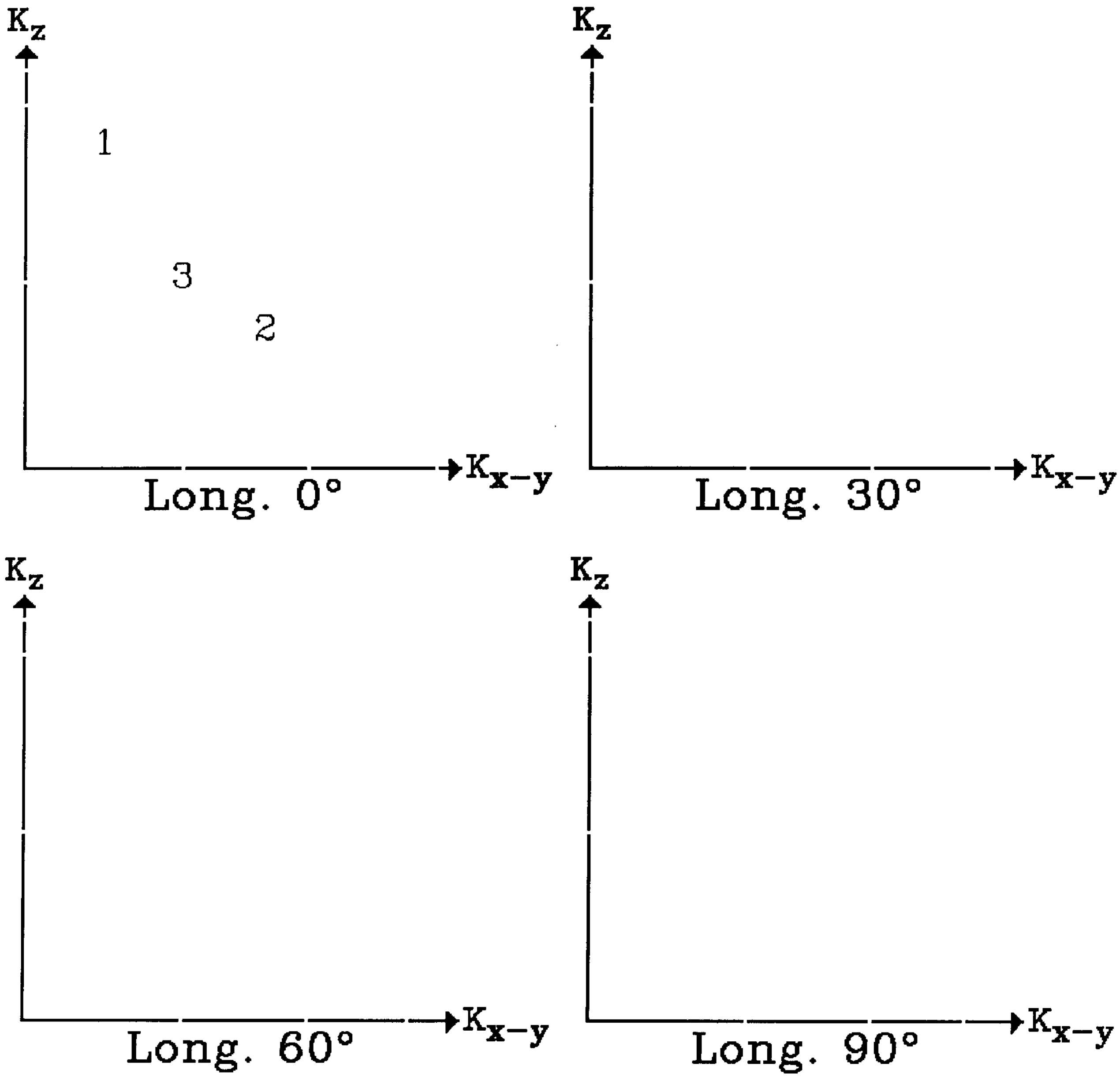
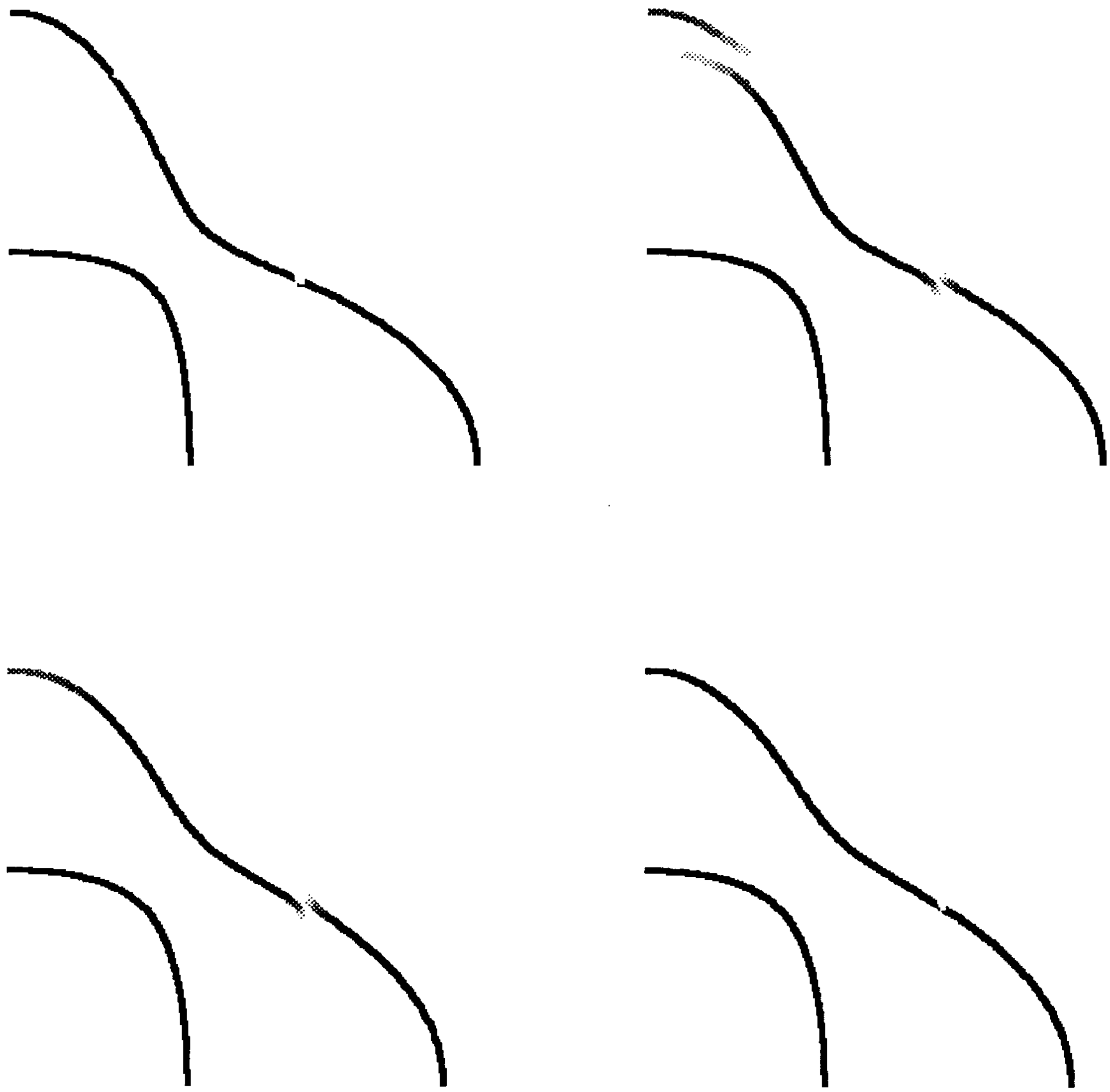
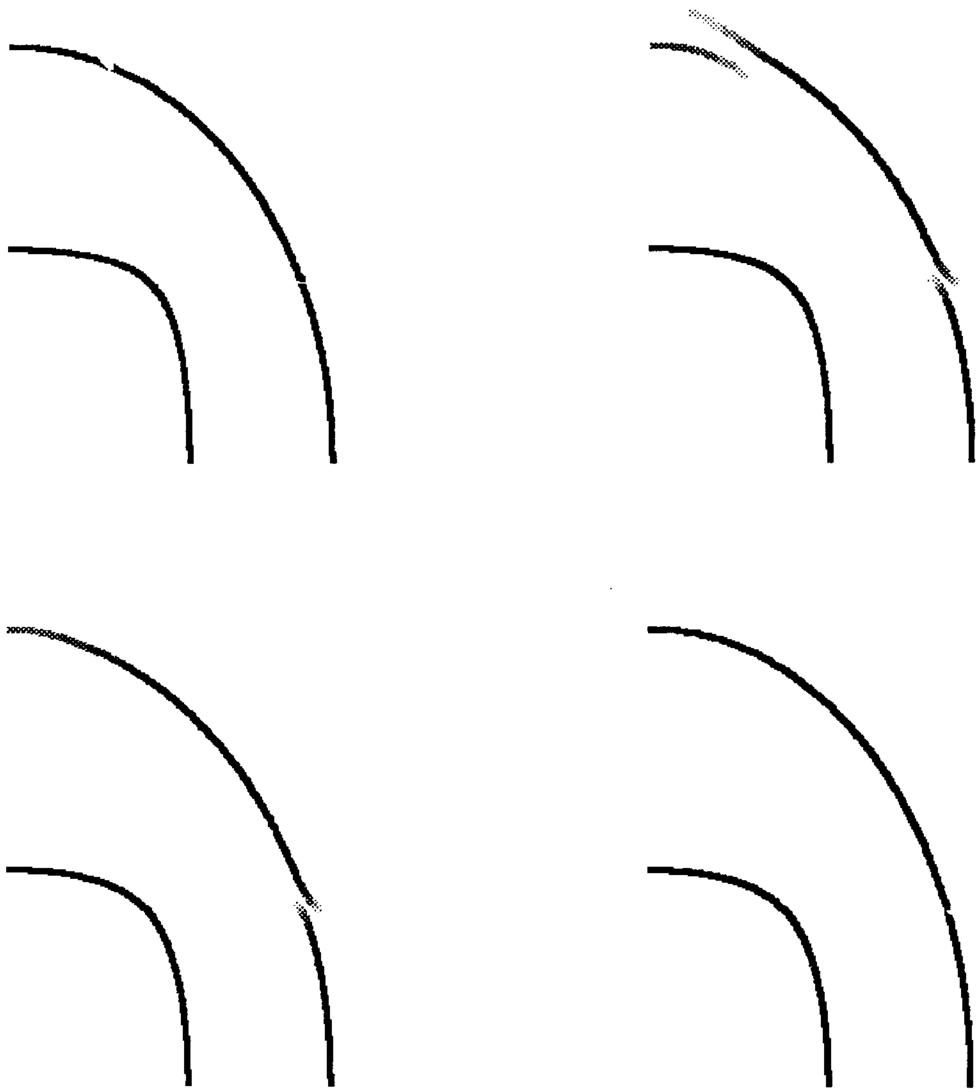
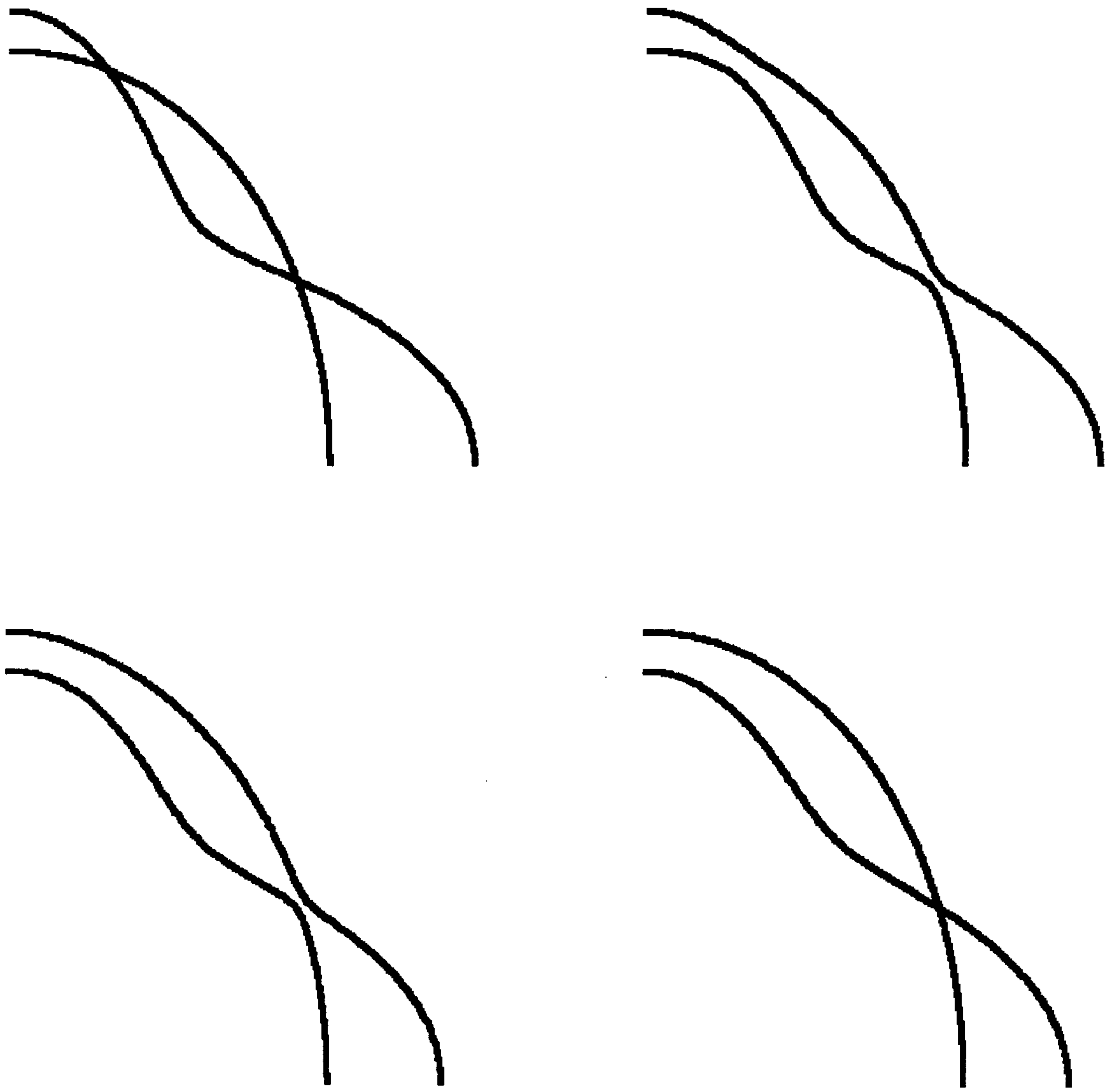


FIG. 5. Four cuts through an orthorhombic slowness surface along lines of longitude. The longitude 0° plot is the *xz* plane, and the longitude 90° plot is the *yz* plane. Color is used to indicate the particle motion type; pure SV is green, pure SH is red, and pure P is blue. Intermediate colors correspond to intermediate particle motion directions. Note how the inner and outer S surfaces abruptly exchange colors without quite touching. The numbered features on the upper left plot correlate with features in Figure 6.







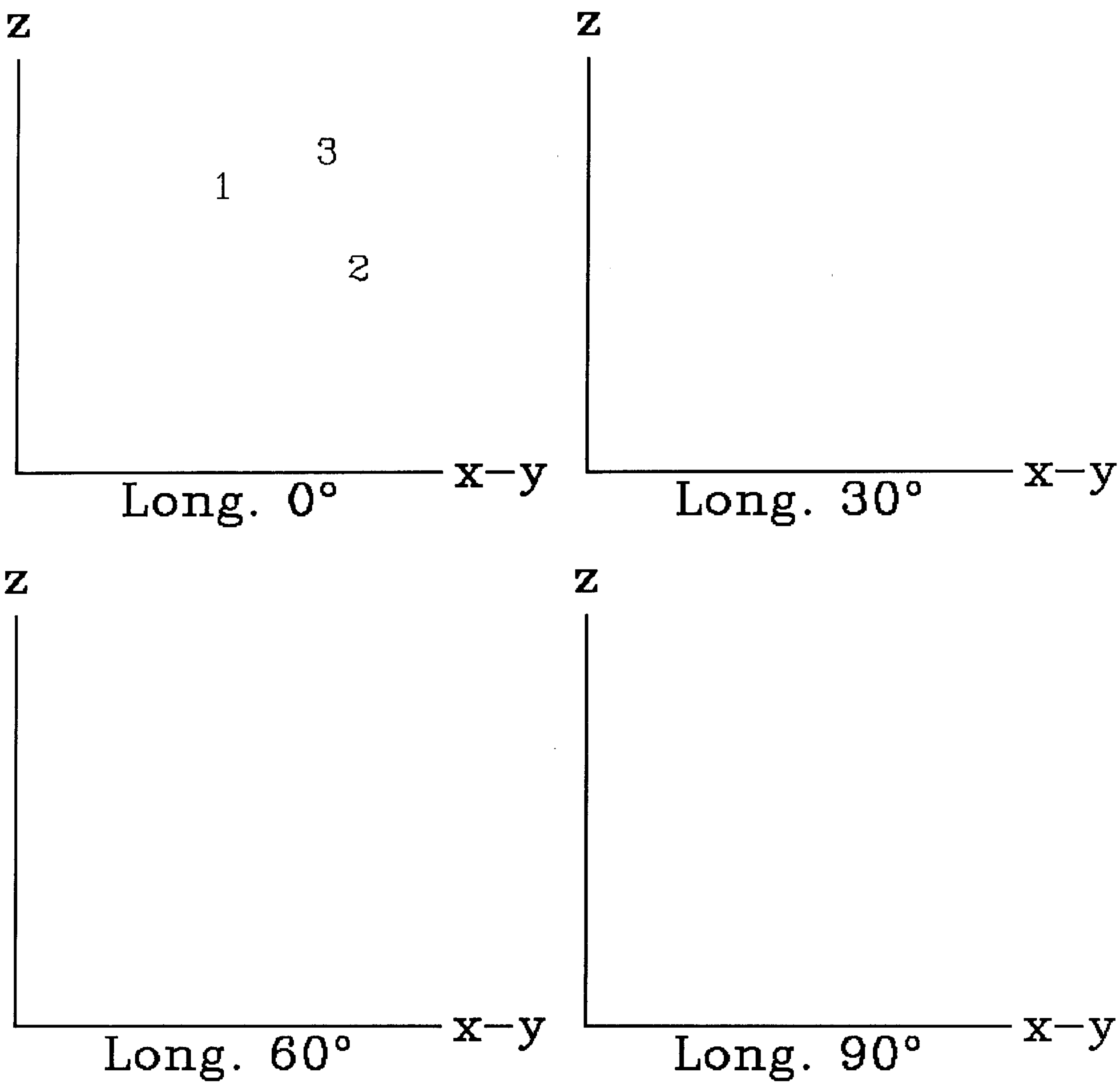


FIG. 6. Impulse responses associated with the slowness surfaces from Figure 5. The same color scheme is used. Only the shear surfaces are shown (the P surface is much bigger.) Numbered features 1 and 2 on the upper left plot point out the connection events associated with the shear surface intersections. Feature 3 is a "normal" triplication.

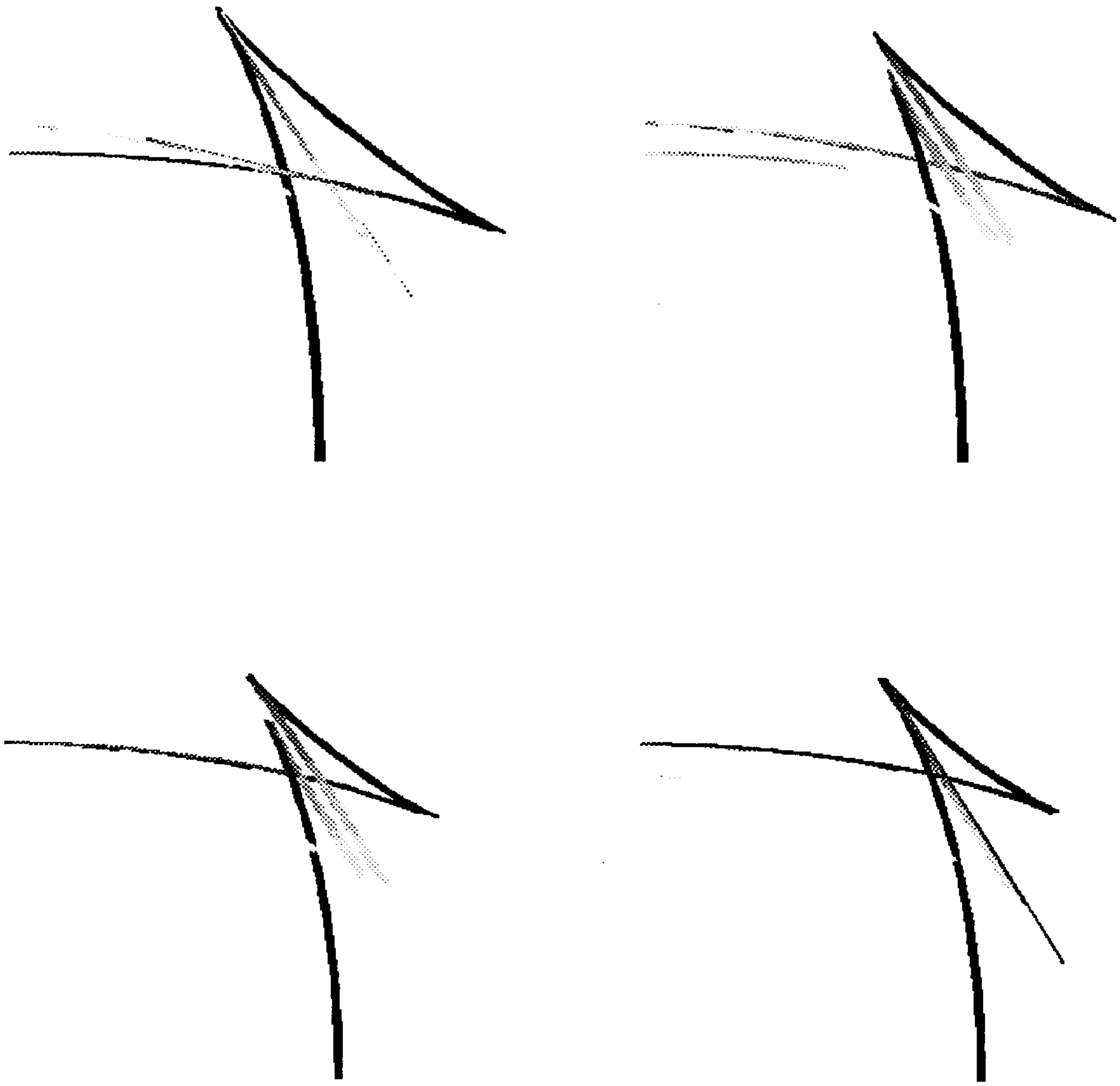
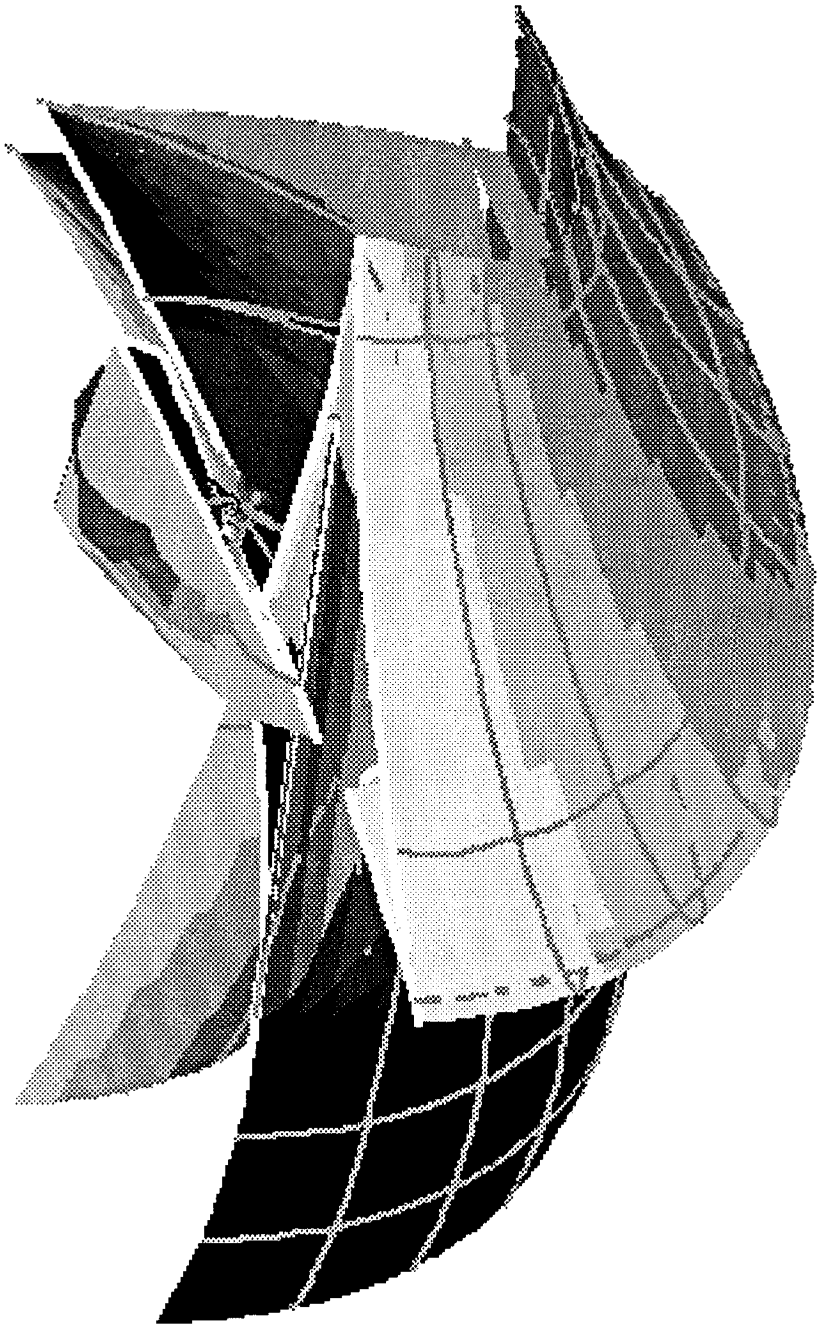
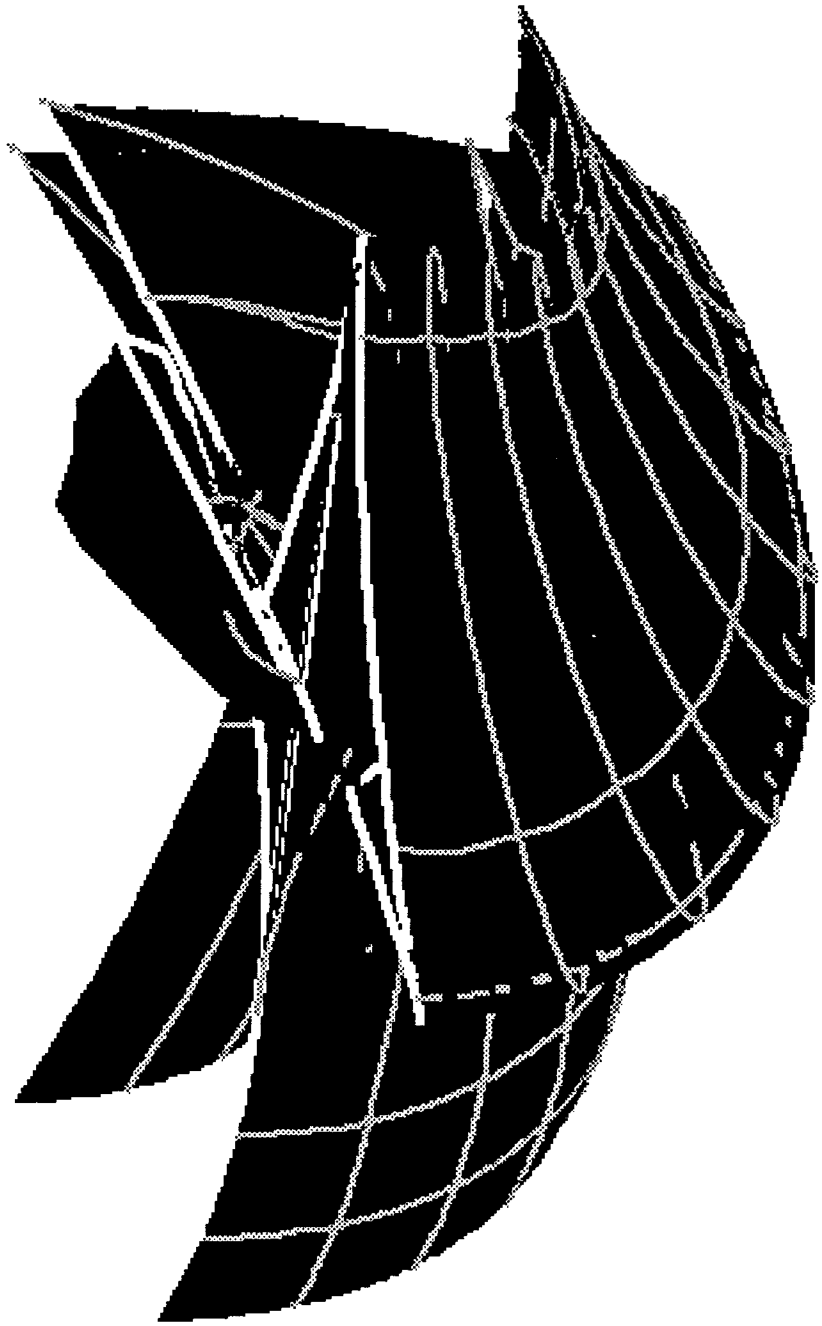


FIG. 7. A shaded three-dimensional "cartoon" view of the same impulse response in Figure 6. The same color scheme is used: Green for SV, Red for SH, intermediate colors for intermediate particle motions. Shading has been added to (try to) create the illusion of depth. Note the fan-like projection sticking out. This is associated with a dimple in the slowness surface at the shear wave singularity (intersection).





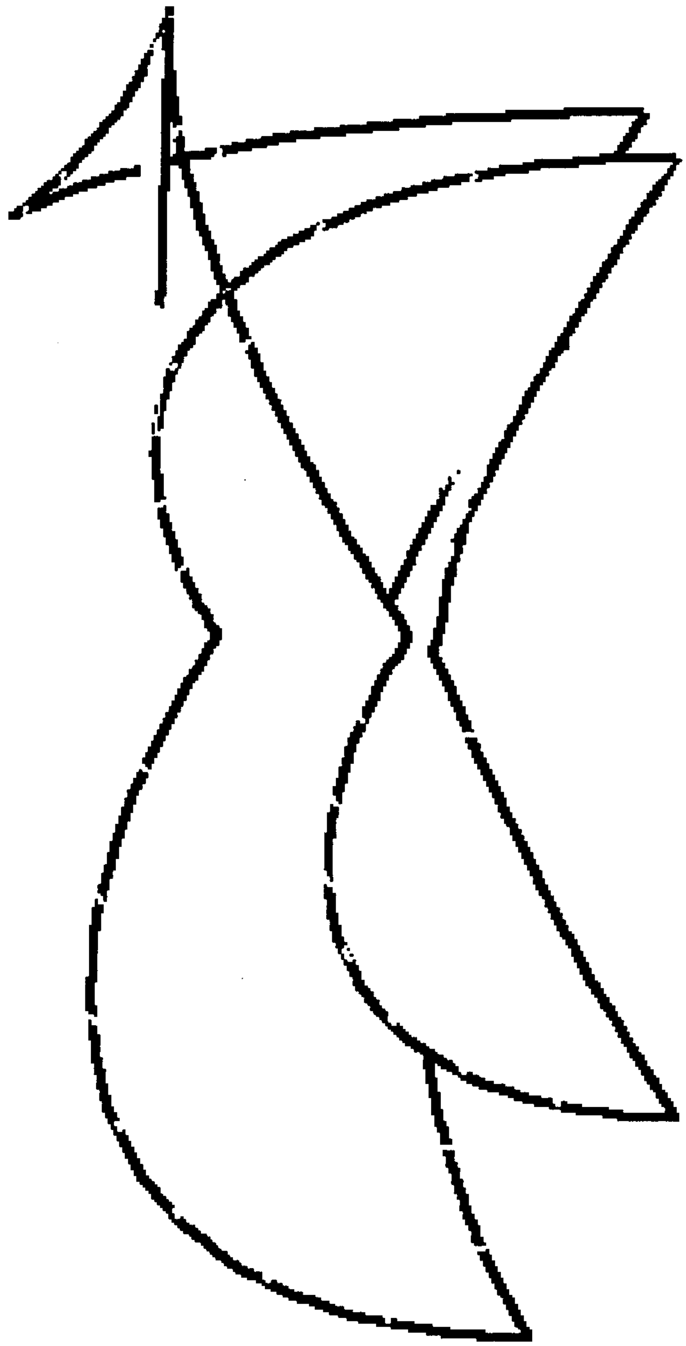


FIG. 8. Another view of the same object in Figure 7. This time we are looking at the inside. The inside is shaded darker than the outside.