The partial Fourier transform

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INTRODUCTION

The discrete Fourier transform transforms one discrete series into another. If the discrete series is written as a vector, then the discrete Fourier transformation (DFT) can be written as a matrix, and the transformed vector equals the DFT matrix times the original vector. This is nothing new. An interesting problem, however, is to consider fractional powers of the DFT matrix. That is, it is possible to find another matrix that, when multiplied by itself a certain number of times, gives the DFT matrix. If we multiply our input vector times this fractional-power DFT matrix, what sort of output do we get? A related question is, what do the eigenvalues and eigenvectors of the DFT matrix look like? (The eigenvalues and eigenvectors, when known, can be used to compute the fractional-power DFT matrix).

When I started thinking about this problem, I was looking for some way of expressing data in a domain that was somewhere in between the time domain and the frequency domain. I thought that this way of expressing data could be useful in such problems as the construction of time-varying frequency filters, where one is interested in something that is not entirely expressible in either domain. I am not the first to seek an intermediate domain (Goupillaud et al., 1984), nor am I the first to look into eigenvectors and fractional powers of the DFT matrix (McClellan and Parks, 1972; Dickinson and Steiglitz, 1982). In fact, other SEP members have also thought about the eigenproperties of the Fourier transform (Claerbout and Fowler, personal communications).

In the end, I produced some interesting plots and learned some things about the eigenvalues and eigenvectors of the DFT matrix, but I could not find a useful intermediate domain in which to express data. It seems the partial DFT is not useful in solving this problem.

THE DISCRETE FOURIER TRANSFORM MATRIX

As shown in FGDP (Claerbout, 1976, p.9), the discrete Fourier transform (DFT) can be written in the form of a matrix:

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{1}$$

where $\mathbf{b}$ is the original series, $\mathbf{B}$ is the transformed series, and $w \equiv e^{2\pi i/N}$, with N the number of points in the series $\mathbf{b}$ (Equation 1 shows the case of $N = 4$). Let $\overline{\mathbf{Y}}$ be defined as the matrix of w's. Then we can call $\overline{\mathbf{Y}}$ the DFT matrix, and write

$$\overline{\mathbf{Y}}\mathbf{b} = \mathbf{B}. \tag{2}$$

PARTIAL POWERS OF THE DFT MATRIX

Now that $\overline{\mathbf{Y}}$ has been defined, it is possible to imagine a matrix $\overline{\mathbf{Y}}^{1/2}$, where $\overline{\mathbf{Y}}^{1/2}\overline{\mathbf{Y}}^{1/2} = \overline{\mathbf{Y}}$. For that matter, one can imagine the matrix $\overline{\mathbf{Y}}$ raised to any fractional power. If $\mathbf{b}$ is a time series, then $\mathbf{B} = \overline{\mathbf{Y}}\mathbf{b}$ is the discrete Fourier transform of that series, and $\mathbf{B}' = \overline{\mathbf{Y}}^{1/2}\mathbf{b}$ is a partial Fourier transform of $\mathbf{b}$.

One way to find $\overline{\mathbf{Y}}^{1/2}$

There are two relatively easy ways to find $\overline{\mathbf{Y}}^{1/2}$. One is to use a program called 'matlab' on our computer. Simply type *"ysq = y**.5"*, and if you have already defined y to be the DFT matrix, then *ysq* will correspond to $\overline{\mathbf{Y}}^{1/2}$. You probably don't have access to this program, however, and so I will explain what 'matlab' is doing.

The eigenvalues and eigenvectors of $\overline{\mathbf{Y}}$ can be found using standard numerical methods. Once these have been found, the eigenvalues can be put in a diagonal matrix $\overline{\mathbf{\Lambda}}$, and the corresponding eigenvectors can be put in a matrix $\overline{\mathbf{V}}$. Then it can be shown that $\overline{\mathbf{Y}} = \overline{\mathbf{V}}\overline{\mathbf{\Lambda}}\overline{\mathbf{V}}^{-1}$ (where $\overline{\mathbf{V}}^{-1}$ is the matrix inverse of $\overline{\mathbf{V}}$). More importantly, it can be shown (Claerbout, 1976, pp. 94-96) that for any function f,

$$f\,(\overline{\mathbf{Y}}) = \overline{\mathbf{V}} f\,(\overline{\mathbf{\Lambda}})\overline{\mathbf{V}}^{-1}. \tag{3}$$

This means, for instance, that

$$\overline{\mathbf{Y}}^{1/2} = \begin{bmatrix} \\ \overline{\mathbf{V}} \\ \\ \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \cdots & \\ & & & \sqrt{\lambda_N} \end{bmatrix} \begin{bmatrix} \\ \overline{\mathbf{V}}^{-1} \\ \\ \end{bmatrix}, \tag{4}$$

where λ_1, λ_2, $\cdots$, λ_N are the eigenvalues contained in $\overline{\Lambda}$. In a later section I'll discuss what the eigenvalues and eigenvectors look like.

Another way to find $\overline{\mathbf{Y}}^{1/2}$

Shuki Ronen (personal communication) suggested another way to find partial powers of the DFT matrix; his method does not involve eigenvectors, and gives quite a bit of insight into what one should expect from applying partial DFT's to data.

It is possible to use a Taylor expansion to evaluate partial powers of the DFT matrix. Then

$$\sqrt{\overline{\mathbf{Y}}} = \overline{\mathbf{I}} + \frac{1}{2}(\overline{\mathbf{Y}} - \overline{\mathbf{I}}) - \frac{1}{8}(\overline{\mathbf{Y}} - \overline{\mathbf{I}})^2 + \frac{1}{16}(\overline{\mathbf{Y}} - \overline{\mathbf{I}})^3 - \cdots \tag{5}$$

Note that $\overline{\mathbf{Y}}^2 = \overline{\mathbf{H}}$, where

$$\overline{\mathbf{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

(shown here for $N = 5$). This simply shows in matrix form the well-known result that a DFT applied twice produces a backwards time series. Note also that $\overline{\mathbf{Y}}^3 = \overline{\mathbf{Y}}\overline{\mathbf{H}}$ (a backwards DFT matrix), and $\overline{\mathbf{Y}}^4 = \overline{\mathbf{I}}$.

If we substitute these into Equation (5), we obtain the qualitative result that $\overline{\mathbf{Y}}^{1/2}$ looks like $\overline{\mathbf{Y}}$ overlaid with $\overline{\mathbf{H}}\overline{\mathbf{Y}}$ overlaid with $\overline{\mathbf{I}}$ and $\overline{\mathbf{H}}$. So $\overline{\mathbf{Y}}^{1/2}\mathbf{b}$ looks like $\mathbf{b}$ + (time-flipped $\mathbf{b}$) + $\mathbf{B}$ + (flipped $\mathbf{B}$). This suggests a strange-looking result, not much like the one I had originally hoped for.

NUMERICAL RESULTS

In this section I will show the result of applying partial DFT's to some simple time functions. First, though, it is interesting to examine what the DFT matrix looks like. The real and complex parts of the DFT matrix $\overline{\mathbf{Y}}$ are shown in Figure 1, with black representing positive values and white representing negative values. Similarly, Figure 2 shows the partial DFT matrix $\overline{\mathbf{Y}}^{1/2}$. The black diagonal line corresponds to the identity matrix, so this Figure gives graphical expression to the assertion made above, that $\overline{\mathbf{Y}}^{1/2}$ is in some sense a superposition of $\overline{\mathbf{I}}$ (the identity matrix), $\overline{\mathbf{Y}}$ (the DFT matrix), and their reversed forms. Figure 3, which shows $\overline{\mathbf{Y}}^{.01}$, has been enhanced (the black-white contrast increased) by about a hundred times to show that $\overline{\mathbf{Y}}^{.01}$ equals $\overline{\mathbf{I}}$ plus a slight

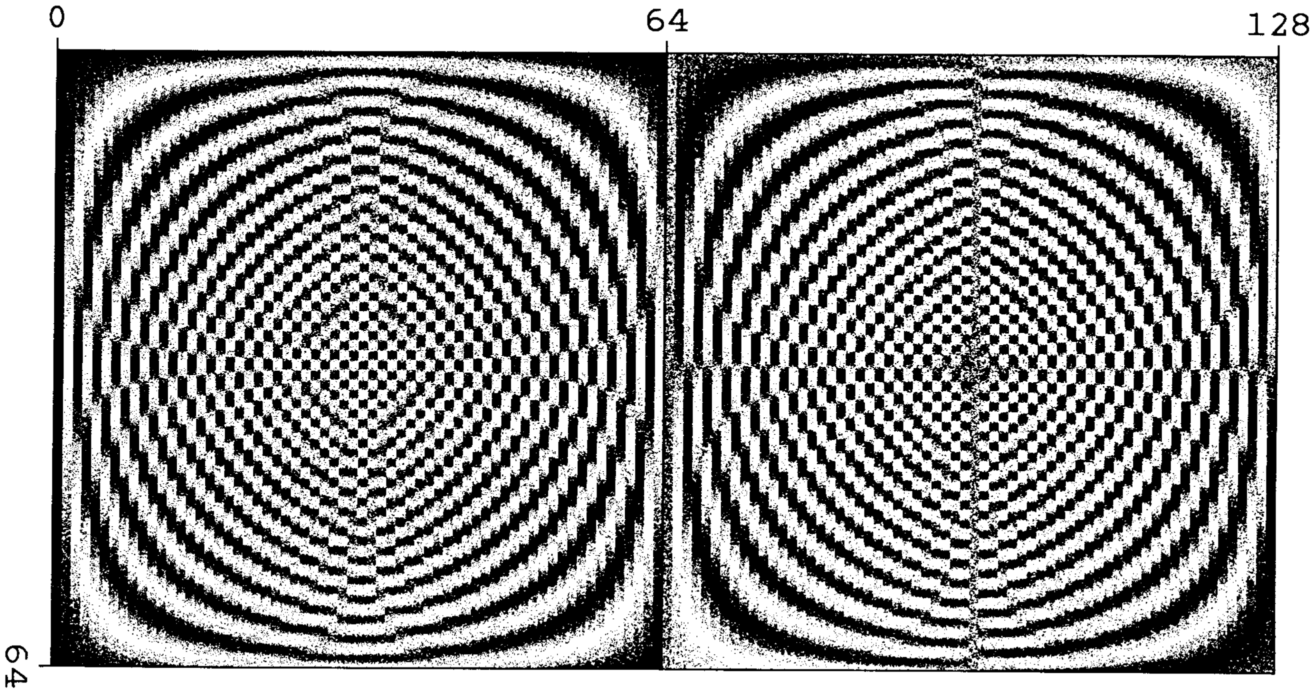

FIG. 1. The real and complex parts of the DFT matrix $\overline{\mathbf{Y}}$ (see Equation (1)). Black represents positive values and white negative values.

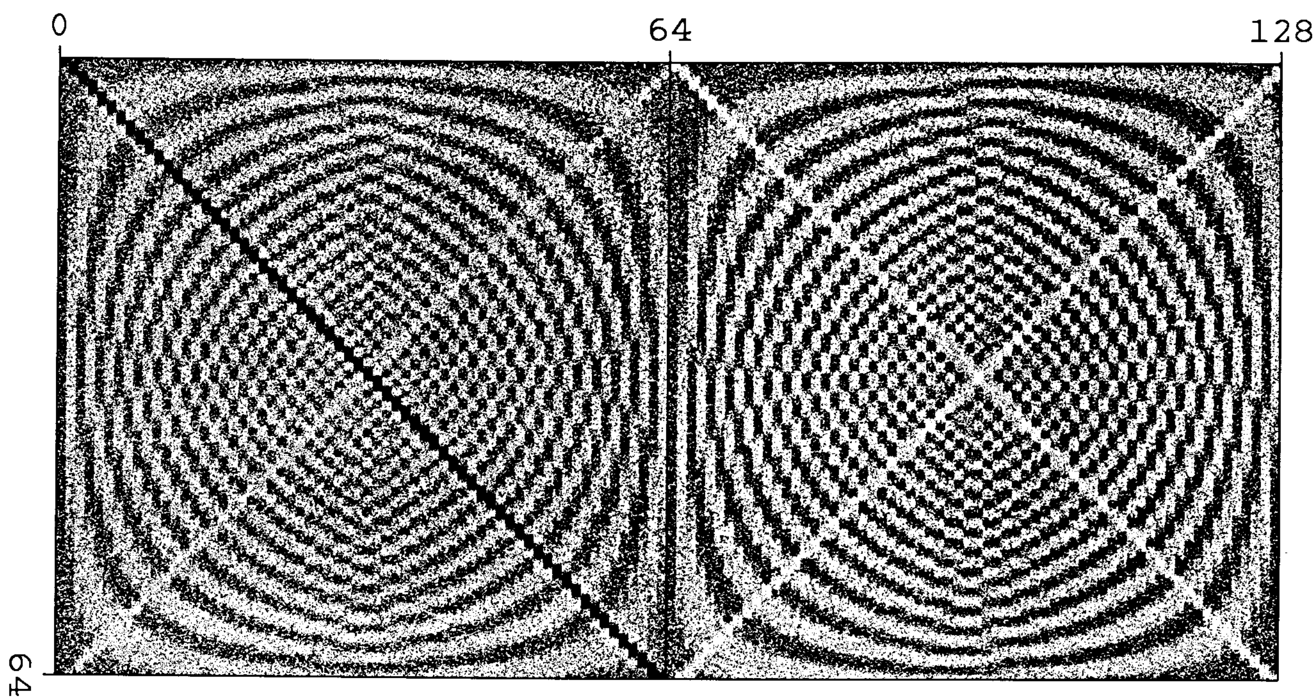

FIG. 2. The real and complex parts of the partial DFT matrix $\overline{\mathbf{Y}}^{1/2}$. Notice the black diagonal line corresponding to the identity matrix, and the line with the opposite slope, which corresponds to the reversed identity matrix.

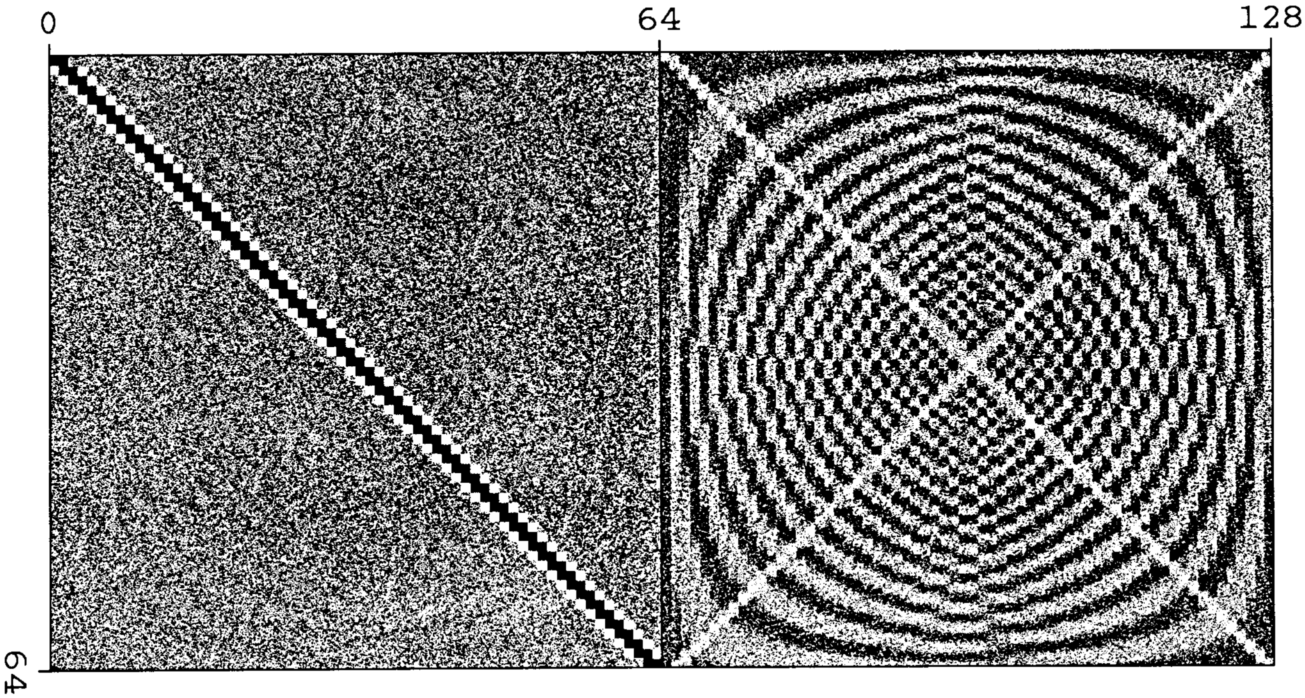

FIG. 3. The real and complex parts of the partial DFT matrix $\overline{\mathbf{Y}}^{.01}$. The black-white contrast has been enhanced a hundred times over that in Figures 1 and 2, in order to bring out the weak imaginary component. Note the strong real component, which is essentially the identity matrix $\overline{\mathbf{I}}$.

imaginary component.

In Figure 4 I show the result of applying some partial DFT's to a particular time series. Fig. 4a shows the original time series, $\mathbf{b}$. Figures 4b-e show, respectively, $\overline{\mathbf{Y}}^{1/5}\mathbf{b}$, $\overline{\mathbf{Y}}^{2/5}\mathbf{b}$, $\overline{\mathbf{Y}}^{3/5}\mathbf{b}$, and $\overline{\mathbf{Y}}^{4/5}\mathbf{b}$. Figure 4f shows $\overline{\mathbf{Y}}\mathbf{b}$, that is, $\mathbf{B}$, the Fourier transform of $\mathbf{b}$.

When I began this project I was naively hoping that I could find a reasonable intermediate stage between a time series and its Fourier transform. I thought that the wavelet in Figure 4a, for instance, might slide over gradually until it reached its final position (or positions) in Figure 4f. Instead, the intermediate stage turned out to be an arithmetic combination of the original series plus the transformed series, with some time-reversal thrown in (note the highlighted bump in Figure 4c). If there is a sort of intermediate stage of representing data, partly in the time domain and partly in the frequency domain, it seems that partial DFT's are not the right way to seek it.

EIGENVALUES AND EIGENVECTORS OF THE DFT MATRIX

There are only four distinct eigenvalues of the DFT matrix: ± 1, and $\pm i$. Since there are, in all, N eigenvalues for the $N \times N$ DFT matrix, there is a great deal of

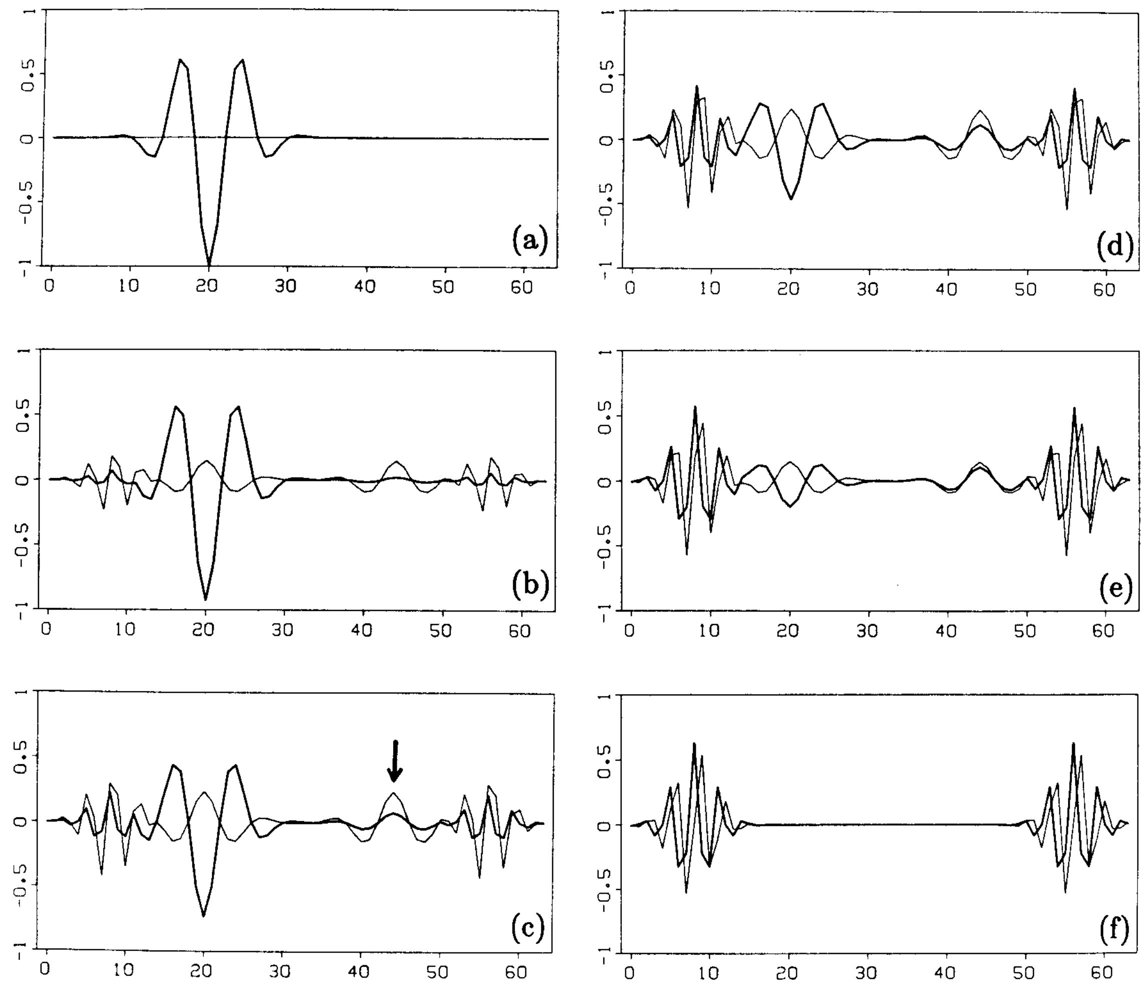

FIG. 4. Partial DFT's applied to a particular time series. Figures 4a-f show, respectively, $\overline{\mathbf{Y}}^{0}\mathbf{b}$ (**b**), $\overline{\mathbf{Y}}^{1/5}\mathbf{b}$, $\overline{\mathbf{Y}}^{2/5}\mathbf{b}$, $\overline{\mathbf{Y}}^{3/5}\mathbf{b}$, $\overline{\mathbf{Y}}^{4/5}\mathbf{b}$, and $\overline{\mathbf{Y}}^{1}\mathbf{b}$ (**B**, the Fourier transform of **b**). Heavy lines show the real components, and light lines show the imaginary. In 4c an arrow highlights the bump due to the reversed-time form of the original time-series **b**.

degeneracy. Thus one cannot speak of a definite set of eigenvectors; there are an infinite number of different possible sets of orthonormal eigenvectors. After I worked for a while on the eigenvector problem I came across a reference (Dickinson and Steiglitz, 1982) to an earlier paper that contained most of the results I sought. This paper (McClellan and Parks, 1972) shows, among other interesting things, that if **b** is any even time series, then $\overline{\mathbf{Y}}\mathbf{b} \pm \mathbf{b}$ is an eigenvector, with an eigenvalue of ± 1, while if **b** is any odd time series, then $\overline{\mathbf{Y}}\mathbf{b} \mp \mathbf{b}$ is an eigenvector, with an eigenvalue of $\pm i$. The problem of constructing an orthonormal set of eigenvectors is discussed in (Dickinson and Steiglitz, 1982).

CONCLUSIONS

It is possible to find partial DFT's, which are defined as partial powers of the DFT matrix $\overline{\mathbf{Y}}$. Analysis by Taylor-series expansion shows that the partial DFT matrix can be represented in the form of a sum of the DFT matrix, the identity matrix, and their reversed forms. I must admit that I see no use for the partial DFT in seismic processing, although I entertained some hopes at the beginning of the project.

ACKNOWLEDGEMENTS

The main acknowledgement should go to Shuki Ronen, who suggested using Taylor-series expansions to look at partial powers of the DFT matrix. Doug Jones, a graduate student at Rice University, provided the main references ((Dickinson and Steiglitz, 1982) and (McClellan and Parks, 1972)) that I used. Joe Dellinger introduced me to his friend Doug, and we corresponded electronically via the ARPAnet. I would also like to thank Paul Fowler for some helpful discussions.

REFERENCES

Claerbout, J.F, 1976, Fundamentals of geophysical data processing: New York, McGraw-Hill.

Dickinson, B.W., and Steiglitz, K., 1982, Eigenvectors and functions of the discrete Fourier transform: IEEE transactions on acoustics, speech, and signal processing, ASSP-30 , 25-31.

McClellan, J.H., and Parks, T.W., 1972, Eigenvalue and eigenvector decomposition of the discrete Fourier transform: IEEE transactions on audio and electroacoustics, AU-20 , 66-74.

Goupillaud, P.L., Grossman, A., and Morlet, J., 1984, A simplified view of the cycle-octave and voice representations of seismic signals: Society of Exploration Geophysicists expanded abstracts with biographies, 1984 technical program, 379-382.

Are you ready to enter Moscow State University? (Answers)

In the last SEP report (SEP-41, p. 326) I gave an example of the entrance exam that would be taken by applicants wishing to be admitted to Moscow State University as undergraduates in geophysics. Applicants have four hours to complete the test. You have had more than four months. Here are the answers (taken from the back of the book; I haven't checked them for accuracy).

Answers to geophysics exam

1. $x = -1;\ x = 4$
2. $|BL| > |BG|$
3. $x = \frac{1}{2};\ 1 \le x \le 2;\ x = \frac{5}{2}$
4. $x = \frac{\pi}{2} + 2k\pi,\ y = \pi - 2k\pi,$ for all $k = 0, \pm 1, \pm 2, \ldots$
5. Point A
6. $\frac{3\pi}{10} \cdot \frac{1}{1+\sqrt{1+\frac{6}{\sqrt{10}}}}$