Layered models, equivalence & Abelian groups

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ABSTRACT

The Dix and Backus models are shewn to be equivalent to algebraic groups, which leads to simple computational schemes for manipulating these models and their extensions. These ideas are illustrated by a lithologic modeling example.

INTRODUCTION

Seismic data analysis leans heavily on two well-established simplifying concepts: the layered earth and the equivalent medium. These notions come together in two quite different contexts: the Dix, high-frequency, ray-theoretic model and the Backus, low-frequency, wave-theoretic model.

By a happy coincidence, these schemes share a commutative group property which illuminates their structure and leads to an operational calculus which translates otherwise quite formidable algebra into simple, modular and maintainable computer code.

THE ABELIAN GROUP

An Abelian group is an abstract algebraic notion which I shall put to practical use.

The group defined

An algebraic system $[\mathcal{G}, *, e]$ is called an Abelian group (Gaal, 1979) if $\mathcal{G}$ is a set of elements on which a binary operation $*$ is defined and e is an element of $\mathcal{G}$ so that the folowing conditions are satisfied for all $x, y, z \in \mathcal{G}$.

closure $x * y \in \mathcal{G}$

association $x * (y * z) = (x * y) * z$

commutation $x * y = y * x$

identity element a unique e such that $x * e = x$

inverses for every x an $xinv$ such that $x * xinv = e$

A calculus

A type family of Abelian groups which will apply to all of the examples in this paper is the set of real n-vectors under addition, with the null vector as the identity element. For all examples of this type, a practical calculus can be built from just two "bullet-proof" constructs, corresponding to *combination*

```
SUBROUTINE combine(n,gin1,gin2,gout)
INTEGER n
REAL    gin1(n),gin2(n),gout(n)
INTEGER j
DO j = 1,n
   gout(j) = gin1(j) + gin2(j)
ENDDO
RETURN
END
```

and *inversion*

```
SUBROUTINE invert(n,gin,gout)
INTEGER   n
REAL      gin(n),gout(n)
INTEGER j
DO j = 1,n
   gout(j) = - gin(j)
END DO
RETURN
END
```

Geophysical relevance

Textbooks on algebra give examples of groups from within their own experience, and it is not always clear why they are important to practical persons like ourselves. It is nice to know that the integers modulo 6 under addition form a group, but it does not seem very useful. For both of our layered regimes the two properties of the Abelian group that I shall be particularly interested in are closure and commutation, and the first will rely on the paradox of the equivalent medium: a sandwich of heterogeneous layers forms a homogeneous layer; not absolutely true, of course, but true in a restricted way that makes sense.

THE DIX MODEL

Dix' Equation (Dix, 1955) provides a method for decomposing average velocities into interval velocities. It recognizes that NMO velocity may be quite inappropriate as a time–to–depth conversion factor. In effect it says, in our group language, that isotropic layers do not form a group under addition as regards their travel times, since, if they did, there would be a concordance between vertical velocities and move-out velocities. However, if we think about layers with an elliptic dispersion relation (at least in paraxial approximation), where there is one velocity (vertical) controlling time-to-depth and another velocity (horizontal) controlling move-out, then we are in business, at least as regards closure: we add two elliptic layers and get a third elliptic layer as the result.

Commutation depends on the well-known property of layered media; viz. rays travel with a constant horizontal velocity through all the layers, and, consequently, travel-time is independent of the ordering of the layers. Alternatively we could appeal directly to Dix' formula, which is a summation independent of ordering. In the end, we have modified Dix, and it is now a method of decomposing aggregate NMO velocities into component NMO velocities, and asserts nothing about isotropy or time-to-depth conversion velocities, which are, in any case, not available from surface-to-surface measurements (Muir & Dellinger, 1985).

Layer properties

Since we know how to add thicknesses of vertical layers, and since we also know how to add vertical travel times, we are now in a position to describe the material parameters of our layers and state the mapping that will carry us over into the group. In all of the models $\mathbf{m}$ is a vector of material layer properties, and $\mathbf{g}$ is the corresponding (vector) group element. For Dix we have:

$\mathbf{m}(1)$ thickness

$\mathbf{m}(2)$ vertical slowness (for time to depth conversion)

$\mathbf{m}(3)$ moveout parameter (square velocity)

From the model world to the group and back

Mapping from a vector of material properties, $\mathbf{m}$, to the corresponding group element, $\mathbf{g}$, is straightforward:

```
SUBROUTINE mtog(m,g)
REAL     m(3),g(3)
g(1) = m(1)
g(2) = m(1)*m(2)
g(3) = m(1)*m(2)*m(3)
RETURN
END
```

Notice particularly that there are no problems in this forward mapping; if the material parameters vector is finite, then so are the corresponding elements of the group member, and, more importantly, if the material is physical, so that the material parameters are all positive, then the elements of the group member are also all positive. The inverse mapping operation is also straighforward but less secure, with the possibility of zero divides and non-physical solutions, and this is typical:

```
SUBROUTINE gtom(g,m)
REAL     g(3),m(3)
m(1) = g(1)
m(2) = g(2)/g(1)
m(3) = g(3)/g(2)
RETURN
END
```

while group elements must be allowed to range over the field of real vectors, material properties will generally have some (positivity) constraints. For Dix, positivity of the elements of the group member ensures the physical reality of the material layer.

Contractions and extensions

Most of the time we may not have access to vertical velocities, in which case there is still a group operation available, and this now corresponds to everyday NMO operation:

m(1) time thickness

m(2) moveout parameter (square velocity)

and the forward and inverse mapping operations:

```
SUBROUTINE mtog(m,g)
REAL     m(2),g(2)
g(1) = m(1)
g(2) = m(1)*m(2)
RETURN
END

SUBROUTINE gtom(g,m)
REAL     g(2),m(2)
m(1) = g(1)
m(2) = g(2)/g(1)
RETURN
END
```

So far we have used the paraxial approximation that the elliptic model inplies, but there may be good reason to add one more degree of freedom—an anelliptic factor. This could be used to better model non-hyperbolic effects that might be either extrinsic and due to layering, or intrinsic, in case the layered material was in fact anisotropic. Our material parameters are now:

m(1) thickness

m(2) vertical slowness (for time to depth conversion)

m(3) moveout parameter (square velocity)

m(4) is $4 \cdot q_w^2 - 4 \cdot q_w - 1$

where q_w, the anelliptic factor, has been previously defined (Muir & Dellinger, 1985). The corresponding forward and backward mapping operations are now:

```
SUBROUTINE mtog(m,g)
REAL     m(4),g(4)
g(1) = m(1)
g(2) = m(1)*m(2)
g(3) = m(1)*m(2)*m(3)
g(4) = m(1)*m(2)*m(3)*m(3)*m(4)
RETURN
END
```

and

```
SUBROUTINE gtom(g,m)
REAL     g(4),m(4)
m(1) = g(1)
m(2) = g(2)/g(1)
m(3) = g(3)/g(2)
m(4) = (g(4)*g(2))/(g(3)*g(3))
RETURN
END
```

Method of development and a check

In the case of the Dix schemes, the mappings were developed by expanding travel-times in a Taylor series about zero offset and then matching coefficients. A useful check is to map over to the group symbolically, average the group elements, and then map back. In the case of the first, regular Dix scheme, this gives:

- $M(1) = Av(g(1)) = Av(m(1))$
- $M(2) = Av(g(2))/Av(g(1)) = Av(m(1) \cdot m(2))/Av(m(1))$
- $M(3) = Av(g(3))/Av(g(2)) = Av(m(1) \cdot m(2) \cdot m(3))/Av(m(1) \cdot m(2))$

the last line being the familiar RMS average.

THE BACKUS MODEL

Backus' paper (Backus, 1962) is a classic, but does not seem to have generated the programing attention that it deserves. The Backus regime is at the other end of the spectrum; a low frequency model that would be particularly appropriate for coalescing seismic data recorded at different scales, such as sonic logs and VSP's—particularly with the newly developed interest in elastic waves and anisotropy. Backus teaches how transverse isotropic media average to form an equivalent medium in case the layers themselves are very fine, and the aggregate layer has the component layers evenly distributed—a stationary material sequence.

Specifying the layer

The material properties for a Backus layer are as follows:

m(1) layer thickness

m(2) density

m(3) C66

m(4) 1/C44

m(5) 1/C33

m(6) C13

m(7) C11

From the model world to the group and back

As with the Dix scheme, mapping from a vector of material parameters to the corresponding group member is straightforward:

```
SUBROUTINE mtog(m,g)
REAL    m(7),g(7)
g(1) = m(1)
g(2) = m(1)*m(2)
g(3) = m(1)*m(3)
g(4) = m(1)*m(4)
g(5) = m(1)*m(5)
g(6) = m(1)*m(5)*m(6)
g(7) = m(1)*(m(7) - m(5)*(m(6)**2))
RETURN
END
```

but the inverse mapping operation, like the Dix scheme, should incorporate a check the group element for physical realizability (not included in this code):

```
SUBROUTINE gtom(g,m)
REAL    g(3),m(3)
m(1) = g(1)
m(2) = g(2)/g(1)
m(3) = g(3)/g(1)
m(4) = g(4)/g(1)
m(5) = g(5)/g(1)
m(6) = g(6)/g(5)
```

```
m(7) = (g(7)*g(5) + g(6)**2)/(g(1)*g(5))
RETURN
END
```

A modified model

If the information to which we have access is seismic alone, then, if the density tensor is isotropic, we can remove the density term from the vector of material parameters, and modify the rest of the parameters so that they are the elastic moduli divided by the density, with units of velocity squared.

m(**1**) layer thickness

m(**2**) C66

m(**3**) 1/C44

m(**4**) 1/C33

m(**5**) C13

m(**6**) C11

From the model world to the group and back

Mapping from a vector of material parameters to the corresponding group member is again straightforward:

```
SUBROUTINE mtog(m,g)
REAL    g(6),m(6)
g(1) = m(1)
g(2) = m(1)*m(2)
g(3) = m(1)*m(3)
g(4) = m(1)*m(4)
g(5) = m(1)*m(4)*m(5)
g(6) = m(1)*(m(6) - m(4)*m(5)**2)
RETURN
END
```

as is the inverse mapping operation:

```
SUBROUTINE gtom(g,m)
REAL     g(6),m(6)
m(1) = g(1)
m(2) = g(2)/g(1)
m(3) = g(3)/g(1)
m(4) = g(4)/g(1)
m(5) = g(5)/g(4)
m(6) = (g(6)*g(4)  + g(5)**2)/(g(1)*g(4))
RETURN
END
```

Groups within groups NOT sub-groups

A nice feature of the Backus group is that the group formulation also encompasses some incomplete parameter sets, where elements from one or more sets may be taken together. These lesser groups are

- thickness
- thickness, density
- thickness, C66
- thickness, C44
- thickness, C33
- thickness, C33, C13
- thickness, C33, C13, C11

and, for example, the SH wave moduli, C44 and C66, can be treated quite independently of the others. Another set of groups comes from factoring out density from all of the parameters by dividing the moduli through by density to produce a set of parameters with dimension velocity squared. This last would be particularly useful in case the data was exclusively seismic.

Method of development and a check

It took me some time to develop the inverse mapping for the Bachus model; for a time I was not even certain that there was a group. Backus had taught how to average, but not how to unaverage. Once the work is done, however, it is very simple to check by the method outlined under Dix: viz. symbolically map from $\mathbf{m}$ to $\mathbf{g}$, average, and map back to $\mathbf{m}$ again.

An extension

The Backus model proper is confined to transverse isotropic materials, but recently his notion as been extended (Schoenberg & Muir, 1987) to include arbitrary anisotropy (up to 21 independent elastic moduli) in each layer, and also a certain model of fracturing. For this scheme the group and model elements are 5-vectors, whose components are 3-by-3 matrices.

AN EXAMPLE IN LITHOLOGY

A Backus example. We have a clastic section of uniform thickness consisting of finely interdigitating sands and shales. There is a continuous and linear facies change in this section from all sand to all shale. The sand is isotropic whereas the shale is very anisotropic, but only in its rigidity moduli. The sand and shale have the same vertical compressional and shear velocities. We have no information on their densities, but we suppose that the sands and shales are equally dense. Give the appropriate component velocities, the following FORTRAN program calculates all the equivalent medium (Backus) velocities at each of one hundred divisions of the transition zone, and is very suitable for inputing directly into an anisotropic elastic modeling program such as Etgen has discussed (Etgen, 1987). Units are the foot and the second.

```
PROGRAM litholog
REAL    m1(6),m2(6),m3(6),g1(6),g2(6),g3(6)

m1(2) = 2.5E+07
m1(3) = 4.0E-08
m1(4) = 1.0E-08
m1(5) = 5.0E+07
m1(6) = 1.0E+08

m2(2) = 5.0E+07
m2(3) = 4.0E-08
m2(4) = 1.0E-08
m2(5) = 5.0E+07
m2(6) = 1.5E+08

DO j = 0,100
   m2(1) = REAL(j)/REAL(jmax)
   m1(1) = 1.0 - m1(1)
```

```
      CALL mtog(m1,g1)
      CALL mtog(m2,g2)
      CALL combine(g1,g2,g3)
      CALL gtom(g3,m3)
      WRITE m3
   END DO
   STOP
   END
```

CONCLUSION

The Dix and Backus models—particularly in their extensions—are most simply manipulated, and their computer codes most simply maintained, when they are mapped over into their group representations.

REFERENCES

Backus, George E., Long-Wave anisotropy produced by horizontal layering: JGR 66 4427-4440 1962

Dix, Charles H., 1955, Seismic velocities from surface measurements: GEOPHYSICS Vol 20, No.1, pp68–86

Etgen, John, 1987, Finite-difference elastic anisotropic wave propagation: SEP–56

Gaal, Lisl, 1979, Classical Galois theory: Chelsea Publishing Company, New York, NY

Muir, Francis and Dellinger, Joe, 1985, A practical anisotropic system: SEP–44 pp 55-58

Schoenberg, Michael and Muir, Francis, 1987, Group theoretic formulation for elastic anisotropic layers: submitted to Geophysical Prospecting

FIRST CIRCULAR

WORKSHOP MEETING ON SEISMIC WAVES IN LATERALLY INHOMOGENEOUS MEDIA III

Castle of Liblice near Prague

June 13 - 18, 1988

Under the auspices of the European Seismological Commission (ESC), the Geophysical Institute of the Czechosl. Acad. Sci., Prague together with the Faculty of Mathematics and Physics, Charles University, Prague will organize a workshop „Seismic waves in laterally inhomogeneous media III", in June 1988.

The workshop will have a similar programme as those organized under the same name at Liblice in February - March 1978 and June 1983. It will mainly deal with the theoretical and computational aspects of the seismic wave propagation in laterally inhomogeneous media. Attention will be also paid to the applications involving these aspects in the Earth's crust and uppermost mantle studies as well as in seismic prospecting.

The following two main topics are recommended for the papers:

I. Numerical modelling of seismic wave fields in laterally inhomogeneous media with curved interfaces and block structures, including anisotropic and dissipative media.
II. Inversion of seismic data for laterally inhomogeneous structures.

Papers devoted to the observations of lateral inhomogeneities and anisotropy of the Earth's crust and the upper mantle and the physical modelling of seismic wave fields in laterally inhomogeneous structures are welcome.

The workshop will be divided into sessions on different subjects. Each session will begin with an invited paper which will be followed by short papers (not longer than 15 minutes). Advanced instructions will be provided on available program packages. A poster session will also be organized.

Emphasis will be put on free and informal discussions and exchange of ideas during the meeting. Suitable conditions for this purpose will certainly be found in the Castle of Liblice near Prague, where the meeting will take place. The capacity of the Castle, however, permits accommodation of only a limited number of persons (70 to 80).

Participants are requested to kindly fill in the enclosed application form and to return it by November 30, 1987 at the latest. The second circular with detailed information will be distributed only to those who have returned the application form.

The conference fee of about 180 US dollars includes accommodation and full board for 5 days at the Castle, social programme and conference materials. For accompanying persons the fee is about 120 US dollars. The participants from Czechoslovakia and other socialist countries will pay a conference fee of 750,- Kčs (500,- Kčs for accompanying persons), which does not include the costs of board and accommodation. The participants from the socialist countries arriving within reciprocity programmes have to ask their authorities to include the conference fee in the programme of their stay in Czechoslovakia.

Any request for additional information and application forms should be sent to the following address: **Ivan Pšenčík, Geophysical Institute of the Czechoslovak Academy of Sciences, Boční II, 141 31 Praha 4 - Spořilov, Czechoslovakia. Telex No. 121 330 SEIS C.**