A simple geometric derivation of the ray spreading factor

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INTRODUCTION

Zero-th order asymptotic ray theory, or geometric ray theory, predicts ray amplitudes on the assumption that no energy passes through boundaries of ray tubes. It is a high-frequency approximation that is accurate in the "far field". The merits of ray theory are that it gives a good intuitive first-order understanding of wave phenomenon and it provides an analytic expression for amplitude which can be computed rapidly. The term "ray spreading factor" is used to denote the geometric factor that quantifies the amount of ray tube spreading between some initial and final locations on a given raypath. This paper gives a simple derivation of the ray spreading factor for a medium consisting of homogeneous regions separated by smooth two dimensional boundaries. The derivation is based on elementary calculus, geometry and algebra and is therefore easy to understand intuitively. The expressions obtained match those found in the literature and have been used in this SEP report (Mora, Elastic inversion using ray theory). Refer to Cerveny and Ravindra (1971) or Hubral and Krey (1980) for a different derivation that allows three dimensional boundaries. The worth of the current paper is the simplicity of the derivation.

There are two main parts to the ray spreading derivation: (i) determining the effect of a curved interface on the wavefront radius of curvature and (ii) finding how the ray spreading factor changes along a raypath passing though a series of homogeneous regions. Given the result of (i) it is easy to derive (ii), which is the final result.

EFFECT OF A CURVED INTERFACE ON WAVEFRONT CURVATURE

Consider a ray of known radius of curvature incident upon a curved interface. The wavefront radii of curvature of the scattered rays required in the ray spreading computations can be derived from the geometry and using Snell's Law. For convenience, the following derivation will be restricted to a refracted ray, although later the results will be generalized to include reflected modes.

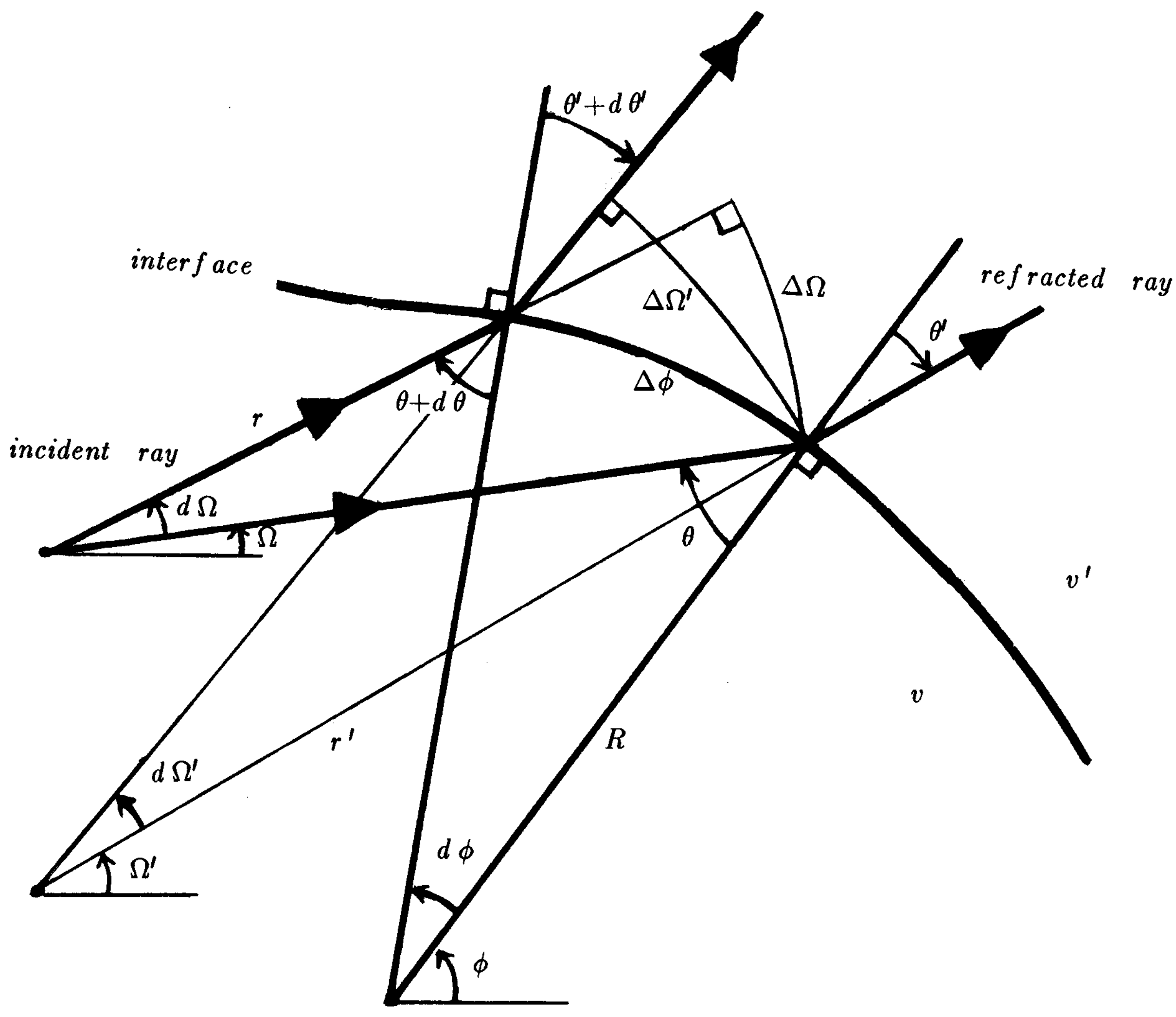

FIG. 1. Geometry of a ray tube incident on a curved interface.

Observe the geometry in figure 1, which depicts a curved interface with an incident ray and a single refracted ray. The incident and scattered ray tubes are drawn with a finite width but all infinitesimal angle increments should be considered in the limit as tending to zero in the proceeding development. Unprimed quantities in the following equations refer to the incident ray, while primed quantities indicate the refracted ray. The following expressions are apparent from the geometry and Snell's Law.

$$r' d\Omega' = \Delta\Omega' \tag{1a}$$

$$r d\Omega = \Delta\Omega \tag{1b}$$

$$r d\phi = \Delta\phi \tag{1c}$$

$$\theta = \phi - \Omega$$

$$\theta + d\theta = \phi + d\phi - \Omega - d\Omega$$

From the above equations

$$d\theta = d\phi - d\Omega \tag{2a}$$

Similarly,

$$d\theta' = d\phi - d\Omega' \tag{2b}$$

Also, from the geometry in figure 1

$$\Delta\Omega = \Delta\phi\cos\theta \tag{3a}$$

$$\Delta\Omega' = \Delta\phi\cos\theta' \tag{3b}$$

And from Snell's Law

$$\frac{\sin\theta}{\sin\theta'} = \frac{\sin(\theta + d\theta)}{\sin(\theta' + d\theta')} = \frac{v}{v'} \tag{4}$$

Equations (1) through (4) summarize the geometry and provide the basis for the ray spreading derivation. An equation for the wavefront radius of curvature of the refracted ray r' in terms of that of the incident ray r is obtained by rearranging (1a) and using (3b), (3a) and (1b).

$$r' = \frac{\Delta\Omega'}{d\Omega'} = \frac{\Delta\phi\cos\theta'}{d\Omega'} = \frac{\Delta\Omega\cos\theta'}{d\Omega'\cos\theta} = r\frac{d\Omega\cos\theta'}{d\Omega'\cos\theta} \tag{5}$$

The infinitesimal angle $d\Omega'$ may be written in terms of $d\Omega$ by rearranging equations (2b) and using (1c), (3a) and (1b).

$$d\Omega' = d\phi - d\theta' = \frac{\Delta\phi}{R} - d\theta'$$

$$= \frac{\Delta\Omega}{R\cos\theta} - d\theta' = \frac{r\,d\Omega}{R\cos\theta} - d\theta' \tag{6}$$

The remaining unknown $d\theta'$ is eliminated by expanding Snell's Law (equation (4)) as a truncated Taylors series in order to remove angular infinitesimals from the trigonometric arguments.

$$\frac{v}{v'} = \frac{\sin(\theta+d\theta)}{\sin(\theta'+d\theta')} = \frac{\sin\theta + d\theta\cos\theta}{\sin\theta' + d\theta'\cos\theta'}$$

This equation can be rearranged to give an equation for $d\theta'$ in terms of $d\theta$. The resulting expression for $d\theta'$ is now substituted into equation (6) followed by successive use of equations (4), (2a), (1c), (3a) and (1b).

$$d\theta' = \frac{\frac{v'}{v}(\sin\theta + d\theta\cos\theta) - \sin\theta'}{\cos\theta'}$$

$$= \frac{v'}{v\cos\theta'}(\sin\theta + d\theta\cos\theta - \sin\theta)$$

$$= \frac{v'\cos\theta d\theta}{v\cos\theta'}$$

$$= \frac{v'\cos\theta}{v\cos\theta'}\left(d\phi - d\Omega\right)$$

$$= \frac{v'\cos\theta}{v\cos\theta'}\left(\frac{\Delta\phi}{R} - d\Omega\right)$$

$$= \frac{v'\cos\theta}{v\cos\theta'}\left(\frac{\Delta\Omega}{R\cos\theta} - d\Omega\right)$$

$$= \frac{v'\cos\theta}{v\cos\theta'}\left(\frac{r}{R\cos\theta} - 1\right)d\Omega \tag{7}$$

Substituting equation (7) into equation (6) to eliminate $d\theta'$ gives

$$d\Omega' = d\Omega\left(\frac{r}{R\cos\theta} - \frac{v'}{v}\frac{r}{R\cos\theta'} + \frac{v'}{v}\frac{\cos\theta}{\cos\theta'}\right) \tag{8}$$

The refracted radius of curvature r' in terms of the incident radius of curvature r is obtained by substituting equation (8) into (5)

$$r' = r\left(\frac{v'}{v}\frac{\cos^2\theta}{\cos^2\theta'} + \frac{r}{R\cos^2\theta'}\left(\cos\theta' - \frac{v'}{v}\cos\theta\right)\right)^{-1} \tag{9}$$

Equation (9) gives the radius of curvature of a refracted wavefront in terms of that of the incident wavefront. Note the convention that the interface radius of curvature R is positive when the interface is concave relative to the incident ray, and negative when the interface is convex relative to the incident ray.

For a reflected ray the geometry and equations remain identical except that the reflected scattering angle would be negative of the refracted angle so equation (2b) would be replaced by

$$d\theta' = d\Omega' - d\phi$$

so equation (6) becomes

$$d\Omega' = \frac{r\,d\Omega}{R\cos\theta} + d\theta'$$

This leads to the equivalent of equation (9), for radius of curvature of reflected wavefronts

$$r' = r\left(-\frac{v'}{v}\frac{\cos^2\theta}{\cos^2\theta'} + \frac{r}{R\cos^2\theta'}\left(-\cos\theta' - \frac{v'}{v}\cos\theta\right)\right)^{-1}$$

The negative must be taken of the above expression because the reflected ray reverses direction. This ensures the correct sign for the wavefront radius of curvature is retained, namely positive for an expanding wavefront.

Therefore, the equation describing the effect of a two dimensional curved interface on wavefront radius of curvature is

$$r' = r\left(\frac{v'}{v}\frac{\cos^2\theta}{\cos^2\theta'} - \frac{r}{R\cos^2\theta'}\left(\frac{v'}{v}\cos\theta +/- \cos\theta'\right)\right)^{-1} \qquad (10)$$

where the plus sign is used for reflected rays while the minus sign is for refracted rays. The equation requires the following quantities:

r = incident wavefront radius of curvature where positive represents an expanding wavefront and negative a collapsing wavefront

R = interface radius of curvature, respectively positive or negative when the interface is concave or convex relative to the incident ray

θ = incident angle

θ' = scattering angle

v = incident medium velocity

v' = scattering medium velocity

A special case of (10) occurs when the ray is reflected without changing mode type. In this case $v' = v$ and $\theta' = \theta$ so the equation for wavefront radius of curvature is

$$r' = r\left(1 - \frac{2r}{R\cos\theta}\right)^{-1} \qquad (11)$$

RAY SPREADING THROUGH A SERIES OF HOMOGENEOUS LAYERS

Consider a ray propagating through a series of homogeneous layers. As it travels through each layer, the ray tube either spreads as the wavefront diverges or contracts as the wavefront converges. The ray spreading parameter S is directly related to the cross sectional area of the ray tube. For a two dimensional medium, S may be expressed as the product of an "in plane" and "out of plane" spreading parameters, respectively denoted S_{in} and S_{out}. ("in plane" refers to the ray spreading being in the plane of the two dimensional structure.)

Because each layer is homogeneous, the wavefront radius of curvature after the ray has propagated through i layers is

$$r_i = r'_{i-1} + d_i \qquad (12)$$

where d_i is the distance between the $(i-1)$-th and the i-th intersection points (see figure 2). From equation (10) we have the scattered wavefront radius of curvature at the i-th ray-interface intersection,

$$r'_i = r_i / \Delta_i \qquad (13)$$

where Δ_i is defined as

$$\Delta_i = \frac{v'_i \cos^2\theta_i}{v_i \cos^2\theta'_i} - \frac{r}{R\cos^2\theta'_i}\left(\frac{v'_i}{v_i}\cos\theta_i \; +/- \; \cos\theta'_i\right) \qquad (14)$$

From figure 2, the ray spreading parameter at the n-th intersection point is given by

$$S_{in} = r_n \, d\Omega_n / d\Omega_0 \qquad (15)$$

where $d\Omega_n$ refers to the infinitesimal angular width of the ray tube at the n-th intersection point. By recursive substitutions using equations (12) and (13), the wavefront radius of curvature at the n-th intersection point r_n can be obtained

$$\begin{aligned} r_n &= r_{n-1}/\Delta_{n-1} + d_n \\ &= (r_{n-2}/\Delta_{n-2} + d_{n-1})/\Delta_{n-1} + d_n \end{aligned} \qquad (16)$$

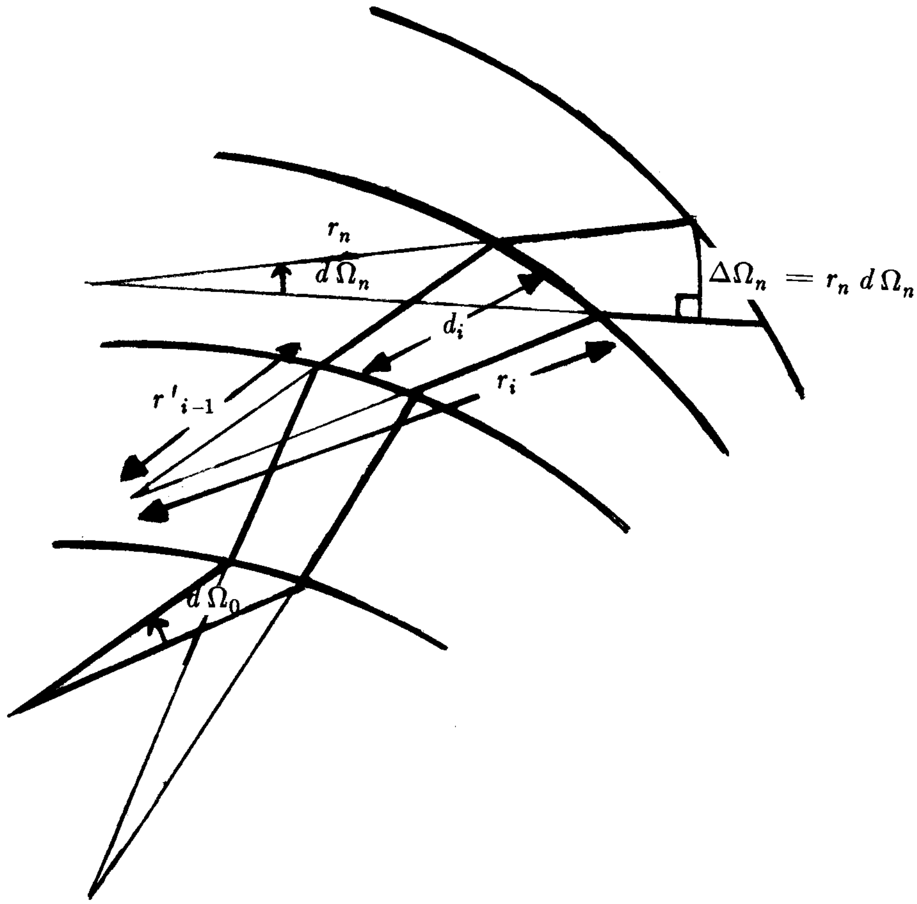

FIG. 2. Geometry of a ray tube passing through a series of homogeneous layers separated by by curved interfaces

$$= \cdots = d_n + \sum_{i=1}^{n-1} d_i \prod_{j=i}^{n-1} 1/\Delta_j$$

From equations (1) and (3) we have

$$\frac{d\Omega'}{d\Omega} = \frac{r\cos\theta'}{r'\cos\theta} = \Delta\frac{\cos\theta'}{\cos\theta}$$

Hence, by recursive substitution, the angular ratio in equation (15) is

$$\frac{d\Omega_n}{d\Omega_0} = \Delta_{n-1}\frac{\cos\theta'_{n-1}}{\cos\theta_{n-1}}\frac{d\Omega_{n-1}}{d\Omega_0}$$

$$= \cdots = \prod_{k=1}^{n-1} \Delta_k \frac{\cos\theta'_k}{\cos\theta_k} \tag{17}$$

Substituting equations (17) and (16) into (15) yields the desired expression for the in plane spreading parameter S_{in} .

$$S_{in} = \left\{ d_n + \sum_{i=1}^{n-1} d_i \prod_{j=i}^{n-1} 1/\Delta_j \right\} \prod_{k=1}^{n-1} \Delta_k \frac{\cos\theta'_k}{\cos\theta_k}$$

Multiplying out the Δ products yields

$$S_{in} = \left\{ d_1 + \sum_{i=2}^{n} d_i \prod_{j=1}^{i-1} \Delta_j \right\} \prod_{k=1}^{n-1} \frac{\cos\theta'_k}{\cos\theta_k} \qquad (18)$$

A useful special case of equation (18) occurs when a two way path has coincident upgoing and downgoing paths so the cosine product in (18) is unity. This special case would apply to the simulation of zero-offset wavefields.

The out of plane spreading parameter can be obtained from equation (18) by setting θ and θ' equal to zero and the interface radius of curvature R , to infinity. Thus

$$\Delta_i = \frac{v'_i}{v_i} \qquad (19)$$

so

$$S_{out} = \frac{1}{v_1} \sum_{i=1}^{n} d_i \, v_i \qquad (20)$$

Finally, the cross sectional area of a ray tube is proportional to the product of the in plane and out of plane spreading parameters, so

$$S = S_{in} \, S_{out} \qquad (21)$$

This cross sectional area is termed the "ray spreading parameter" and is inversely proportional to the energy density. Hence, the ray amplitude is proportional to the inverse of the square root of the ray spreading parameter.

$$ray \;\; amplitude = \frac{constant}{\sqrt{S}} = \frac{constant}{\sqrt{S_{in} \, S_{out}}} \qquad (22)$$

The meaning of a negative value for S_{in} is that the ray has passed through a two dimensional focus. This negative value for S_{in} implies a purely imaginary ray amplitude (in the frequency domain) and hence a 90^0 phase shifted signal in the time domain.

NOTE ON TPOW

The wave divergence correction is not the first power of time unless velocity is constant (see Newman, 1973). This correction is the factor that removes the effect of wave divergence from the seismic amplitudes. Hence, from equation (22), it is clear that the ray-theoretical spreading correction is the square root of the ray spreading parameter S. The formulas in this paper give the ray spreading parameter in the presence of curved interfaces. For the special case of a vertical ray passing through a medium in which the velocity depends only on z, we have from equation (20)

$$\sqrt{S} = S_{out} = \frac{1}{v(0)} \int_0^T v^2(t)\, dt \tag{23}$$

Therefore, the spreading correction is proportional to $v_{rms}^2 t$. This is in agreement with Newman's results.

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