Mass-spring networks to simulate the 4-th order Laplacian

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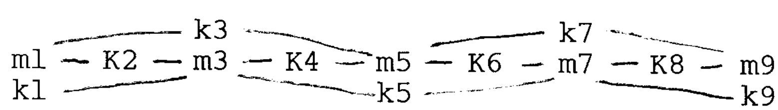
Consider a one dimensional network of masses and springs. Let

v = velocities of masses, m

p = pressures in reservoirs, spring constants K

Here is the layout of the network. (It seems to be two-dimensional, but that is only because the labels needed to be separated for printing purposes.)



Masses are at odd numbered points. Springs at all points, K=strong, k=weak. It is the weak, odd-numbered springs that enable 4-th order accuracy.

```
                                                   | |
                                                   | |
m d/dt |v|  =  |      ++0--                    |  |p|
       | |     |         ++0--                 |  | |
               |            ++0--              |  | |
                                                  | |

        | |     |  +0-          |
        | |     |    +-         |
        | |     |    +0-        |
1       | |     |      +-       |   | |
-  d/dt |p|  =  |      +0-      |   |v|
K       | |     |        +-     |   | |
        | |     |        +0-    |
        | |     |          +-   |
        | |     |          +0-  |
```

Magically you observe a transpose relationship in the above two matrices. The two matrix equations above may be written as the following two equations.

$$m \frac{dv}{dt} \quad = \quad -\mathbf{A}\, p \tag{1a}$$

$$K^{-1}\, \frac{dp}{dt} \quad = \quad \mathbf{A}^{\mathrm{T}} v \tag{1b}$$

$$\begin{bmatrix} \mathbf{m} & 0 \\ 0 & \mathbf{K}^{-1} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} \quad = \quad \begin{bmatrix} 0 & -\mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} \tag{1c}$$

ENERGY AND STABILITY

Define the total energy E as kinetic plus potential

$$E \quad = \quad v^{\mathrm{T}} \mathbf{m} v \ + p^{\mathrm{T}} \mathbf{K}^{-1} p \tag{2}$$

By differentiation of (2) and substitution of (1), dE/dt is identically zero, meaning that total energy, kinetic plus potential, is constant.

((((The following inequality says that the energy decreases with time.

$$\frac{dE}{dt} \quad = \quad - \textit{positive quadratic form} \quad \leq \quad 0 \tag{3}$$

I imagined that I could find the positive quadratic form in (3) and that it would be a diagonal matrix containing the real part of the spring constant impedances.))))

NUMERICAL ANALYSIS

Numerical solution 1-D

Equation (1) may be extrapolated directly via Crank Nicolson. In one dimension, interlacing (1a) with (1b) gives a matrix of 5 bands.

I contemplated a numerical experiment adjusting the ratio of k/K and watching the dispersion go thru a minimum at the best value.

Numerical solution 2-D

Crank-Nicolson style implicit methods are too costly in two dimensions. On a Cartesian mesh we have ADI (alternating direction implicit). This deserves further investigation. Meanwhile, ADI isn't applicable to an icosahedral mesh. (The icosahedral mesh is a prototype for irregular meshes in 3-D surveys). So let us look at the irregular mesh.

Explicit in 2-D

We have

$$p_{t+1} - 2p_t \ + p_{t-1} \quad = \quad - \frac{1}{\Delta t^2} \, \mathbf{K} \, \mathbf{A}^{\mathrm{T}} \, \mathbf{m}^{-1} \, \mathbf{A} \, p_t \tag{4}$$

The right hand side may be called the discrete Laplacian. Chuck has a representation of this Laplacian. His early work did not have it in the factored form (2). I think it is in factored form now, but to see it, you need to see his papers also to appear in this volume.

SYSTEM IDENTIFICATION

I don't know how to use physical principles to determine the masses and spring constants from the geometry of the nodes. It seems that the masses should be proportional to the surrounding area, say the square of the average surrounding edge length. But I don't know how to choose the springs to model an isotropic medium, to say nothing of a homogeneous one.

USE OF INVERSE THEORY TO DETERMINE MESH ELEMENTS

In principle we can use geophysical inverse theory to find the masses at the nodes and the spring constants on the connecting edges. To see how this might be done, consider two cases.

Pole to pole

Put a point source at one pole and observe a messy wave at the antipode. Make the claim that you received a theoretical impulse at the antipode. Use geophysical inverse theory to perturb the mesh elements to bring the mess as close as possible to the theoretical wave.

Pole to equator

Put a point source at one pole and observe an imperfect plane wave on the equator. Define the spatially averaged plane wave by

$$\overline{p}(t) = \frac{1}{N} \sum_{i \ on \ equator}^{N} p_i(t)$$

Minimize

$$\sum_{equator} \sum_{t} (p_i - \overline{p})^2$$

The idea is to back project the residuals... somehow... but I am afraid it may be more sensitive to reflectivity than to velocity inhomogeneity or anisotropy. That too should be expressible in terms of inverse theory. Just now, I don't see how. Pete could become famous in applied mechanics by solving these problems.