# Two Domains of Anisotropy

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## INTRODUCTION

This is a recapitulation of appendix 2 of "The Kinematics of Axisymmetric Anisotropy" (Dellinger and Muir, SEP–42), presented in a somewhat more intuitive form.

In that paper, the concept of phase and group velocity were discussed, and equations were given which related these two domains. Passing notice was given to certain special symmetries of these equations. In this paper, these symmetries will be examined more carefully.

## THE GROUP AND PHASE DOMAINS

The group velocity curve is a graph of $V_r$, the velocity of energy propagation, versus $\phi_r$, the angle of energy propagation. The phase velocity curve is a graph of $V_w$, the velocity of plane wave propagation, versus $\phi_w$, the angle of plane wave propagation. Since it is possible to construct $V_r(\phi_r)$ given $V_w(\phi_w)$, and vice-versa, these two alternative forms express the same information, and can be considered as two different "domains" for representing the same thing.

These two domains are closely allied to the two domains of Fourier analysis. The "group" domain is that of time and space. A graph of the group velocity as a function of angle traces out the outer surface of the impulse response, also known as the ray surface.

The "phase" domain is that of plane waves. For a given wave equation, it is the phase velocity of a wave traveling in a given direction that is easy to solve for. This information is contained in the dispersion relation.

Instead of velocities, it is also possible to express these quantities in terms of slowness, which is simply the inverse of velocity. We will represent slowness with an upside-down $V$, $\Lambda$. A dispersion relation as normally plotted is simply a graph of phase slowness, $\Lambda_w(\phi_w)$. In figure 1 the equations relating the group and phase domains are given, both in terms of velocity and slowness.

Notice the symmetry of these formulas. There are two transformations involved here. Let us call the one which transforms from group velocity to phase velocity "$T$", and the one which

$$V_w = \frac{{V_r}^2}{\sqrt{{V_r}^2 + (\frac{dV_r}{d\phi_r})^2}} \quad , \quad \phi_w = \phi_r - \text{Arctan}(\frac{dV_r/d\phi_r}{V_r})$$

$$\varLambda_r = \frac{\varLambda_w^2}{\sqrt{\varLambda_w^2 + (\frac{d\varLambda_w}{d\phi_w})^2}} \quad , \quad \phi_r = \phi_w - \text{Arctan}(\frac{d\varLambda_w/d\phi_w}{\varLambda_w})$$

$$V_r = \sqrt{{V_w}^2 + (\frac{dV_w}{d\phi_w})^2} \quad , \quad \phi_r = \phi_w + \text{Arctan}(\frac{dV_w/d\phi_w}{V_w})$$

$$\varLambda_w = \sqrt{\varLambda_r^2 + (\frac{d\varLambda_r}{d\phi_r})^2} \quad , \quad \phi_w = \phi_r + \text{Arctan}(\frac{d\varLambda_r/d\phi_r}{\varLambda_r})$$

FIG. 1: Equations relating the group and phase domains, in terms of both velocity $V$ and slowness $\varLambda$. The symmetry evident in these equations is there because the transformation from the group velocity curve (ray surface) to the phase slowness curve (dispersion relation) is its own inverse.

transforms from velocity to slowness "$X$". Trivially, $X$ is its own inverse. The interesting symmetry in the formulas arises because the composite transformation $TX$ is also its own inverse (as is $XT$). $TX$ transforms from the ray surface to the dispersion relation (and back again).

In figure 2 all four representations are plotted together for comparison, with the arrows showing how the four figures are related by the two transformations.

Note that corresponding regions on the dispersion relation and ray surface are always either both concave or both convex.

### The stretch theorem

The composite transformation has one more important symmetry. If either the ray surface or the dispersion relation is stretched by a constant factor, then the other domain is compressed by that factor along the same direction, point by point. This can be proven algebraically. Intuitively, it is a manifestation of the "stretch theorem" of Fourier analysis.

Since circles map onto circles, this shows that ellipses map onto ellipses. More importantly, the *best fitting* ellipse at a point in one domain can be easily mapped onto the the *best fitting*

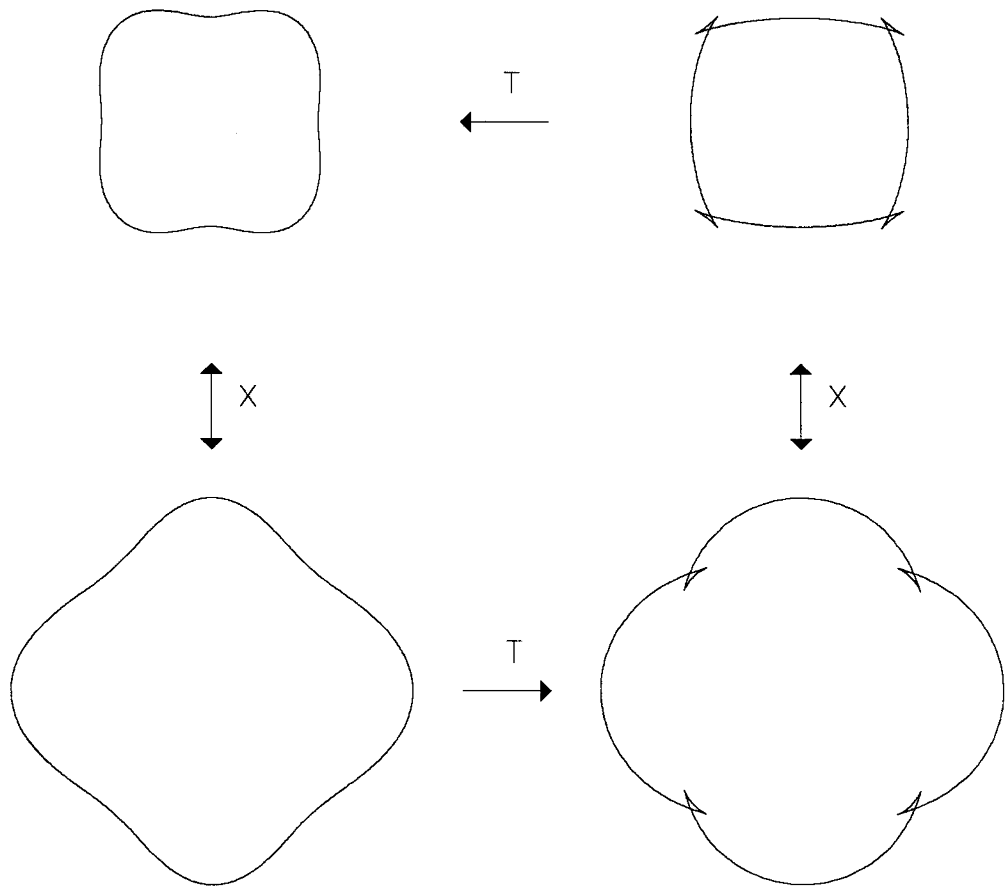


FIG. 2: Counterclockwise, starting from the upper right: ray surface (group velocity), phase velocity, dispersion relation (phase slowness), group slowness. Put another way, the right column is the "group" domain, and the left column is the "phase" domain. The top row is velocity, and the lower row is slowness.

ellipse at the corresponding point in the other domain.

## APPROXIMATIONS

The operator $TX$ does not have a convenient form, since given $V_w(\phi_w)$ we can only obtain $V_r(\phi_r)$ parametrically, as a function of $\phi_w$. It is therefore convenient to make some sort of approximation which *can* be transformed "exactly" from one domain to the other.

Given axisymmetry, there are two symmetry planes present in both the ray surface and the dispersion relation, a horizontal one and a vertical one. At these symmetry planes $\phi_w$ and $\phi_r$ are equal, and locally the operator $T$ does nothing. Since the horizontal and vertical velocities are important geophysical parameters, a reasonable approximate scheme should handle these two points correctly.

The horizontal axis of the best fitting ellipse for small offsets determines the *apparent* (NMO) velocity. This is also an important geophysical parameter. A good approximate scheme should also at least correctly map this ellipse.

Francis Muir's "nemesis" approximation described in the previous paper in this report hopefully should prove useful in anisotropic studies because it possesses these essential properties and is also quite simple.