# A review of some seismic inversion methods

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## ABSTRACT

In this paper, we will review some seismic inversion methods. These methods vary in their approaches and the assumptions they make. Some results will be given for further illustration of the differences.


## INTRODUCTION

In the seismic experiment, the energy source used at the surface of the earth is ideally represented by a delta function $\delta(\mathbf{r}, t)$, where $r^2 = x^2 + y^2 + z^2$, and we measure the field $\psi(\mathbf{r}, t)$ at the surface $z = 0$ over a range of time and offset. The physical quantity $\psi(\mathbf{r}, t)$ that is measured can be pressure, velocity, or acceleration. Our task is to interpret these measurements to determine the structure of the subsurface. Migration is one method that is commonly used to interpert the data: it moves (or *migrate*) events to where they belong, and thus provides a better image of the subsurface than that of the unmigrated section. We know that each method of migration involves few approximations; but we also know from experience that they are usually good approximations. We should remember first, that migration *needs* the velocity of the subsurface and it does not provide it, and, second, that other parameters of the subsurface, such as density and bulk modulus, cannot be deduced from migration. (Al-Yahya and Muir, (1984) and Yilmaz and Chambers (1984) attempted to use migration itself as a means to estimate the velocity of the subsurface.)

Other methods of interpretation must be used to deduce the density and other elastic (or acoustic) parameters of the subsurface. The ideal method would be to start with the differential equation governing the propagation of waves inside the earth and work out the distribution of the earth's parameters (density, velocity, or bulk modulus for example). This procedure is referred to as inversion. There are various inversion methods that can be classified as follows: direct methods and iterative methods.

In an iterative method, an initial model of the medium is used to produce a synthetic seismogram. The model is then modified to produce a seismogram that better resembles the data. The model can be modified with generalized linear inversion, which eventually leads to a least-squares solution (Cooke and Schneider, 1983). The least-squares solution is the *best* solution from a probabilistic point of view.

Direct methods can be divided into two groups: exact methods and approximate methods. It should be noted, first, that both of these methods involve approximations. The difference between them is that approximate methods uses approximated equation from the beginning, whereas exact methods use approximations at a late stage; and, in the exact method, the approximations are used only in *solving* the equations not in the equations themselves.

A few SEP papers have discussed inversion, especially the Born inversion. This paper discusses the Born inversion with equations that are cast in a different form, which hopefully makes them more understandable. We will look at the Born inversion in the general context of inversion and see its relation to migration.

## THE RELATION BETWEEN MIGRATION AND INVERSION

Both migration and inversion start with same physical law that governs the propagation of waves in the medium, namely the wave equation. From this point they use different methods to reach their final goal. It is interesting to see what relationship of any exists between the two. Cheng (1984) studied this relationship between migration and one kind of inversion, the Born inversion (discussed below). He showed that they are related mathematically by

$$\frac{\partial}{\partial z}\alpha(x,y,z) = h_r(x,y,z) *** R(x,y,z) \tag{1a}$$

$$\frac{\partial}{\partial z}\alpha(x,y,z) = Mig[p_{mod}(x,y,z=0,t)] \quad , \tag{1b}$$

where

$\alpha(x,y,z)$ is the velocity function defined by equation (3)

$***$ denotes three-dimensional convolution

$h_r(x,y,z) = \frac{1}{\pi v}\frac{\partial^3}{\partial z^3}(\frac{z}{r})$

$Mig$ is the migration operator

$p(x, y, z = 0, t)$ is the recorded data

$p_{mod}(x, y, z = 0, t)$ is the modified data

The modification of the data is done by convolving the recorded data with $h_d(x, y, t)$, where

$$h_d(x,y,t) = \frac{1}{(2\pi)^2} \frac{H(\frac{\rho}{v} - t)}{\rho v} \left[ \frac{\partial^2}{\partial t^2} \frac{4}{3} (1 - \frac{v^2 t^2}{\rho^2})^{\frac{3}{2}} + \pi (1 - \frac{v^2 t^2}{\rho^2}) + 2 (1 - \frac{v^2 t^2}{\rho^2})^{\frac{1}{2}} \right]$$

$$\rho = \sqrt{x^2 + y^2}$$

Equations (1a) and (1b) say that we can get the Born inversion result by convolving the reflectivity, which is the result of migration, with the spatial operator $h_r(x, y, z)$; alternatively, we can get it by modifying the recorded data and then migrating the modified data.

## THE BORN INVERSION: a direct and approximate method

Assuming a constant density, the one-dimensional acoustic wave equation for the scattered field in the frequency domain is

$$\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2(z)} \} \psi^{sc}(z, \omega) = 0 \tag{2}$$

If we define

$$\alpha(z) = \frac{v_0^2}{v^2(z)} - 1 \quad , \tag{3}$$

where $v_0$ is a reference velocity (usually the velocity at the surface), equation (2) can be written as

$$\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_0^2} \} \psi^{sc}(z, \omega) = - \frac{\omega^2}{v_0^2} \alpha(z) \psi^{sc}(z, \omega) \tag{4}$$

Clayton (1981) showed that this reference (or background) velocity can be made to be a slowly varying function of depth.

Let's now consider the equation

$$\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_0^2} \} G(z, \omega, \xi) = -\delta(z - \xi) \tag{5}$$

Equation (2) is similar to equation (5). The difference between them is that the latter has a source (of unit strength) located at $z = \xi$. $G(z, \omega, \xi)$ in equation (5) will be recognized as the Green's function

$$G(z, \omega, \xi) = \frac{e^{-i \frac{\omega}{v_0} |z - \xi|}}{2i \frac{\omega}{v_0}} \tag{6}$$

As shown in the appendix, equations (4) and (5) can be combined to give

$$\psi(z,\omega) = e^{-i\frac{\omega}{v_0}z} + \frac{\omega^2}{v_0^2}\int_0^\infty \alpha(\xi)\psi(\xi,\omega)G(z,\omega,\xi)d\xi \tag{7}$$

where we decomposed the total field $\psi(z,\omega)$ into an incident field $e^{-i\frac{\omega}{v_0}z}$ and a scattered field $\psi^{sc}$. Equation (7) is an integral equation where the unknown is $\alpha(\xi)$. Note that the equation we arrived at is exact. No approximations have been made so far. The Born approximation is an attempt to make this equation easy to solve.

## The first Born approximation

For the first Born approximation, the total field $\psi(\xi,\omega)$ in the integrand of equation (7) is replaced by the incident field $\psi^{in}(z,\omega) = e^{-i\frac{\omega}{v_0}\xi}$, which means that this approximation does not hold for media with strong contrasts. As shown in the appendix, this approximation leads to

$$\alpha(z) = -4\int_0^{\frac{2z}{v_0}} R(t)dt \quad , \tag{8}$$

from which the velocity function can be obtained as

$$v(z) = v_0\sqrt{\frac{1}{\alpha(z)+1}} \quad ,$$

which imposes the restriction $\alpha(z) > -1$.

## The second Born approximation

A second (and better) approximation can be obtained by the following change of variables

$$\frac{dz}{d\xi} = v(z) \quad \rightarrow \quad \xi = \int_0^z \frac{dz'}{v(z')} \tag{9}$$

From the appendix, we see that this leads to

$$v(\xi) = \exp\left[2\int_0^{2\xi} R(t)dt\right] \tag{10}$$

and the depth $z$ can be found from equation (9)

$$z(\xi) = \int_0^\xi v(\xi')d\xi'$$

Note that in the first Born approximation we replaced the field by $e^{-i\frac{\omega}{v_0}}$, but in the second we replaced it by $\exp\left[-i\omega \int_0^z \frac{1}{v(\xi)} d\xi\right]$, which is a WKJB approximation.

We see from Figure 1 that the second Born approximation yields a better result than does the first. The difference between the two methods is made very clear when the velocity contrast is big. In fact, if we used a larger contrast, we would not be able to reconstruct the velocity function using the first approximation, because of the restriction that $\alpha$ should be greater than -1. Another difference between the two approximations, is that the first one requires a background velocity and thus is sensitive to the choice of this velocity. Weglein and Gray (1983) showed that when the first Born approximation is used, there is a tradeoff in accuracy between the velocity and the position of the reflector. We see in Figure 1 that the first Born inversion places layer boundaries before their actual position.

The two approximations we have discussed are the first in a series. Clayton (1981) showed that it is possible to go to higher-order approximations by using more terms in the Born series; adding terms in the series means including higher-order multiples.

## AN EXACT INVERSION METHOD

We now look at an exact inversion method. The reader will remember that approximate methods of solutions are used. The main advantages of direct methods are their speed and independence of an initial guess; their main problem is stability.

The most important shortcoming of particular this method is that it needs all of the low frequencies, so it is useless for ordinary seismic surveys in which low frequencies are missing. Also, because this method is nonlinear, scaling the data is a very important step.

### The algorithm

Starting with the observed field $p(t)$, we scale it by $-2v_0^2$ to obtain $B(t)$.

$$B(t) = -\frac{p(t)}{2v_0^2}$$

Weidelt (1972) suggested the following algorithm to reconstruct the velocity function of the subsurface from $B(t)$.

1. $A(\xi,\eta) = B(\xi+\eta) + \int_{-\xi}^{\xi} A(\xi,t)[B(\xi+t) + B(\eta-t)]dt, \quad |y| \le x, x \ge 0$

2. $G(\xi) = 1 + \int_{-\xi}^{\xi} A(\xi,t)dz$

3. $v(z) = \frac{v_0}{G^2(\xi)}$ and $z(\xi) = \int_0^{\xi} \frac{1}{G^2(\xi)} d\xi$

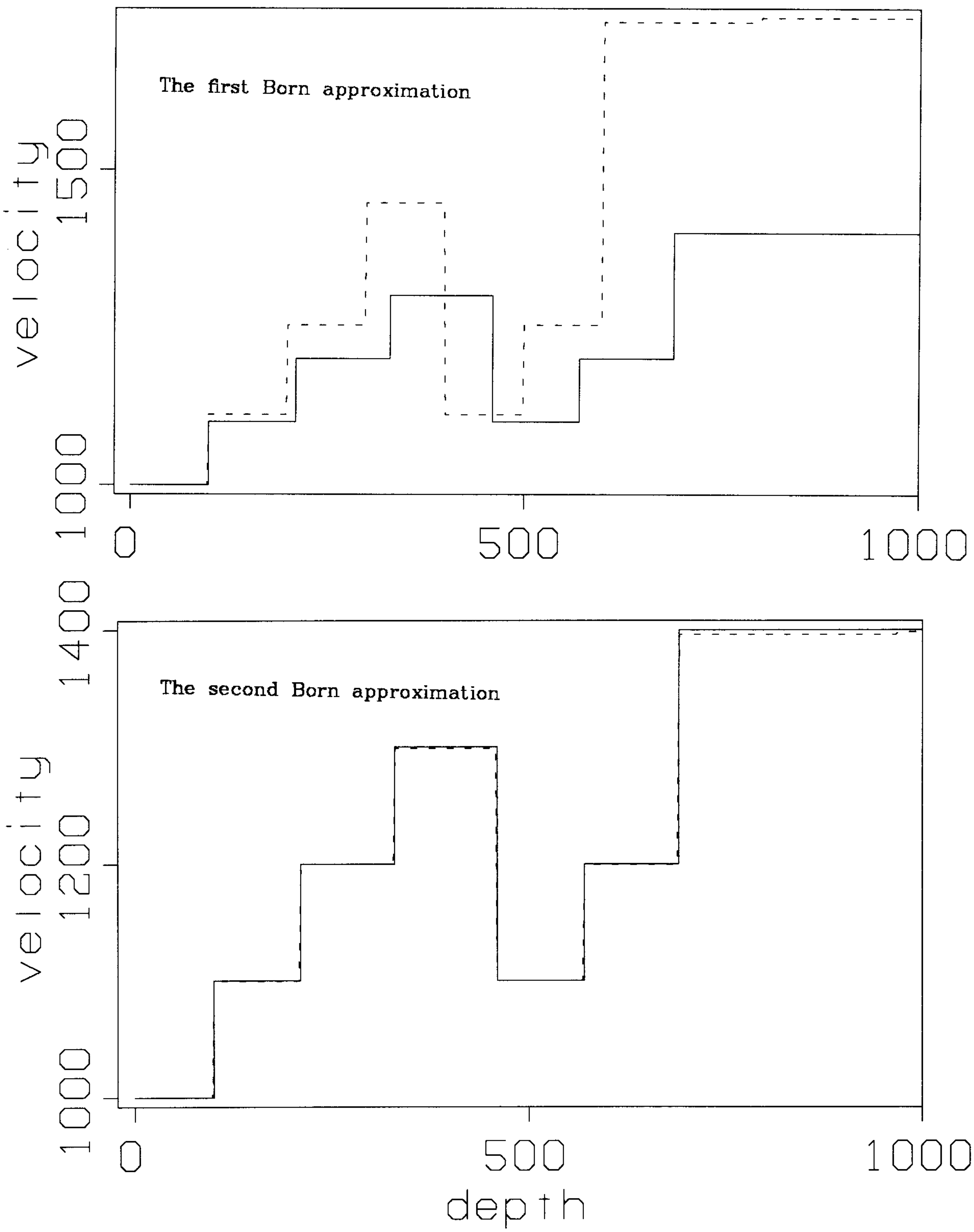


**FIG. 1.** The result of Born inversion. The model in both cases is the same (the solid line). Notice that the scale is different in each case.

The crucial step of the algorithm is the first one, which involves solving an integral equation for $A(\xi, \eta)$. The last two steps involve direct and easy computations, giving the velocity $v(z)$ and the corresponding depth $z$. The proof of this theorem can be found in Weidelt's paper (1978).

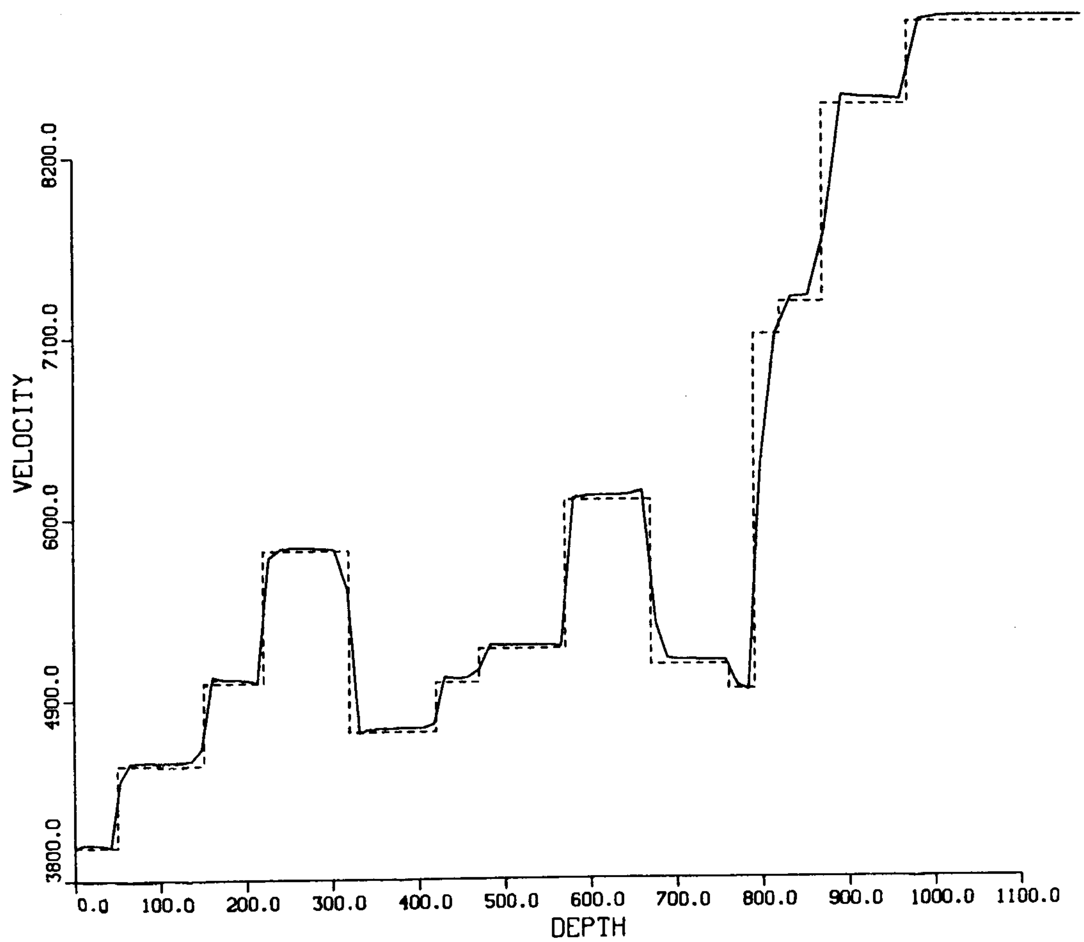


FIG. 2. The true (dashed line) and the reconstructed (solid line) velocity obtained by using Weidelt's theory.

I have tested this algorithm on different models. Figure 2 shows a synthetic seismogram for a model consisting of 16 layers. It also shows the actual and the reconstructed velocity functions. The result can be seen to be very satisfactory.

## CONCLUSIONS

We have seen that inversion can be direct or iterative and that direct methods can be exact or approximate. The direct methods have the advantage that they do not need an initial guess; whereas direct methods might suffer from stability problems. To alleviate the stability problem and to make the inverse problem easier to solve, approximations are made to the direct methods; the Born inversion is a popular one.

We also have seen that migration and inversion are related processes. In particular, there exists a mathematical relationship between the Born approximation and migration.

## REFERENCES

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## APPENDIX

### Deriving the first Born approximation

Multiply equation (4) by $G(z,\omega,\xi)$ and equation (5) by $\psi(z,\omega)$ and subtract to get

$$G(z,\omega,\xi)\frac{\partial^2}{\partial z^2}\psi(z,\omega)-\psi(z,\omega)\frac{\partial^2}{\partial z^2}G(z,\omega,\xi)=\psi(z,\omega)\delta(z-\xi)-\frac{\omega^2}{v_0^2}\alpha(z)\psi(\omega,z)G(z,\omega,\xi)$$

The left-hand side can be shown to vanish upon integration, and we are left with

$$\psi(z,\omega)=\frac{\omega^2}{v_0^2}\int_0^\infty \alpha(\xi)\psi(\xi,\omega)G(z,\omega,\xi)d\xi \qquad (A-1)$$

Now $\psi(z,\omega)$ can be thought of to consist of an incident field $\psi^{in}(z,\omega)$ plus a scattered field $\psi^{sc}(z,\omega)$. The incident field is a plane wave that can be written as $e^{-i\frac{\omega}{v_0}z}$. Putting these substitutions in equation (A-1) leads to equation (7).

Substituting for $G(z,\omega,\xi)$ from equation (6), we obtain

$$\psi(z,\omega)=e^{-i\frac{\omega}{v_0}z}+\frac{\omega}{2iv_0}e^{i\frac{\omega}{v_0}z}\int_0^\infty \alpha(\xi)e^{-2i\frac{\omega}{v_0}\xi}d\xi \qquad (A-2)$$

Therefore, the reflection coefficient is the coefficient of $e^{i\frac{\omega}{v_0}z}$ in the second term on the right-hand side of equation (A-2).

$$R(\omega)=\frac{\omega}{2iv_0}\int_0^\infty \alpha(\xi)e^{-2i\frac{\omega}{v_0}\xi}d\xi \qquad (A-3)$$

Integrating the right-hand side by parts,

$$R(\omega)=\frac{\omega}{2iv_0}\left[\alpha(\xi)\frac{e^{-2i\frac{\omega}{v_0}\xi}}{-2i\frac{\omega}{v_0}}\right]_0^\infty-\frac{\omega}{2iv_0}\frac{1}{-2i\frac{\omega}{v_0}}\int_0^\infty \frac{\partial\alpha(\xi)}{\partial\xi}e^{-2i\frac{\omega}{v_0}\xi}d\xi$$

Now $\alpha(0)=0$ and assuming that $\alpha(\infty)=0$, then

$$R(\omega)=-\frac{1}{4}\int_{-\infty}^\infty \frac{\partial\alpha(\xi)}{\partial\xi}e^{-2i\frac{\omega}{v_0}\xi}d\xi$$

Taking the inverse Fourier transform gives

$$R(2t)=-\frac{1}{8}\frac{\partial}{\partial t}\alpha(v_0t)$$

from which we derive at equation (8).

## Deriving the second Born approximation

In terms of the travel time $\xi$ defined in (9), equation (2) can be written as

$$\psi_{\xi\xi}(\xi,\omega) + \omega^2\psi(\xi,\omega) = \frac{v_\xi}{v}\psi_\xi(\xi,\omega)$$

Using the same method of used in obtaining (A-1), we arrive at

$$\psi(\eta,\omega) = e^{-i\omega\eta} - \int_0^\infty \left[\frac{d}{d\xi}(\log\frac{v(\xi)}{v_0})\right]\left[\frac{\partial}{\partial\xi}\psi(\xi,\omega)\right]G(\eta,\xi,\omega)d\xi$$

Substituting for the Green's function and evaluating at $\eta = 0$, we obtain the reflection coefficient

$$R(\omega) = \frac{1}{2}\int_0^\infty f(\xi)e^{-2i\omega\xi}d\xi \qquad (A-4)$$

where

$$f(\xi) = \frac{d}{d\xi}\log\frac{v(\xi)}{v_0}$$

Equation (A-4) defines a Fourier transform pair from which we can write

$$f(\xi) = 4\int_{-\infty}^{\infty} R(w)e^{i2\omega\xi}d\omega$$

or

$$f(\xi) = 4R(2\xi)$$

Substituting for $f(\xi)$, we arrive at equation (10).