Polyspectral phase estimation and deconvolution

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INTRODUCTION

Standard deconvolution methods estimate the unknown seismic source wavelet using only second order statistics, that is spectrum or autocorrelation functions, and cannot determine the phase of the wavelet without further assumptions, such as minimum phase. Lii and Rosenblatt (1982) have described a method, which could be applied to the identification of the phase of the seismic wavelet. Unlike Minimum Entropy Deconvolution methods (Walden, 1985), the method of Lii and Rosenblatt does not require prior assumptions about the distribution of the reflection coefficients. It is applied in the frequency domain, and is based on the estimation of polyspectra, which are generalizations of the spectrum, sensitive to the phase of the wavelet.

In this paper the methods of bispectral and trispectral estimation and deconvolution are applied to synthetic examples with both asymmetric and symmetric distributions. The results are examined for evaluation of the number of data which would be needed for seismic applications.

THE CONVOLUTIONAL MODEL

Linearity of the acoustic response of the subsurface implies that the seismic trace is the convolution of the source wavelet with the impulse response of the medium. When multiple reflections are neglected, the impulse response becomes equal to the reflectivity series (Bamberger and al., 1980). A noise free seismogram can then be modeled as a convolution of a source wavelet and the reflectivity sequence:

$$x_t = w_0 r_t + w_1 r_{t-1} + \cdots = (w * r)_t , \tag{1}$$

where (x_t) is the observed signal and (w_t) is the source wavelet and (r_t) is the reflectivity sequence.

In the other limiting case, when strong multiple reflections create time delayed version of the seismogram of primary reflections, the appropriate convolutional model is:

$$x_t = v_1 x_{t-1} + v_2 x_{t-2} + \cdots + r_t . \tag{2}$$

In the above model, $v = (1.,-v_1,-v_2,..)$ is interpreted as a reverberation filter and r_t as a reflection coefficient.

More generally a combination of models (1) and (2) can be considered:

$$x_t - v_1 x_{t-1} - \cdots = w_0 r_t + w_1 r_{t-1} + \cdots \tag{3}$$

Model (3) is an example of an ARMA process (Robinson, 1980), or of a feedback-feedforward system. Theoretically it is possible to represent models (2) and (3) as a moving average (equation (1)) by multiplying these equations by the inverse of the filter $v = (1.,-v_1,-v_2,..)$. However the correct choice of the model is important for computations, since the representation of the inverse filter may require infinitely many terms at positive and negative times.

In the rest of the paper, we will discuss a deconvolution model based on the convolutional equation (1) for a moving average process, with a reflectivity series modeled as a white noise process with unknown distribution. It will be shown however, that the method can be adapted for ARMA models (2) or (3).

MINIMUM PHASE DECONVOLUTION

As an introduction to the method of polyspectral deconvolution, we recall briefly a method for solving equation (1) using only second order statistics of the data. Examples of second order statistics are the autocorrelation and the spectrum (Claerbout, 1976). Let the symbol $E(.)$ represent averaging over disjoint windows of the data (x_t). Estimates of the second order statistics of (x_t) can be obtained either in the time domain by:

$$r_t = \frac{1}{L} E\left(\sum_{s=0}^{L} x_s \, x_{t+s} \right) , \; 0 \le t < L \le T ,$$

or in the frequency domain by:

$$p(\omega_k) = \frac{1}{L} E\left(X_L(\omega) \overline{X}_L(\omega)\right) ,$$

where $X_L(\omega)$ is the discrete Fourier transform of a window of L data samples,

$$X_L(\omega_k) = \sum_{t=0}^{L-1} x_t \, e^{i \omega_k t} , \text{ with } \omega_k = \frac{2\pi k}{L} , \; 0 \le k < L ,$$

and $\overline{X}_L(w)$ the complex conjugate of $X_L(\omega)$. The spectrum is the discrete Fourier transform of the autocorrelation, therefore the autocorrelation and the spectrum contain exactly the same statistical information about the sequence (x_t). Also the estimate of the spectrum $p(\omega)$ is the discrete Fourier transform of the estimate of the autocorrelation r_t.

In the spectral domain the convolutional equation (1) becomes:

$$X(\omega) = W(\omega)R(\omega), \tag{4}$$

with $X(\omega)$ the Fourier transform of the data, $W(\omega)$ the Fourier transform of the wavelet, and $R(\omega)$ the Fourier transform of the reflectivity. The assumption of a white reflectivity series allows solving equation (3) for the spectrum of the wavelet, since

$$E(R(\omega)\overline{R}(\omega)) = c = constant \tag{5}$$

and

$$E(X(\omega)\overline{X}(\omega)) = W(\omega)\overline{W}(\omega)E(R(\omega)\overline{R}(\omega)) = cW(\omega)\overline{W}(\omega),$$

Equation (5) shows how to obtain the spectrum of the wavelet up to a constant, but gives no indication about its phase, since second order statistics do not depend on the phase of the wavelet. The standard way of solving (3) is to make the hypothesis of a minimum phase wavelet. The phase of the minimum phase wavelet can be computed from its spectrum by a Hilbert transform (Claerbout, 1976) and the wavelet will then be determined up to a real scale factor. The minimum phase hypothesis is supported by simple physical models with explosive sources as well as indirectly by its widespread use in seismic data processing (Robinson, 1980). Alternative methods, are of interest however, since they could provide a test for the minimum phase hypothesis and a method for deconvolution of data generated with non-minimum phase sources.

POLYSPECTRAL ESTIMATION

An alternative to the minimum phase hypothesis is to use higher than second order statistics, which depend on the phase of the wavelet (Lii and Rosenblatt, 1982 and Rosenblatt, 1985).

For three zero mean random variables, (U_1, U_2, U_3), the third order cumulant is defined as the expected value of their product:

$$cum_3(U_1, U_2, U_3) = E(U_1 U_2 U_3).$$

The third order cumulant is the third order statistic, analogous to the autocorrelation function. The bispectrum is the two-dimensional Fourier transform of the third order cumulant. For a zero mean process, the third order cumulant can be estimated from a sample (x_t) by:

$$b(t_1, t_2) = \frac{1}{L} E\left(\sum_{s=0}^{L-1} x_s \, x_{t_1+s} \, x_{t_2+s}\right),$$

where the symbol $E(.)$ represents averaging over windows of length L. Similarly, the bispectrum can be estimated by:

$$B(\omega_1,\omega_2) = \frac{1}{L}E(X(\omega_1)X(\omega_2)\overline{X}(\omega_1+\omega_2)). \tag{6}$$

For a symmetric reflectivity series, the third order statistics are zero, making it necessary to work with fourth-order statistics. For zero mean random variables (U_1,U_2,U_3,U_4), the fourth order cumulant is defined as:

$$cum_4(U_1,U_2,U_3,U_4) = E(U_1U_2U_3U_4) - \Big(E(U_1U_2)E(U_3U_4) + \cdots\Big)$$

The three-dimensional Fourier transform of the fourth order cumulant is the trispectrum. The trispectrum can be estimated by:

$$C(\omega_1,\omega_2,\omega_3) = \frac{1}{L}E(X(\omega_1)X(\omega_2)X(\omega_3)\overline{X}(\omega_1+\omega_2+\omega_3))$$

Computationally, the estimation of the spectrum, bispectrum or trispectrum, proceeds by breaking the data into disjoint windows, Fourier transforming the data, computing products of the Fourier transform at different frequencies, smoothing over frequencies and then averaging over the time windows. The only step in this estimation procedure not apparent from the definitions is the smoothing over frequencies. The reason for smoothing is well known for spectral estimation and is the same for polyspectral estimation (Robinson, 1980, Rosenblatt, 1985). The estimators defined above are unbiased, but inconsistent, that is, the variance of the estimates does not tend to zero as the size of the sample is increased, unless there is smoothing over frequencies. Another important result about these estimators is that the estimation error decreases as the inverse of the length of the time window.

POLYSPECTRAL DECONVOLUTION

For non-zero third order statistics, combining equations (4) and (6) for a window of data of length L, we obtain:

$$X(\omega_1)X(\omega_2)\overline{X}(\omega_1+\omega_2) = W(\omega_1)W(\omega_2)\overline{W}(\omega_1+\omega_2)R(\omega_1)R(\omega_2)\overline{R}(\omega_1+\omega_2) \tag{7}$$

The bispectrum of the white noise process is constant, therefore taking averages over disjoint windows of data leads via equation (7) to:

$$E(X(\omega_1)X(\omega_2)\overline{X}(\omega_1+\omega_2)) = c \;\; W(\omega_1)W(\omega_2)\overline{W}(\omega_1+\omega_2), \tag{8}$$

where

$$c = E\,(R\,(\omega_1) R\,(\omega_2) \overline{R}\,(\omega_1 + \omega_2)).$$

Equation (8) is interesting because it provides a relation between the unknown phase of the wavelet, $\phi(\omega)$, and the phase $\psi(\omega_1,\omega_2)$ of the bispectrum, which can be computed from the data. Since we equate arguments of complex numbers, the relation holds only modulo π:

$$\psi(\omega_1,\omega_2) = \phi(\omega_1) + \phi(\omega_2) - \phi(\omega_1 + \omega_2) \quad [\,\pi\,] \tag{9}$$

When the trispectrum is used the development proceeds in a similar way and the equation analogous to (9) is:

$$\psi(\omega_1,\omega_2,\omega_3) = \phi(\omega_1) + \phi(\omega_2) + \phi(\omega_3) - \phi(\omega_1 + \omega_2 + \omega_3) \quad [\,\pi\,]. \tag{10}$$

In addition to equations (9) or (10) the phase function has to satisfy the following general constraints: $\phi(\omega)$ is defined only modulo 2π and

$$\phi(\omega) + \phi(-\omega) = 2n\,\pi\ ,$$

with n integer, since the wavelet is real in the time domain.

Equation (8) can be derived only if the bispectrum is non-zero, that is the trispectrum of the white noise process (denoted in the above equation by c) is non zero. Gaussian processes are such that all cumulants of order higher than two vanish. Therefore this method of phase estimation is not applicable to Gaussian input processes.

Recall also that the phase of the minimum phase wavelet can be computed from the spectrum of the observed process by a Hilbert transform. Then the right hand side term of equation (9) could be evaluated and compared to the phase of the estimated bispectrum. This could serve as a test for the minimum phase assumption.

Finally, if the initial convolutional model was not a moving average, but a AR or ARMA process as defined by equations (2) or (3), the Fourier transform of the wavelet $W(\omega)$ would be replaced by the transfer function of the corresponding system and its phase could be identified in the same way.

PHASE IDENTIFICATION

The method for phase identification, given by Lii and Rosenblatt, 1980, was implemented for the computations of the next section and is outlined below. The paper by Matsuoka and Ulrych 1982, discusses in detail this and two other methods for solving equation (8) given bispectral estimates.

The set of equations below follows from equation (9):

$$\begin{cases} \phi(\omega_1) + \phi(\omega_1) - \phi(\omega_2) & = \psi(\omega_1,\omega_1) \\ \phi(\omega_1) + \phi(\omega_2) - \phi(\omega_3) & = \psi(\omega_2,\omega_1) \\ \ldots\ldots & \ldots \\ \phi(\omega_1) + \phi(\omega_{L-1}) - \phi(\omega_L) & = \psi(\omega_{L-1},\omega_1) \end{cases} \tag{11}$$

where $\omega_k = \frac{2\pi k}{L}$. There are $(L-1)$ equations, each modulo π, and L unknowns. The additional constraints are the requirement of a smoothly varying phase, $\phi(0) = 0$ or π, and $\phi(0) + \phi(2\pi) = 2n\,\pi$. For small ω_1, the sign of the real part of the bispectrum $Re\,(B\,(\omega_1,\omega_1))$ determines the value of $\phi(0)$. In fact $\phi(0)$ can be set equal to zero, by multiplying if necessary all bispectral estimates by -1 or equivalently subtracting π from both sides of equation (11). Setting $\phi(0) = 0$ allows us also to choose $\psi(\omega_k\,,\omega_1)$ in the interval $[-\frac{\pi}{2}\,,\frac{\pi}{2}]$, when the phase function is smoothly varying and ω_1 is small. Finally, we solve the system of equations (11) for several values of $\phi(\omega_L)$, $\phi(\omega_L) = 2n\,\pi$, and choose the smoothest solution.

A serious limitation of the method for phase computation presented above is that it uses the bispectral information only along a line. This makes it unsuitable for bandlimited processes (Matsuoka and Ulrych, 1984). There are also simple examples of processes which could not be distinguished from bispectral values given only along a line. On the other hand this method provides a simple solution to the system of equations (8), which are defined only modulo π.

NUMERICAL EXAMPLES

In this section we present examples of deconvolution using both bispectra and trispectra. The first series of examples in Figures 1,2, and 3 illustrate the bispectral deconvolution of an asymmetric input process. The distribution of the input process and the wavelets used for the convolution are the same as those used by Lii and Rosenblatt (1980). The input white noise process is simulated by drawing 12800 numbers from an exponential distribution with mean 1. The histogram and a window of the input process are shown in Figures 1.a and 1.b. For the examples of Figure 2 the input white noise was convolved with the three point minimum phase wavelet given in figure 1.c. The z transform of this wavelet is

$$w\,(z\,) = 1 - \frac{5}{6}z + \frac{1}{6}z = \frac{1}{6}(z - 2)(z - 3).$$

The results of bispectral deconvolution and of standard minimum phase deconvolution are shown in Figures 2.c and 2.d. The length of the data windows for the estimation of the spectrum and the bispectrum were 128 samples and the averaging was done over 18 frequencies. Both methods reconstruct well the original white noise process.

In the examples of Figure 3 the exponential process was convolved with a non-minimum phase wavelet, with z transform:

$$w(z) = 1 - \frac{7}{3}z + \frac{2}{3}z = -\frac{2}{3}(z - 2)(z - \frac{1}{3}).$$

Figure 3.c shows that bispectral deconvolution has reconstructed the input process well, while the minimum phase deconvolution has failed.

Figure 4 shows a similar example for a white noise process with a uniform distribution with mean zero and variance one. For such a symmetric distribution the bispectrum is identically zero and it is necessary to estimate the trispectrum for the deconvolution. The input process consists of 64000 samples, the length of the data windows is 512 and the smoothing is done over 64 frequencies. The deconvolution is still accurate although not as good as the one for the bispectrum, despite the increased number of samples. The computation of the phase of the wavelet is done using the values of the phase of the trispectrum along the line $\psi(\omega_1, \omega_2, \omega_k)$, with $\omega_k = \frac{33\pi k}{512}$.

DISCUSSION

We have not yet tried the estimation of trispectra from seismic data. The numerical examples given in this paper illustrate the feasability of a non-minimum phase, non-parametric identification of a moving average process. They indicate that the estimation of fourth order statistics by the methods described above requires roughly 100 times more data than the estimation of second order statistics. Such volumes of data are available in seismic exploration, however they are not likely to fit the stationary model. Further, the computation of the phase from trispectral estimates requires "phase unwrapping." These remarks suggest that a formulation of non-minimum phase deconvolution in the time domain, possibly combined with parametric estimation techniques, should be investigated.

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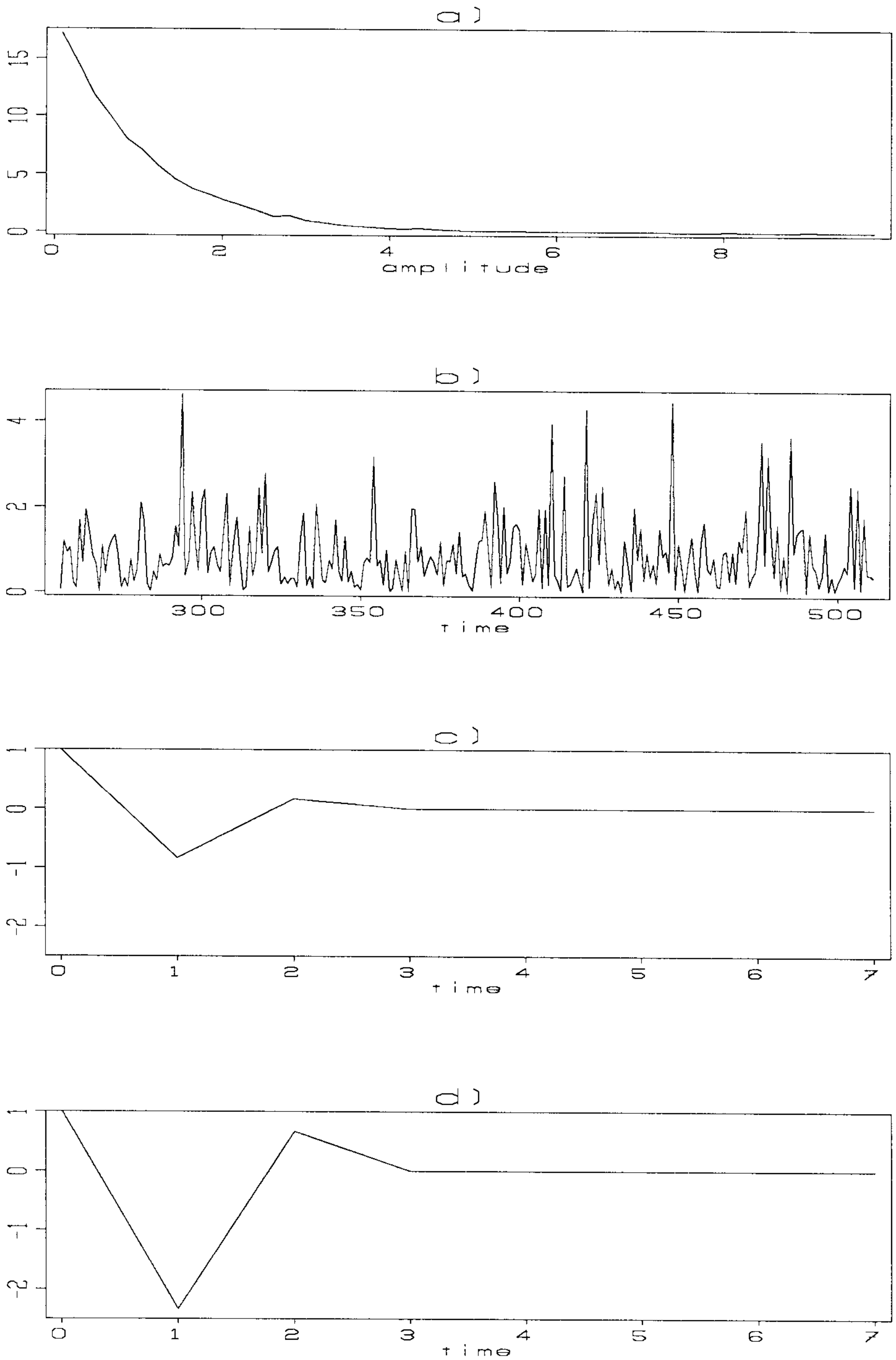

FIG. 1.
1.a) Histogram of 12800 random numbers drawn from an exponential distribution with mean 1
1.b) window of input process
1.c) minimum phase wavelet, $w = (1., -0.866, 0.167)$
1.d) mixed phase wavelet, $w = (1., -2.33, 0.667)$

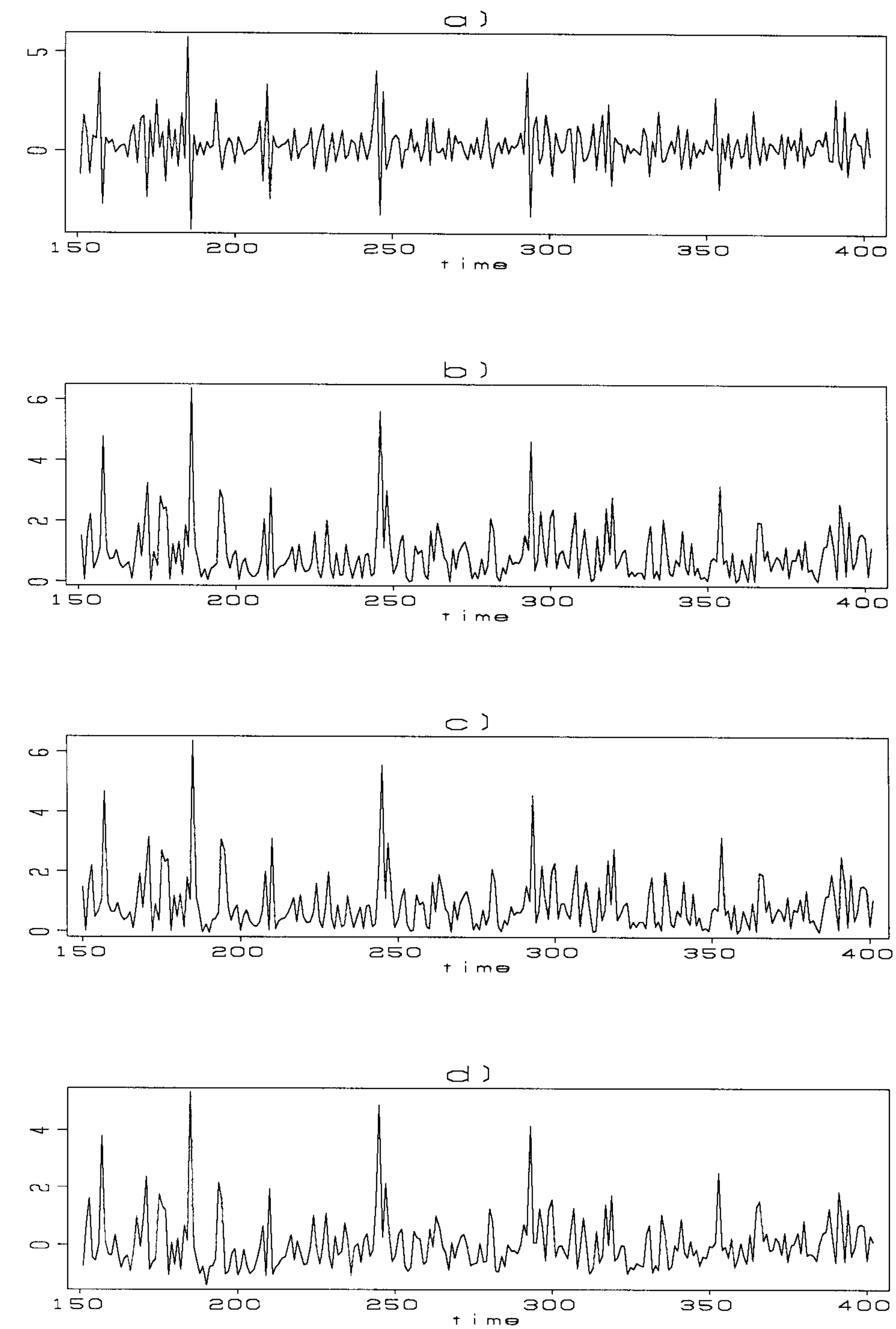

FIG. 2.
2.a) window of process convolved with the minimum phase wavelet shown in Figure 1.c
2.b) window of input white noise process
2.c) result of bispectral deconvolution
2.d) result of minimum phase deconvolution (residuals of mean zero)

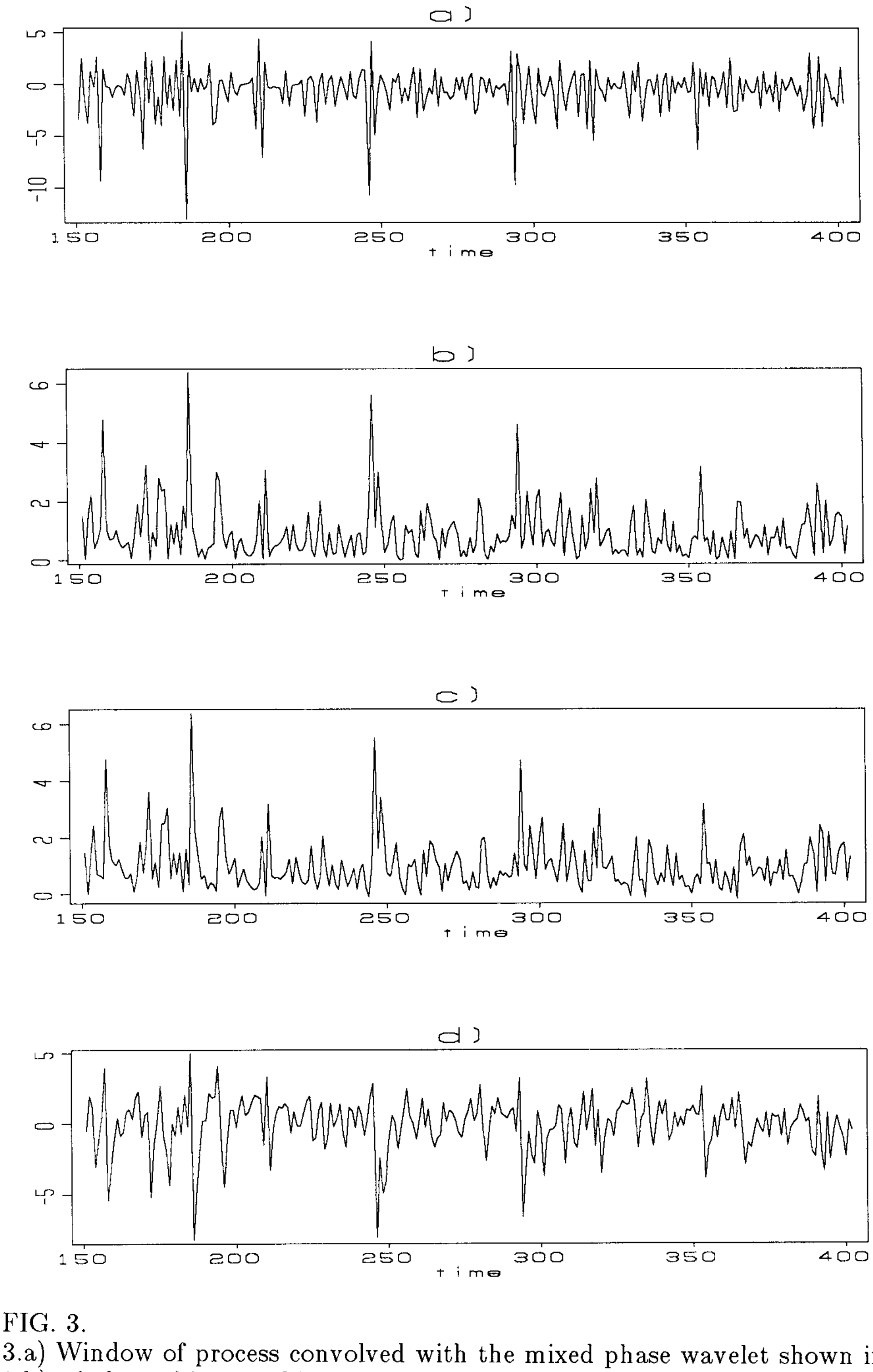

FIG. 3.
3.a) Window of process convolved with the mixed phase wavelet shown in Figure 1.d
3.b) window of input white noise process
3.c) result of bispectral deconvolution
3.d) result of minimum phase deconvolution

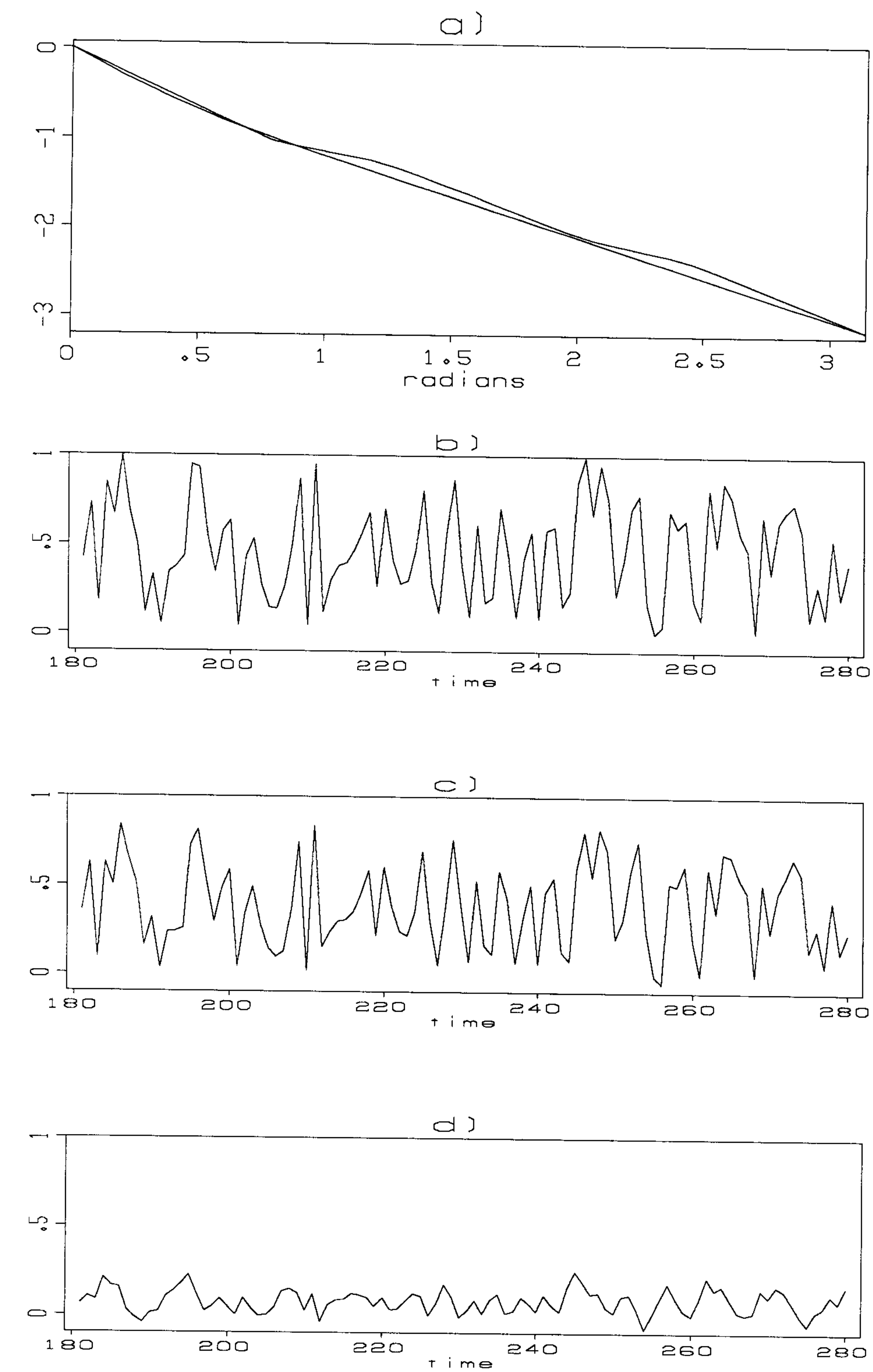

FIG. 4.
4.a) difference between the true phase and the estimated phase for the mixed phase wavelet of Figure 1.d
4.b) window of the input process
4.c) window of deconvolved process
4.d) window of the deconvolution error