Cascaded 15 degree equations simplified

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INTRODUCTION

At the 1985 Washington, D.C. SEG meeting, Beasley and Larner applied principles of residual migration to produce inexpensive, wide angle migrations by cascading 15 degree migrations with small velocities. In *Velocity extrapolation by cascaded 15 degree migration* (this volume), Jon Claerbout goes them one better by deriving a simple equation to allow incremental migration velocity steps to be applied directly to a section rather than indirectly via 15 degree migration with a tiny velocity. I was dissatisfied with the derivations for both of these equations and set out to find simpler ones. Lastly, while generating figures for another article in this volume I happened on a novel instance of Li's linearly transformed wave equation.

BEASLEY AND LARNER

Stolt migration (Stolt, 1978) consists of frequency downshift in the (ω,k_x) plane plus scaling by the Jacobian of the coordinate change. (See also Levin, *Test your migration IQ*, this volume.) $15\,^\circ$ constant velocity migration may be similarly described but with a different frequency downshift (see, e.g. section 5 of Jakubowicz and Levin, 1983). Specifically, the frequency downshifts are given by

$$\begin{array}{ll} 90\,^\circ: & sgn\,(\omega)\,\sqrt{\omega^2 - v^2 k_x^{\,2}} \\ 15\,^\circ: & \omega - v^2 k_x^{\,2}/2\omega \end{array} \tag{1}$$

Letting $k_\tau(\alpha) = sgn\,(\omega)\sqrt{\omega^2 - \alpha v^2 k_x^{\,2}}$ so that $\omega \rightarrow k_\tau(1)$ represents $90\,^\circ$ migration, one way to evaluate k_τ is to integrate the first order differential equation

$$\frac{dk_\tau}{d\,\alpha} = -\frac{v^2 k_x^{\,2}}{2k_\tau} \tag{2}$$

with initial condition $k_\tau(0) = \omega$. Discretizing this with $\Delta\alpha = 1/N$ gives a cascade of frequency downshifts that take the form

$$k_\tau \;\rightarrow\; k_\tau - v^2 k_x^{\,2}/2Nk_\tau \tag{3}$$

which is $15\,^\circ$ migration with velocity $v/\sqrt{N}$ as used by Beasley and Larner. They also noted that the discretization of α is not required to be uniform.

Actually, the above argument has only shown that the frequency downshift of cascaded 15 ° migrations approximates that of 90 ° migration. There are also the Jacobian scaling factors to consider. For 90 ° migration this factor is k_τ/ω and for 15 ° migration it is $(2 - k_\tau/\omega)^{-1}$. Thus for cascaded 15 ° migrations we divide by

$$2 - \frac{k_\tau + \Delta k_\tau}{k_\tau} = 1 - \frac{\Delta k_\tau}{k_\tau} \tag{4}$$

at each step. Taking logarithms and dropping terms higher than first order as $\Delta k_\tau \to 0$, converts the product to the integral

$$\int_0^\alpha \frac{1}{k_\tau} \frac{dk_\tau}{d\alpha}\, d\alpha \tag{5}$$

which is $\ln(k_\tau(\alpha)/k_\tau(0))$. Exponentiation therefore gives the cascaded scale factor as k_τ/ω, matching 90 ° migration.

There is nothing magic about using 15 ° migration in the cascade. One may use higher order migrations as well, such as 45 ° and 60 ° schemes. While they tend in practice to be slower than 15 ° algorithms, this is compensated by their higher order fit to the 90 ° semicircle which allows larger velocity steps, thus fewer intermediate migrations. The limiting case is to use 90 ° migration where anything more than one step is overkill.

CLAERBOUT

Claerbout realized that an essential observation of Beasley and Larner was that for small velocity 15 ° migration is an excellent approximation to 90 ° migration. Using this idea he deduced a 15 °-like equation describing the evolution of migrated sections as a function of v^2. His equation is

$$\frac{\partial^2 U}{\partial \mathrm{w}\, \partial t} = -\frac{t}{2} \frac{\partial^2 U}{\partial x^2} \tag{Jon-8}$$

where $\mathrm{w} \equiv v^2$ and $U(t, x; \mathrm{w})$ is the time section migrated at squared-velocity w. Actually, since we are discussing the (velocity) evolution of migrated data, t, unmigrated time, should be τ, migrated time, in the above to agree with conventional usage.

In terms of the previous section, there is a short derivation of Claerbout's equation when the Jacobian scaling is omitted. Later I will derive the correct equation to use when scaling is included. Take the double Fourier transform of migrated data to be $U(k_\tau, k_x) = P(\omega(k_\tau), k_x)$, where, from equation (1),

$$\omega(k_\tau) = sgn(k_\tau) \sqrt{k_\tau^2 + \mathrm{w} k_x^2} \tag{6}$$

and P is the double Fourier transform of the unmigrated time section. Compute $\partial U/\partial \mathrm{w}$ and $\partial U/\partial k_\tau$ using the chain rule:

$$\begin{aligned} \frac{\partial U}{\partial \mathrm{w}} &= \frac{\partial P}{\partial \omega} \times \frac{k_x^2}{2\omega} \\ \frac{\partial U}{\partial k_\tau} &= \frac{\partial P}{\partial \omega} \times \frac{k_\tau}{\omega} \end{aligned} \quad . \tag{7}$$

Eliminating $\omega^{-1}\,\partial P/\partial\omega$, yields

$$k_\tau \frac{\partial U}{\partial \mathrm{w}} = \frac{k_x^{\,2}}{2} \frac{\partial U}{\partial k_\tau} \tag{8}$$

which, when inverse transformed, is Claerbout's equation.

A different equation arises when the Jacobian scaling is included. In this event

$$U = \frac{k_\tau P}{\omega} = \frac{\partial}{\partial k_\tau} \int^{\omega} P(\omega,k_x)\, d\omega \tag{9}$$

and $\partial U/\partial \mathrm{w}$ may be written as

$$\frac{\partial U}{\partial \mathrm{w}} = \frac{\partial}{\partial k_\tau}\left\{ \frac{k_x^{\,2} P}{2\omega} \right\} \quad . \tag{10}$$

Eliminating P/ω gives

$$\frac{\partial U}{\partial \mathrm{w}} = \frac{\partial}{\partial k_\tau}\left\{ \frac{k_x^{\,2} U}{2k_\tau} \right\} \tag{11}$$

which inverse transforms to

$$\frac{\partial U}{\partial \mathrm{w}} = -\frac{\tau}{2} \int^{\tau} \frac{\partial^2 U}{\partial x^2} \quad . \tag{12}$$

The difference between this equation and (Jon-8) is that the order of the time recurrence and the multiplication by $-\tau/2$ are interchanged.

Both Li and I have gone back over Jon's original derivation from 15 degree migration and come up with a simple reason to prefer formulation (12) over (Jon-8). The idea of Jon's derivation is that extrapolation with a tiny velocity does little to the data. Writing 15 degree extrapolation as

$$\frac{\partial P}{\partial \tau} = -\frac{\Delta \mathrm{w}}{2} \int^{t} \frac{\partial^2 P}{\partial x^2} \quad , \tag{13}$$

this idea has two consequences:

(1) A crude finite difference operator is a good approximation to the $\partial/\partial\tau$ derivative, and

(2) an initial τ derivative may be used to extrapolate P a long ways along the τ axis.

Therefore we approximate (13) with

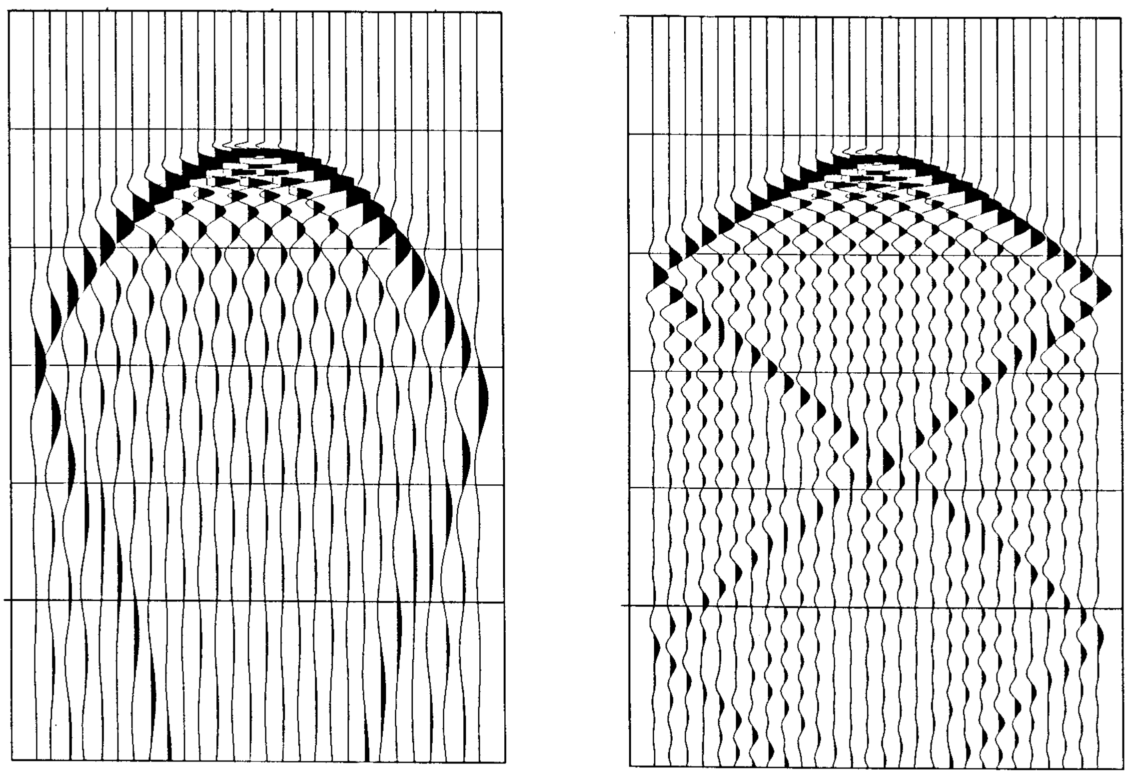
$$\frac{P(\tau{=}t) - P(0)}{t} = -\frac{\Delta \mathrm{w}}{2} \int^{t} \frac{\partial^2 P}{\partial x^2} \tag{14}$$

or

$$\frac{P(\tau{=}t) - P(0)}{\Delta w} = -\frac{t}{2}\int^{t} \frac{\partial^2 P}{\partial x^2} \tag{15}$$

which is equation (12) in the limit $\Delta w \to 0$.

Claerbout points out that equations (Jon-8) and (12) are transposes of each other. This also follows from Harlan (1983) who showed that the adjoint of Stolt migration is Stolt modeling without the Jacobian of the frequency shift. Since Jon uses both (Jon-8) and its transpose it follows that, at least in one direction, he does use (12).



(a) **(b)**

FIG. 1. Impulse responses for 15 ° modeling with (a) upward continuation and (b) forward time evolution. The apex curvatures are the same but the slopes of the flanks are markedly different.

LI

An open question with cascaded migration is how to properly incorporate nonconstant velocity. The frequency-wavenumber arguments given above provide no clue how or even if the process can be generalized to variable velocity. In this section I will show a quite different 15 ° cascade, unrelated as far as I can tell to that discussed previously, in which I am able to incorporate variable velocity. While this new cascade does not solve the underlying problem of how to compensate for an incorrect migration velocity, it raises the hope that the problem may be tractable.

Figures 1a and 1b show impulse responses for two 15 ° modeling methods: upward-continuation and forward-time evolution. The former starts below the deepest reflector with an empty time section and upward continues through the model adding each reflecting point in at $t=0$ as the depth of that point is reached. The latter uses the reflector image as a $t=0$ snapshot and forward steps it in time, stripping off the resulting wavefield at the surface of the earth to form the modeled time section. You may be struck as I was by how different the shapes of the curves are away from their apices. Indeed you may speculate that I forgot to halve the migration velocity in the latter computation. This was not the case. The differences are real. The dispersion relation for the former was given in equation (1)

$$k_\tau = \omega - \frac{v^2 k_x^2}{2\omega} \quad . \tag{16}$$

The dispersion relation for the latter is given by

$$\omega = k_\tau + \frac{v^2 k_x^2}{2k_\tau} \quad . \tag{17}$$

Stationary phase calculations produce the moveout equations

$$t = \tau + \frac{x^2}{2\tau v^2} \tag{18}$$

for 15 ° upward-continuation modeling and

$$t = \tau + \frac{x^2}{2tv^2} \tag{19}$$

for 15 ° forward-time modeling. The former describes the $x-t$ parabola classically associated with the 15 ° equation. The latter may be rewritten

$$(t - \frac{\tau}{2})^2 = (\frac{\tau}{2})^2 + \frac{x^2}{2v^2} \tag{20}$$

showing it describes a hyperbola.

Suppose now we cascade these two different 15 ° modeling schemes by taking the output ω of modeling with equation (16) and substituting it for k_τ in equation (17). Denoting this intermediate frequency $\hat{\omega}$, the final frequency ω is related implicitly to k_τ by the pair of relations

$$\omega = \hat{\omega} + \frac{v^2 k_x^2}{2\,\hat{\omega}} \tag{21}$$

and

$$k_\tau = \hat{\omega} - \frac{v^2 k_x^2}{2\,\hat{\omega}} \quad . \tag{22}$$

Adding and subtracting equations (21) and (22) produce respectively

$$\omega + k_\tau = 2\,\hat{\omega} \tag{23}$$

and

$$\omega - k_\tau = 2\frac{v^2 k_x^2}{2\,\hat{\omega}} \quad . \tag{24}$$

Eliminating $2\,\hat{\omega}$ gives

$$\omega - k_\tau = 2\frac{v^2 k_x^2}{\omega + k_\tau} \tag{25}$$

or

$$\omega^2 - k_\tau^2 = 2\,v^2 k_x^2 \tag{26}$$

which is 90 ° modeling with velocity $\sqrt{2}\,v$.

So here we have another way of getting high accuracy with the 15 ° equation. Simply cascade two different 15 ° modeling programs each using half the original squared velocity. A most intriguing alternative to residual modeling with two passes of 90 ° modeling with the same reduced velocity as discussed by Rothman, Levin and Rocca (1985).

We will now see that this is really a special case of linearly transformed wave equation extrapolation (LITWEQ) for constant velocity (Li, 1984). Figure 2 is Li (1984) Figure 1 rotated 90 ° clockwise. Modeling is performed on this grid by placing the subsurface section on the right-hand edge of the triangular grid and propagating down and to the left using a 15 ° differencing star with velocity v and sampling intervals $\Delta t' = \Delta\tau' = \Delta\tau/\sqrt{2}$ or equivalently velocity $v' = v/\sqrt{2}$ and sampling intervals $\Delta t = \Delta\tau$. The center column thus contains a section modeled by 15 ° upward-continuation with velocity $v/\sqrt{2}$. Computations continue past the center column until the left-hand edge is filled in to produce the final output time section. This second half is clearly forward-time modeling with a 15 ° algorithm and velocity $v/\sqrt{2}$ and completes the cascade to produce a 90 ° migration with velocity v .

Now that I've shown the similarity to Li's method, it's only fair to point out some differences. First, Li's method places zeros above the right-hand edge, which I've shown in *Test your migration IQ* integrates flat reflectors on the input to form step functions on the intermediate result in the central column. Li also places zeros above the left-hand edge which I've shown differentiates flat reflectors in the central column. Thus Li's method preserves flat reflectors by first integrating them and then differentiating them. Cascaded 15 ° modeling would preserve them throughout unchanged.

A second difference is that Li's method works for nonconstant velocity, the cascaded 15 ° method only handles velocity variation properly in the second half of the cascade, 15 ° forward modeling, but not in the first half of transposed 15 ° migration.

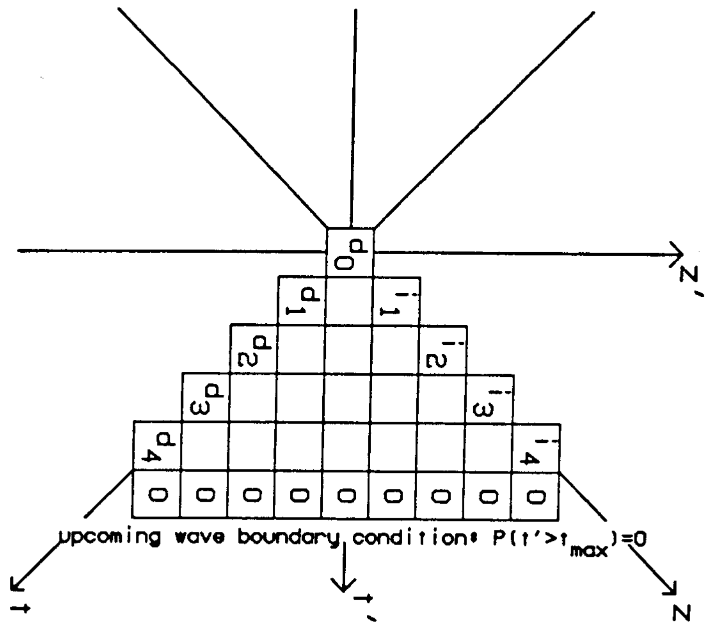


FIG. 2. Full wave equation finite difference grid in (t', z'). The subsurface model is on the right-hand edge, the surface time section on the left-hand.

Indeed Li's result shows the upward-continuation modeling must be done with a time-variable velocity $v(\tau+2t)$. This is in qualitative agreement with Rothman, Levin and Rocca (1985) where they showed that if you use instead a 90 ° algorithm with RMS velocities, then an impulse in the subsurface model maps to the intermediate curve

$$t^2 = \tau^2 + \frac{x^2}{v^2(\tau) - 1/2\, v^2(t)} \tag{27}$$

a mixture of velocities at t as well as τ.

SUMMARY

Using frequency-wavenumber arguments I've given some simplified derivations for three useful ways to cascade 15 ° equations. The drawback of these arguments is they are strictly constant velocity and give no clue about how variable velocity might fit in. However Li's independently developed linearly-transformed wave equation does show how to generalize one of these cascades to nonconstant velocity and thereby offers some hope that the other cascades might generalize equally well.

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