Wave field extrapolation

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ABSTRACT

The downward extrapolation of a zero-offset (stacked) time section by use of the finite difference method in the 2D Cartesian space coordinates and frequency domain, plus the concept of the exploding reflector model, gives a depth image of the earth's reflectivity interior, because the time events are migrated to their subsurface location. This processing is designed to image time patterns when there are velocity variations in both depth and lateral direction.

The one-way wave equation is extracted from the scalar wave equation first introduced by Claerbout in 1972. We derive extrapolation operators by performing a matrix finite periodic fraction expansion of the "square root operator". This square root appears naturally in the one-way wave equation.

The extrapolation operators that handle severe velocity variations in both depth and lateral directions are split in 45-degree operators. The stability analysis, made with absorbing side boundary conditions, gives a condition sufficient to guarantee the absolute stability of the extrapolation scheme.

Reflectors dipping steeply up to 55-degrees are properly migrated by the extrapolation operator generated by the first order of the finite periodic expansion, the so-called 45-degree approximation of the square root.

For complex structures, the conventional processing is abandoned for the prestack migration, processing which requires wide-angle approximations of the square root. The second order of the periodic fraction expansion, called the 65-degree approximation of the square root, is then used to downward extrapolate the common shot gathers.

INTRODUCTION

Migration methods to image the earth reflectivity interior have had constant developments since the introduction of the scalar wave theory in reflection seismology by Jon Claerbout (1970, 1976). The finite difference method to migrate time patterns (Claerbout and Doherty 1972, Claerbout 1976) was derived to solve the paraxial wave equation, commonly known as the 15-degree equation, in a moving coordinate frame.

This one-way wave equation is used to downward continue the wavefield in a medium whose velocity varies both with depth and with offset. The migrated section is then obtained by applying the imaging concept of the exploding reflector model when stacked seismic data are extrapolated. Stolt (1978) described a Fourier method to directly map a zero-offset stacked section in a migrated section for a medium with constant velocity. This scheme migrates steeply dipping beds and is limited to the sparseness of lateral sampling of the seismic data.

Stolt has proposed a coordinate transform to extend use of this Fourier method to a medium with velocity variations in depth. He also extended the implicit expansion of the square root operator, approximation first used by Claerbout to derive the paraxial equation, to the 45-degree equation in the frequency-wavenumber domain.

The same year, two depth extrapolation schemes to downward continue the wavefield were introduced. The first of them (Kjartansson, 1978) solves the 45-degree equation by using a finite difference method in the frequency-space domain. He properly migrates in depth, time patterns in inhomogeneous media whose steep dips are less than 50 degrees. At the same time, Jacobs solved the 45-degree equation by using the finite difference method in the time domain for a laterally homogeneous medium.

The second method (Gazdag, 1978) is the migration by phase-shift, which downward continues the wavefield in a laterally homogeneous medium. Gazdag has extended the phase-shift method to a medium with lateral velocity variations, by using the mean of the phase-shift plus interpolation.

Francis Muir (1980) suggested approximations of higher orders than the 15- and 45-degree equations by representing the square root as a continued fraction expansion. Ma (1981, 1983) derived wide-angle one-way wave equations in the time domain from the scalar wave equation in a way similar to Claerbout and Doherty's (1972) method to find the 15- degree equation, Stolt's method for the 45-degree equation and Berkhout's (1980) wide-angle convolution operators.

Ma split the wide-angle approximations in 15- and 45-degree type equations by first splitting in the frequency-wavenumber domain, and then coming back into the time-space domain. Clayton (1981) extended the finite difference method first used by Kjartansson for the 45-degree equation, to higher order approximations of the continued fraction square root recursion. Jacobs (1983) used this approach for the pre-stack migration of profiles in the Cartesian coordinate space and frequency domain.

In this paper, we first extract one-way wave equations from the scalar wave equation in the space frequency domain. Extrapolation operators are then derived by a periodic continued fraction expansion of the focusing operator that appears after the splitting of the one-way wave equations into two partial differential equations.

A causal dip-filter is implemented in the extrapolation operators in order first to attenuate the evanescent waves, and second, to filter steep dips that give rise to migration artifacts (sparse sampling of the midpoint axis). An important by-product of using the causal dip-filter is that the approximants of the recursive relation used to generate the periodic continued fraction will better match the relation dispersion.

The extrapolation scheme, with the splitting of the extrapolation operators in 45-degree type operators, is then described.

The stability analysis deserves much care in use, and a sufficient stability condition is demonstrated for z-data independent absorbing side boundary conditions. The numerical analysis of the extrapolation procedure emphasizes the implementation of the 45- and 65-degree extrapolation operators, the two first orders of the finite periodic continued fraction.

Examples using synthetic and field data examples are given for the depth migration of stacked seismic cross-sections in inhomogeneous media. The continued fraction expansion of the square root is derived from the scalar wave equation; we demonstrate that the implicit and explicit expansions give the same approximation of the square root.

Finally, we build time extrapolation operators in the time domain from the explicit expansion of the square root, which is the relevant continued-fraction expansion in the time domain.

We demonstrate that the three first approximants of this continued fraction expansion, the 15-, 45- and 60- degree time extrapolation operators, are the three operators of interest for the migration of seismic data.

EXTRAPOLATION OPERATORS

We extract the one-way wave equation that downward continues an upgoing wave from the scalar wave equation. The state variable that must be extrapolated is the pressure wavefield over the square root of the impedance.

One way-wave equation and extrapolation operators

The acoustic wave equation simultaneously supports two solutions, a downgoing wave and an upcoming wave. The downward continuation of the wavefield recorded at the earth's surface is solved in the time-space domain either by the "non-reflective full wave equation" (Kosloff) or by the time migration described by Claerbout (1976). The downward continuation of the wavefield in the frequency-space domain is done either by using the full scalar wave equation (Baysal and Kosloff) or by solving a one-way wave equation, which supports only one wave.

We assume that the density model is a constant. Clayton (1981) has shown that the density variations modify the amplitudes of the wavefield but not the phase, so that the kinematics of the imaging are not affected by this approximation. The pressure wavefield $\psi(x,z,w)$ in the space-frequency domain is the solution of the acoustic wave equation:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2} = - \frac{w^2}{v(x,z)^2} \psi \tag{1}$$

where $v(x,z)$ denotes the velocity of the medium. In the discrete domain, the acoustic wave equation becomes:

$$- D_x^H D_x \psi - D_z^H D_z \psi = -\frac{w^2}{v^2} \psi \tag{2}$$

where D_{x_i} and $D_{x_i}^H$ are, respectively, the discrete causal and anti-causal partial derivatives with respect to x_i ($x_i = x$ or z). In term of matrix operators, D_{x_i} is a bidiagonal matrix and $D_{x_i}^H$ is its transposed form:

$$D_{x_i} = \frac{1}{\Delta x_i^2} (0,-1,1,0,0) \tag{3}$$

where (0,-1,1,0,0) denotes a pentadiagonal matrix whose diagonal coefficients are 1s. We define a new state variable by dividing the pressure wavefield by the velocity, and multiplying each element in the left side of equation (2) by the velocity, to get:

$$\left(- v \; D_z^H D_z \; v \right) \frac{\psi}{v} = \left(-w^2 + v \; D_x^H D_x \; v \right) \frac{\psi}{v} \tag{4}$$

The way to derive a one-way wave operator is to take the square root of both the matrix operators encountered to the right and to the left of the equation (4). Because these matrices are symmetric, they have positive eigenvalues, and it makes sense (as far as mathematics is concerned) to take the square root of them. To downward continue the wavefield, we have to use a spatial anti-causal derivative with respect to the depth variable z.

We first notice that the square of the spatial anti-causal operator D_z is the pentadiagonal matrix:

$$D_z \; D_z \;\; = \frac{1}{\Delta z^2} \, (0,0,1,-2,1) \tag{5}$$

Here, the matrix operator $- D_z^H D_z$ is the pentadiagonal matrix $\frac{1}{\Delta z^2}(0,1,-2,1,0)$. When this matrix is applied to the wavefield, we get the partial second derivative with respect to the vertical variable at the depth z, while the square-matrix of the anti-causal operator D_z gives the same derivative but at the depth $z + \Delta z$. This feature explains the following approximation:

$$D_z \; D_z \;\; = \; - D_z^H D_z \tag{6}$$

This approximation may seem to have an accuracy-costing counterpart, but this loss will appear to be of third order in Δz: we will later use the Crank-Nicholson transform to both stabilize and improve the downward continuation scheme.

We now have to take the square root of the operator $v \;\; D_z D_z \;\; v$. When the velocity has no depth variations, the velocity and the partial derivative with respect to z commute, and :

$$\left(\, v \;\; D_z \;\; D_z \;\; v \;\; \right)^{1/2} \;\; = \;\; v^{1/2} D_z \;\; v^{1/2} \tag{7}$$

When the velocity has vertical variations, the above relation is no longuer true, because the partial derivative with respect to the depth and the square root of the velocity don't commute. The physics of the problem show that we can neglect the correction term, which takes into account the non commutativity of these operators.

The downward continuation of the pressure wavefield is done by using depth increments of Δz. To avoid aliasing in the z-axis direction, the depth increment Δz must have a magnitude less than the wavelength of the pressure wavefield. Thus, neglecting the correction term in the square root operator corresponds to smoothing the variations of the velocity on a wavelength. That smoothing is exactly what the seismic waves do.

Instead of a velocity log, which takes into account high-frequency variations of the velocity because it uses a signal whose frequency is very high, a shot profile can detect only velocity variations on a distance comparable with the wavelength of the seismic waves. That limitation justifies the use of extrapolation operators, whose sensitivity to velocity variations is the same as the sensitivity of the experiment we want to describe with these operators.

The next step in the derivation is to take the square root of each side of the equation (4), to get:

$$\left(v^{1/2} D_z \ v^{1/2} \right) \frac{\psi}{v} = \pm \left(-w^2 + v \ D_x^H D_x \ v \right)^{1/2} \frac{\psi}{v} \tag{8}$$

Let us denote by Ψ the pressure wavefield over the square root of the velocity, and by Λ the slowness of the medium (i.e., the inverse of the velocity). Thus we get from equation (8):

$$D_z \ \Psi = \pm \Lambda^{1/2} \left(-w^2 + v \ D_x^H D_x \ v \right)^{1/2} \Lambda^{1/2} \Psi \tag{9}$$

If we define the Fourier transform of the state variable Ψ by:

$$\Psi(x,z,t) = \int_{-\infty}^{+\infty} dw \ e^{jwt} \ \Psi(x,z,w) \tag{10}$$

then the one-way extrapolator that downward continues an upgoing wave is:

$$\boxed{D_z = +\Lambda^{1/2} \left(-w^2 + v \ D_x^H D_x \ v \right)^{1/2} \Lambda^{1/2}} \tag{11}$$

The same operator can be used to downward continue a downgoing wave, by using a negative integration parameter $-\Delta z$ in the extrapolation scheme or by simply changing the sign of w.

Shortcoming of the wavefield extrapolators

We have so far derived the downward and upward extrapolation operators; i.e., tools to extrapolate the wavefield in depth in both directions. Forward and backward extrapolation operators of the wavefield are also interesting tools in seismic data processing.

These operators extrapolate the wavefield laterally; i.e., in the midpoint-axis direction and in the frequency-space domain. If we define the 2D Fourier transform of the state variable Ψ by:

$$\Psi(x,z,t) = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dw \ e^{\,j\,(wt \,-\, k_x)} \ \Psi(kx,z,w) \tag{12}$$

The sign convention for the Fourier transform with respect to the lateral parameter x is opposite to that with respect to the time t. Thus, the lateral extrapolators are:

$$D_x = -\Lambda^{1/2} \left(-w^2 + v \; D_z^H D_z \; v \right)^{1/2} \Lambda^{1/2} \tag{13a}$$

$$D_x^H = +\Lambda^{1/2} \left(-w^2 + v \; D_z^H D_z \; v \right)^{1/2} \Lambda^{1/2} \tag{13b}$$

where the extrapolation operator D_x forward extrapolates a wave travelling backward, i.e. towards the left. Similarly, D_x^H forward extrapolates a wave travelling forward.

In accordance with Clayton (1981), we assume in what follows that the state variable in a medium with density variations, is the pressure wavefield over the square root of the impedance.

PERIODIC CONTINUED FRACTION EXPANSION

In this section, we split the extrapolation operator into two sub-operators, a phase-shifting and a focusing operator. For 1D vertical propagation, only the phase-shifting operator has to be applied and the solution is WKBJ accurate. Extrapolation operators are then derived by applying a periodic matrix continued fraction expansion to the focusing operator.

Derivation of wide-angle approximations

Let's consider the downward continuation of an upgoing wave travelling at the vertical. The extrapolation operator in equation (11) simplifies and becomes:

$$D_z = \Lambda^{1/2} \; jw \; \Lambda^{1/2} \tag{14}$$

In this case, the downward continuation operator is a phase-shifting operator. At each depth step, we solve the partial differential equation:

$$D_z = \frac{jw}{v} \Psi \tag{15}$$

The analytical solution of this phase-shift equation is given by:

$$\Psi(z + \Delta z) = e^{\frac{jw}{v} \Delta z} \; \Psi(z) \tag{16}$$

An integration with respect to the depth parameter z of the PDE (16), followed by a time-inverse Fourier transform leads to:

$$\psi(t,z) = \psi_0 \sqrt{\frac{Z(z)}{Z(0)}}\, e^{\, j \left[wt + \int_0^z \frac{w}{v(z)} dz \right]} \tag{17}$$

which is the WKBJ solution for the pressure wavefield $\psi(t,z)$ as shown by Clayton (1981). Therefore, the solution does produce the correct amplitude (for 1D vertical propagation) for both smooth and discontinuous velocity variations. The one-way wave extrapolation operators do not incorporate the geometrical spreading. Neither do the full scalar wave extrapolation operators, because they are derived from the 2D scalar wave equation and the waves travel in a 3D media.

Following Claerbout (1976), we write the extrapolation operator as the sum of the phase-shift operator we have previously described and the diffraction operator:

$$D_z = \Lambda^{1/2}\, jw\, \Lambda^{1/2} + \Lambda^{1/2} \left(-jw + \left((jw)^2 + v\, D_x^H D_x\, v \right)^{1/2} \right) \Lambda^{1/2} \tag{18}$$

The downward extrapolation of the state variable Ψ is then performed by solving at each z-step two partial differential equations:

$$D_z\, \Psi = \frac{jw}{v}\, \Psi \tag{19a}$$

$$D_z\, \Psi = -R\, \Psi \tag{19b}$$

where the focusing operator R is defined by:

$$R = -\Lambda^{1/2} \left(-jw + \left((jw)^2 + v\, D_x^H D_x\, v \right)^{1/2} \right) \Lambda^{1/2} \tag{20}$$

Let us introduce the operator S by:

$$S = \Lambda^{-1/2} R\, \Lambda^{-1/2} \tag{21}$$

We define recursively the operator S by use of the finite periodic continued fraction expansion:

$$S_n = \frac{I}{A + \frac{I}{B + S_{n-1}}} \quad ; \; n > 0 \; ; \; S_0 = 0 \tag{22}$$

With a slight change shown in the appendix (A-1), this periodic continued fraction is seen to be the matrix analog of Stieltjes (1894) "preferred continued fraction" Another point is that this expansion is generated by impedance functions (Claerbout, 1976, 1983); we will use this feature in the stability analysis.

The periodic continued fraction used in this paper is related to the continued fraction expansion first used to approximate the square root (equation [13]) by the fact that each order of the periodic continued expansion corresponds to an even order of the other one.

The point is that, in the frequency domain, the odd orders of the continued fraction expansion have the same computation cost as the next higher even order. Therefore, there are never used to build extrapolation operators in that domain.

The Raphson-Newton recursive relation (Dubrulle, 1983) also gives a mean to approximate the square root. This scheme generates approximations which converge quadratically to the square root and one could think that it gives better approximations than the periodic continued fraction expansion.

This scheme is never used because either it matches an approximant of the recursive relation defined in equation (22) (that is the case of the 45-degree approximation as mentioned by Dubrulle, 1983), or it does not but in this case, the similar (in a computational cost point of view) approximant of the periodic continued fraction expansion is more accurate.

We now relate the periodic continued fraction with the matrix fractional transformation that generates it:

$$\boxed{F(S) = \frac{I}{A + \dfrac{I}{B + S}}} \tag{23}$$

The fixed points of this transformation are defined by:

$$F(S) = S \tag{24}$$

This equation has in general two roots S_∞^+ and S_∞^-, which are solutions of the matrix quadratic equation:

$$S^2 + B\,S - \frac{A}{B} = 0 \tag{25}$$

The above equation has been derived for matrices that commute; we will verify it after computing the matrices A and B. The roots of equation (25) are:

$$S_\infty^- = -1/2\,B - \left(\frac{B^2}{4} + \frac{B}{4\,A} \right)^{1/2} \tag{26a}$$

$$S_{\infty}^{+} = -1/2\, B + \left(\frac{B^2}{4} + \frac{B}{4\, A} \right)^{1/2} \tag{26b}$$

The values of the matrices A and B are determined by comparing, the value of S given in equation (21) (where the operator R is defined in equation (20)), with the desired value of the operator; this value is for the downward continuation of an upcoming wave (equation (21a)):

$$S_{\infty}^{-} = +jw - \left((jw)^2 + v\ D_x^H D_x\ v \right)^{1/2} \tag{27}$$

From equations (26a) and (27), we get:

$$A^{-} = \frac{-2jw}{v\ D_x^H\ v} \quad , \quad B^{-} = -2jw \tag{28}$$

Similarly, the desired value of the operator for the downward continuation of a downgoing wave is:

$$S_{\infty}^{+} = -jw + \left((jw)^2 + v\ D_x^H D_x\ v \right)^{1/2} \tag{29}$$

The matrices A^+ and B^+ are the same as in equation (30), but with an opposite sign. In both cases, it is obvious that the matrices A and B commute because the matrix B is the matrix unity I multiplied by a constant.

Because the matrix transformation defined in equation (25) is a linear fraction transformation, reversing the scheme is achieved by simply changing the signs of the matrices A and B.

The demonstration of the convergence of the periodic continued fraction in the propagating region (i.e., when $\mid K_x \mid \leq \frac{w}{v}$) is made in appendix (A-1). The point is that we can use the convergence theorem of periodic continued fractions (Wall, p. 35, 1948). First, when there is convergence, the limit is one of the fixed points of the transformation defined in equation (23); and second, when the recursive relation is generated by the matrices A^- and B^-, the limit (if there is one) is S^- (and there is a similar result for A^+ and B^+).

In practice, only the two first orders of the recursion defined in the equation (23), approximants called the 45- and 65-degree approximations of the square root, are used in the extrapolation processing of seismic data. The important element is not the convergence of the recursive relation (23) itself, but instead the fact that its two first approximants provide sufficiently accurate approximations of the focusing operator.

In Figure 1 are shown the group velocity parametric curves for the 45-, 65-, and 80-degree (third order of the recursive relation) approximations of the square root. These curves represent the theoretical wavefront of waves generated by a point source in a medium with constant velocity. These wavefronts are inscribed within the semi-circle that is the curve of the group velocity vector for the correct dispersion relation.

As described by Claerbout *(Imaging the Earth's Interior,* section 4.2), the evanescent waves are below the semi-circles that indicate waves propagating laterally. We derive these parametric curves in the appendix (A-2) and show, as according to Claerbout (1983) that the extrapolation operators are not frequency dispersive (things are worse) but instead angle dispersive.

These group velocity parametric curves (Figure 1) show first, that the 65-degree approximation is more accurate than the 45- one for angles greater than 45-degree; and second, that the 80-degree approximation (third order of the recursive relation (23)) is slightly better than the 65- one.

A dip filter presented in the next section, also enables to improve the accuracy of the extrapolation operators, i.e., to better match the dispersion relation. In practice, we will show how the 45-degree approximation can match the dispersive relation of the 55-degree approximation in the propagating region (third order of the continued fraction expansion of the square root). Similarly, the 65-degree approximation will match the dispersive relation of the 80- degree one.

Evanescent waves and dip-filtering

The mathematical extrapolation of seismic waves back in time gives rise to evanescent waves because we downward extrapolate the wavefield with a square root operator, i.e. a one-way wave equation instead of the full wave equation (Claerbout, 1976, 1983).

Figure 2 shows two impulse responses corresponding to the downward continuation of a spike in a constant velocity media with respectively the 45- and 65-degree extrapolation operators. These evanescent waves don't give rise to stability problems when appropriate boundaries are used (see the section "Stability and absorbing side boundary conditions").

Nevertheless, they are highly undesirable in this processing because they would give rise to migration artifacts and deteriorate the quality of the migrated section. When running a movie modeling the downward continuation of a wavefront generated by a spike on Claerbout's computers, we realized that these evanescent waves do follow the wavefront and seem to propagate.

The point is that, although these waves are well known not to propagate because they attenuate on a short distance, the extrapolation operator generates constantly evanescent waves at each z-step extrapolation and they do seem to propagate.

The filtering of these waves is therefore a requirement to build accurate extrapolation operators. The full wave migration (Baysal and Kosloff, 1983) also requires a filtering of these waves but for another reason. Here, it is an imperative to filter these waves to guarantee the stability of the scheme itself. They eliminate these waves by laterally Fourier transform the wavefield at each z-step extrapolation and for each frequency and then simply select the appropriate wavenumbers.

This expensive method is avoided in our method by implementing a filter inside the extrapolation operator itself and at no cost. Claerbout (1976) first implemented such a filter in the 15-degree extrapolation operator. He also described the decomposition of the 45-degree equation into effects (1982).

The elimination of the evanescent waves, a dip filtering and a better approximation of the dispersion relation is done by simply implementing a causal function $\gamma_1(jw)^{\gamma_2}$ inside the recursive relation which generates the periodic continued fraction. We modify the recursive relation (equation (22)) as following:

$$S_{n+1} = \frac{I}{A + \dfrac{I}{B + S_n}} \quad ; \quad S_1 = \frac{I}{A + \dfrac{I}{\gamma_1(jw)^{\gamma_2} B}} \quad , n > 0 \tag{30}$$

where γ_1 and γ_2 are two positive real numbers. Kjartansson (1978) has demonstrated that the function $(jw)^{\gamma_2}$ is causal. This function is related to the quality parameter Q which described the seismic attenuation in sedimentary rocks. Therefore, it is natural to use such a function for the attenuation of the evanescent waves. What is really a nice feature of this causal function is the fact that it does not modify the causal properties of the operator S_n defined in the equation (30).

This is immediately verified by first, noticing that the operator B can also be written in a similar form than the dip filter function and second, that the product of two exponentials is an still an exponential. Thus, Kjartansson proof applies and the operator $\gamma_1 (jw)^{\gamma_2} B$ is a causal operator.

Here, the causal function is multiplied by a positive real number γ_2. This does not modify the causal property of $(jw)^{\gamma_2}$ and provides a second degree of freedom which enables us to improve the accuracy of the approximants (equation [30]) to the dispersive relation.

Practically, the two orders of interest for our seismic data are the two first approximants generated by the recursive relation (30). The choice of the parameters γ_1 and γ_2 is discussed in detail in the section "Numerical Implementation".

Two impulse responses generated by the 65-degree extrapolation operator are shown in the Figure 3. One of them (Figure 3a) uses a dip filter which removes the evanescent waves and improves the accuracy of the operator which matches the dispersion relation up to 80 degrees.

We show in Figure 4 how the parameter $\gamma 1$ improves the accuracy of the operators. The 45-degree approximation matches the dispersion relation of the 55-degree one while the 65-degree approximation becomes accurate to 80-degrees. That is the reason why the two first orders of the periodic continued fraction are the operators of interest for the processing of seismic data.

DEPTH EXTRAPOLATION PROCEDURE

We present the downward extrapolation procedure of the state variable. At each z-step extrapolation, a phase-shift operator and a focusing operator are applied to the state variable. The focusing operator is split in 45-degree type focusing operators.

Ma Zaitian (1981, 1983) derives extrapolation operators by splitting the time extrapolation operators in 15-degree type extrapolation operators. These operators will be defined in the section "Derivation of the Continued Fraction Expansion". In the frequency domain, the operators of interest are the 45-degree type extrapolation operators.

The equations to solve at each z-step are (equation (19a) and (19b)):

$$D_z \Psi = \frac{jw}{v} \Psi \tag{31a}$$

$$D_z \Psi = -R \Psi \tag{31b}$$

The downward extrapolation procedure is first to solve analytically the equation (31a) and then to solve the equation (31b) for the phase-shifted wavefield.

The equation (31a) is the phase-shift equation and its analytical solution is:

$$\Psi = e^{\frac{jw}{v}\Delta z} \Psi \tag{32}$$

Therefore, the phase-shift operator is the exponential operator defined in the equation (32). The advantage of splitting the extrapolation operator in the phase-shift and the focusing operators is that the flat reflectors will be correctly migrated (Jacobs, 1983). This not true for the migration with the full acoustic wave equation.

The focusing equation (31b) is solved by applying a Crank-Nicholson transform which leads to the following implicit scheme:

$$\Psi(z+\Delta z) = \frac{I - \frac{\Delta z}{2} R}{I + \frac{\Delta z}{2} R} \Psi(z) \tag{33}$$

where $\Psi(z)$ is the state variable which has been phase-shifted, according to the equation (32). The focusing operator R is then substituted by its approximation R_n defined in equation (21):

$$R_n = \Lambda^{1/2} S_n \Lambda^{1/2} \tag{34}$$

In the above equation, the operator S_n is defined by the recursive relation (30). In a similar way than Ma Zaitian (1981) did to split the extrapolation operator in 15-degree type operators, we first compute the partial fraction decomposition of S_n to get:

$$S_n = jw \sum_{i=1}^{n} \frac{a_{i,n} S_x}{b_{i,n} + S_x} \quad ; \quad S_x = \frac{VD_x^H D_x V}{(jw)^2} \tag{35}$$

where $a_{i,n}$ and $b_{i,n}$ are functions of w because we have implemented a filter inside the recursive relation (equation [30]). If we consider the case where n is equal to one (i.e. the case of the 45-degree approximation), the equation (35) becomes:

$$S_1 = jw \frac{a_{1,1} S_x}{b_{1,1} + S_x} \tag{36}$$

The above equation combined with the equation (35) shows that the operator S_n is the sum of 45-degree type approximations. We now replace the operator R in the equation (33) by its approximation R_n and it yields:

$$\Psi(z+\Delta z) = \frac{I - \frac{jw\Delta z}{2} \sum_{i=1}^{n} \Lambda^{1/2} \frac{a_{i,n} S_x}{b_{i,n} + S_x} \Lambda^{1/2}}{I + \frac{jw\Delta z}{2} \sum_{i=1}^{n} \Lambda^{1/2} \frac{a_{i,n} S_x}{b_{i,n} + S_x} \Lambda^{1/2}} \Psi(z) \tag{37}$$

The numerator and the denominator of the focusing operator defined by the above equation are both split to give:

$$\Psi(z+\Delta z) = \prod_{i=1}^{n} \left(\frac{I - \frac{jw\Delta z}{2} \Lambda^{1/2} \frac{a_{i,n} S_x}{b_{i,n} + S_x} \Lambda^{1/2}}{I + \frac{jw\Delta z}{2} \Lambda^{1/2} \frac{a_{i,n} S_x}{b_{i,n} + S_x} \Lambda^{1/2}} \right) \tag{38}$$

When n is equal to one, the above expression simplified and becomes:

$$\Psi(z+\Delta z) = \frac{I - \frac{jw\,\Delta z}{2}\,\Lambda^{1/2}\,\frac{a_{1,1}\,S_x}{b_{1,1}+S_x}\Lambda^{1/2}}{I + \frac{jw\,\Delta z}{2}\,\Lambda^{1/2}\,\frac{a_{1,1}\,S_x}{b_{1,1}+S_x}\Lambda^{1/2}} \tag{39}$$

which is the 45-degree focusing equation. In what follows, we will call the operator defined in the equation (39) a 45-degree type extrapolation operator.

The equation (39) shows that the extrapolation operator of order n is split in n 45-degree type extrapolation operators. More precisely, this is the focusing operator which is split in 45-degree type focusing operators. In what follows, we will suppose that the state variable is first phase-shifted and that the extrapolation operator is the focusing one.

Again, the operators of interest for our seismic data are the 45- and 65-degree extrapolation operators. The first one is not split because it is a 45-degree type extrapolation operator. Only the 65-degree extrapolation operator (second order of the recursive relation) is concerned by the splitting described above.

This splitting has the nice feature to enable us to code only a 45-degree type operator to derive higher order operators. As shown above, the 65-degree operator is split in two 45-degree operators while the 80- degree operator is split in three 45-degree one.

These 45-degree operators differ only by the coefficients $a_{i,n}$ and $b_{i,n}$. Another point is that the procedure without splitting would require the solution of pentadiagonal systems for the 65-degree operator and larger systems for higher order operators. With the splitting, we only have to solve tridiagonal systems as shown in the section "Numerical Implementation".

The Crank-Nicholson transform has been used because it first enables us to stabilize the scheme and second provides a better accuracy (Claerbout, 1976). It gives a scheme accurate to third order in Δz when there is no splitting of the focusing operators. One can think that the splitting affects the accuracy and that the accuracy is to second order in Δz when one splits the focusing operator in 45-degree type operators.

This is not the case! It comes from the fact that we have applied the splitting on both the numerator and the denominator of the focusing operator in the equation (38). This important property of the splitting demonstrates in the appendix (A-2) comes from the symmetry of the Crank-Nicholson transform. In laterally homogeneous media, the splitting operator is even slightly more precise than the non splitting one.

This is shown in the Figure 5 which displays the impulse response of a point source in a constant velocity medium with the 65-degree extrapolation operator split (Figure 1a) and non split (Figure 1b).

When we use a dip-filter to attenuate the evanescent waves, the split (Figure 4) and non split 65-degree extrapolation operators give a similar impulse response.

Finally, the procedure is completely described by the implementation of the 45-degree type operator. If we denote $(OP_i^{(n)})$ the 45-degree type operator whose coefficients are $a_{i,n}$ and $b_{i,n}$, the equation (39) yields:

$$(OP_i^{(n)}) = \frac{I - \frac{jw\,\Delta z}{2}\,\Lambda^{1/2}\,\frac{a_{i,n}\;S_x}{b_{i,n}\;+S_x}\Lambda^{1/2}}{I + \frac{jw\,\Delta z}{2}\,\Lambda^{1/2}\,\frac{a_{i,n}\;S_x}{b_{i,n}\;+S_x}\Lambda^{1/2}} \tag{40}$$

First, the operator $-D_x^H D_x$ which represents δ_{xx}, second partial derivative with respect to x, is approximated by (Claerbout, 1983):

$$\delta_{xx} = -\frac{1}{\Delta x^2}\,\frac{T}{I-\beta\,T} \tag{41}$$

where the matrix T is the tridiagonal matrix (–1,2,–1) used to discretized the discrete second partial derivative with respect to x.

Second, following Claerbout (1983), we will suppose that the velocity slowness Λ commutes with δ_{xx}. This assumption saves computational time because the slowness matrix multiplies the operator δ_{xx} both to the right and to the left.

Again, one must remember the physics of the problem and not forget that the extrapolation procedure downward continues the state variable Ψ by step Δz more than 5 times smaller than the wavelength of the seismic waves.

Therefore, the velocity can be consider to be laterally constant on such small distances. Another point is that even if there is severe lateral velocity variations, the local operators used in the procedure "almost" commute and no accuracy is lost.

Finally, an algebraic matrix manipulation shows that we can transform the operator $(OP_i^{(n)})$ in the reduced form:

$$\boxed{(OP_i^{(n)}) = \alpha_i^{(n)} + \frac{\mu_i^{(n)}}{\lambda_i^{(n)} + T_i^{(n)}}} \tag{42}$$

where $\alpha_i^{(n)}$, $\mu_i^{(n)}$ and $\lambda_i^{(n)}$ are diagonal matrices and $T_i^{(n)}$ a tridiagonal matrix.

Therefore, the main work is to solve tridiagonal linear systems.

We have not yet addressed the important problem of the stability. It is certainly the most important one. Dubrulle (1983) and Brysk (1983) have demonstrated stability proofs when one uses poor absorbing side boundary conditions, i.e. the zero-slope or zero-value conditions.

The key point of their demonstration (Dubrulle, 1983) is that the eigenvalues of the matrix T (defined above) are real positive numbers. This comes from the fact that the matrix D_x^H is the hermitian matrix of D_x when, again, poor absorbing side boundary side conditions are implemented. When "sophisticated" absorbing conditions are used (such B1, B2 or B3 conditions), this becomes wrong and the scheme can be unstable.

The next section presents a sophisticated absorbing side condition and a stability analysis related to it.

STABILITY AND ABSORBING SIDE BOUNDARIES

We first define z-data independent absorbing side boundary conditions and how these conditions are implemented in the extrapolation operators.

Then, the stability analysis is closely related to the absorbing side conditions used in the algorithm. A sufficient condition is given to guarantee the absolute stability of the scheme. This condition is also a necessary condition when the velocity has no lateral variation. We finally give the physical interpretation of this condition in terms of energy coming in the grid and energy going outside the grid used in the finite difference scheme.

Virtual trace

The finite grid used for the extrapolation scheme requires absorbing side boundaries to minimize reflected waves on the vertical edges of the grid. The trick is to laterally extrapolate the wavefield and use a virtual trace at each vertical boundary of the grid.

Two kinds of lateral extrapolation have been designed by Jon Claerbout and his Stanford Exploration Project team. These methods are a data-dependent absorbing side boundary condition suggested by Clayton (Hale and Toldi, 1982), and a wave-field extrapolation model (Clayton and Engquist, 1977, 1980).

We define z-data independent boundary conditions as the absorbing side conditions derived from the lateral extrapolation of the seismic data (John Toldi, private communication). Practically, this definition means that our stability analysis, which follows below, is not relevant for the B2 and B3 absorbing side boundary conditions.

The wavefield Ψ at the depth z is a vector whose dimension is nx, the number of zero-offset traces or common midpoints we downward continue. Let us define Ψ_1 and Ψ_{nx} as the coordinates of the state variable Ψ, respectively at the left side (and right side) of the grid.

We call Ψ_0 and Ψ_{nx+1}, the virtual traces used to absorb the energy going outside the grid. These traces are related to the wavefield Ψ by:

$$\Psi_0 = p_1 \Psi_1 \tag{43a}$$

$$\Psi_{nx+1} = p_{nx} \Psi_{nx} \tag{43b}$$

where the coefficients p_1 and p_{nx} are determined by a lateral extrapolation algorithm. The coefficient p_1 is implemented in the left corner of the matrix D_x, and the coefficient p_{nx} in the right corner of D_x^H.

The discrete partial derivative D_x with respect to the parameter x at the left vertical boundary of the grid is:

$$D_x \Psi_1 = \frac{\Psi_1 - \Psi_0}{\Delta x} \tag{44}$$

Replacing Ψ_0 with its value given by the equation (43a), we get:

$$D_x \Psi_1 = \frac{1-p_1}{\Delta x} \Psi_1 \tag{45}$$

Therefore, the matrix operator D_x becomes:

$$D_x = \frac{1}{\Delta x} \begin{bmatrix} 1-p_1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \cdots & & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}$$

And similarly for the coefficient p_{nx} that is implemented in the right diagonal corner of the matrix D_x^H.

The data-dependent absorbing boundary condition suggested by Clayton uses a Burg prediction filter to compute these coefficients. The energy to be minimized is defined by:

$$E(p_1) = | \Psi_2 - p_1 \Psi_1 |^2 + | \Psi_1 - p_1^* \Psi_2 |^2 \tag{46}$$

The minimization of the energy $E(p_1)$ with respect to the coefficient p_1 leads to:

$$p_1 = 2 \frac{\Psi_1 \Psi_2^*}{|\Psi_1|^2 + |\Psi_2|^2} \tag{47}$$

And a similar relation can be derived for the right-side boundary coefficient. The physical feature of the prediction absorbing condition is that it adapts to the wavefield (Hale and Toldi, 1982). Let us consider the case of a monochromatic plane wave whose angle with the vertical axis is θ:

$$\Psi(x,z,w) = a\, e^{jw\left(\frac{\cos(\theta)}{v} z + \frac{\sin(\theta)}{v} x - t \right)} \tag{48}$$

Combining the equations (43a) and (47) yields:

$$\Psi_0 = e^{-\frac{jw \sin(\theta)}{v} \Delta x} \Psi_1 \tag{49}$$

This is the expected value of the wavefield according to the equation (48). The pitfall of this method occurs when the reflectors dip upward toward the left side of the section. The migration tends to move the energy into the section while the data-dependent absorbing condition, adapting to the wavefield, lets the energy go inside the grid; these conditions give rise to an unstable scheme. Similarly, when the reflectors dip downward toward the right boundary, the data-dependent absorbing condition lets the energy enter into the grid and also gives rise to an unstable scheme.

Therefore, the stability of the extrapolation scheme must be closely related with the absorbing side conditions used. The next paragraph gives a condition upon the coefficients p_1 and p_{nx}, sufficient to guarantee a stable scheme in the case of z-independent absorbing side boundary conditions.

Causal positive real operators

In this part, we demonstrate that a condition sufficient for a stable extrapolation is that the operator $-D_x^H D_x$ be a causal positive real operator. A positive real operator is a matrix whose eigenvalues have a real positive part. A causal operator is an impedance function (Claerbout, 1983).

The extrapolation will be stable if the diffraction equation leads to a stable scheme. For each z-step extrapolation, after a Crank-Nicholson transform, the equation to solve is:

$$\Psi(z+\Delta z) = \frac{I - \frac{\Delta z}{2} R_n}{I + \frac{\Delta z}{2} R_n} \Psi z \tag{50}$$

where R_n is the diffraction operator and n is the order of the periodic fraction expansion of the square root. The Von Neumann stability criterion applied to the equation (50) requires that each eigenvalue γ_k of the matrix operator R_n have a real positive part.

The matrix R_n is computed from the matrix S_n by the equation (23). We rewrite it as the following:

$$R_n \ = \ \Lambda^{1/2} S_n \ \Lambda^{1/2} \tag{51}$$

where the matrix S_n is defined by the recursive relation equation (24). We first demonstrate how the eigenvalues r_k of the matrix R_n and the eigenvalues s_k of S_n are related.

Let us consider a non-zero eigenvector Ψ_{r_k} of the matrix R_n; this eigenvector is associated with the eigenvalue r_k. We now use a very old mathematical trick that consists of computing the dot product of $<R_n \ \Psi_{r_k} \mid \Psi_{r_k}>$ by two different ways. The dot product $<. \mid .>$ is the usual hermitian dot product associated with the complex space. First, allowing the property of the vector Ψ_{r_k} to be an eigenvector of the matrix R_n leads to:

$$<R_n \ \Psi_{r_k} \mid \Psi_{r_k}> \ = \ r_k \ <\Psi_{r_k} \mid \Psi_{r_k}> \tag{52}$$

The second way is to project the vector $\Lambda^{1/2}\Psi_{r_k}$ on an eigenvector orthogonal basis of the matrix S_n.

$$\Lambda^{1/2} \Psi_{r_k} \ = \ \sum_{i=1}^{i=nx} \alpha_i \ \Psi_{s_i} \tag{53}$$

Then, we compute the vector $S_n \Lambda^{1/2} \Psi_{r_k}$:

$$S_n \Lambda^{1/2} \Psi_{r_k} \ = \ \sum_{i=1}^{i=nx} s_i \ \alpha_i \ \Psi_{s_i} \tag{54}$$

The matrix $\Lambda^{1/2}$ is an hermitian matrix (it is a diagonal matrix with real numbers); therefore, we can compute the dot product as:

$$<R_n \ \Psi_{r_k} \mid \Psi_{r_k}> \ = \ <S_n \Lambda^{1/2}\Psi_{r_k} \mid \Lambda^{1/2}\Psi_{r_k}> \tag{55}$$

From the equation (55), we get the second relation for the dot product:

$$<R_n \ \Psi_{r_k} \mid \Psi_{r_k}> \ = \ \sum_{i=1}^{i=nx} s_i \ \mid \alpha_i \mid^2 <\Psi_{s_i} \mid \Psi_{s_i}> \tag{56}$$

The substitution of the dot product in the equations (52) and (56) gives the relation between the eigenvalue r_k of the matrix R_n and the eigenvalues of the matrix S_n:

$$r_k = \sum_{i=1}^{i=nx} s_i \; | \alpha_i |^2 \frac{|\,|\Psi_{s_i}|\,|^2}{|\,|\Psi_{r_k}|\,|^2} \tag{57}$$

Therefore, the eigenvalues of the matrix R_n have a real positive part if the eigenvalues of the matrix S_n have a real positive part. This is a sufficient condition, but when the velocity has no lateral variations it is also a necessary condition, because the matrices S_n and R_n are related by a real positive coefficient that is the inverse of the velocity.

The next step is to relate the eigenvalues of the matrix S_n to the eigenvalues of the matrix $D_x^H D_x$ and the coefficients p_1, p_{nx}. The matrix S_n is generated by the finite matrix periodic expansion defined in the equation (24), and in the case of the downward extrapolation is:

$$S_n = \frac{I}{A^- + \frac{I}{B^- + \cdots}} \tag{58}$$

In the equation (58), the matrix B^- is a diagonal matrix so that it shares the eigenvectors with the matrix A^-. Therefore, the matrices S_n and A^- share the same eigenvectors. An eigenvalue s_{n_k} of the matrix S_n is related to an eigenvalue d_k of the matrix $VD_x^H D_x\, V$ by the same finite periodic fraction expansion used to define the matrix S_n :

$$s_{n_k} = \frac{1}{\frac{-2jw}{d_k} + \frac{1}{-2jw + \cdots}} \tag{59}$$

We first notice that the complex number $\frac{-2jw}{d_k}$ has a real positive part if d_k has a negative imaginary part. Both the addition and the inverse of complex numbers with a real positive part still gives a complex number with a real positive part. Thus, the eigenvalues of the matrix S_n have a real positive part if the eigenvalues of the matrix $VD_x^H D_x\, V$ have a negative imaginary part.

We now relate the eigenvalues of the matrix $VD_x^H D_x\, V$ to the absorbing side boundaries coefficients p_1 and p_{nx}. Let us consider a non zero eigenvector Ψ_{d_k} associated with the eigenvalue d_k. We compute the dot product $< VD_x^H D_x\, V\, \Psi_{d_k} \,|\, \Psi_{d_k} >$ by two different ways.

First, defining the property of the vector Ψ_{d_k} to be an eigenvector of the matrix $VD_x^H D_x\, V$ yields:

$$< VD_x^H D_x \ V \ \Psi_{d_k} \mid \Psi_{d_k} > \ = \ d_k \ < \Psi_{d_k} \mid \Psi_{d_k} > \tag{60}$$

Then, we rewrite the matrix $VD_x^H D_x V$ as a function of the matrix D_x (where we set p_1 to be zero) and the coefficients p_1 and p_{nx} :

$$VD_x^H D_x \ V \ = \ VD_x^t D_x \ V - p_1 \frac{v_1^2}{\Delta x^2} I_1 - p_{nx} \frac{v_{nx}^2}{\Delta x^2} I_{nx} \tag{61}$$

where the matrix I_1 has a coefficient equal to 1 in the left corner and zeros elsewhere, the matrix I_{nx} has a non zero-coefficient equal to 1 in the right corner and zeros elsewhere; and D_x^t denotes the transposed matrix of D_x.

From the equation (61), we get the second relation for the dot product:

$$< VD_x^H D_x \ V \ \Psi_{d_k} \mid \Psi_{d_k} > \ =$$

$$\mid \mid D_x \ V \Psi_{d_k} \mid \mid^2 - p_1 \frac{v_1^2}{\Delta x^2} \mid \Psi_{1,d_k} \mid^2 - p_{nx} \frac{v_{nx}^2}{\Delta x^2} \mid \Psi_{nx,d_k} \mid^2 \tag{62}$$

The combinations of equations (60) and (62) gives the relation between the boundary parameters p_1, p_{nx} and the eigenvalues d_k of the matrix $VD_x^H D_x V$:

$$\boxed{d_k \ = \ \frac{\mid \mid D_x \ V \Psi_{d_k} \mid \mid^2}{\mid \mid \Psi_{d_k} \mid \mid^2} - p_1 \frac{v_1^2}{\Delta x^2} \frac{\mid \Psi_{1,\, d_k} \mid^2}{\mid \mid \Psi_{d_k} \mid \mid^2} - p_{nx} \frac{v_{nx}^2}{\Delta x^2} \frac{\mid \Psi_{nx,\, d_k} \mid^2}{\mid \mid \Psi_{d_k} \mid \mid^2}} \tag{63}$$

This relation places a condition on the coefficients p_1 and p_{nx}, sufficient to guarantee the absolute stability of the extrapolation scheme. The eigenvalues d_k of the matrix $VD_{x_H} D_x V$ have a negative imaginary part if the following condition is verified:

$$v_1^2 \ Im(p_1) \ \mid \Psi_{1,d_k} \mid^2 + v_{nx}^2 \ Im(p_{nx}) \ \mid \Psi_{nx,d_k} \mid^2 \ \geq \ 0 \tag{64}$$

This equation is a condition sufficient to guarantee the stability of the extrapolation; this condition is necessary when the velocity has no lateral variations. The physical interpretation of this condition is that the energy entering to the section through one absorbing lateral side must be compensated by energy going of the section through the other side. Let us consider a plane wave travelling forward; i.e., from the left to the right of the grid. From the equation (64) we get:

$$- v_1^2 \ \sin\left(\frac{w \, \Delta x}{v_1} sin(\theta)\right) \ \mid \Psi_{1,d_k} \mid^2 + v_{nx}^2 \ \sin\left(\frac{w \, \Delta x}{v_{nx}} sin(\theta)\right) \ \mid \Psi_{nx,d_k} \mid^2 \ \geq \ 0 \tag{65}$$

The stability condition to downward continue a downgoing wave is the same as in the equation (64) but with an opposite sign because we use a negative frequency.

The absorbing side boundary conditions used to generate the different examples shown in this paper are either data-dependent (Burg prediction filter) or data-independent. When the data-dependent condition yields to an unstable scheme, we use instead the lateral extrapolation operator D_x to determine the coefficient p_1. If we assume a wave propagating in the x direction, the extrapolation equation becomes:

$$D_x = \frac{jw}{v_1} \tag{66}$$

Therefore, after a Crank-Nicholson transform, we get:

$$\Psi_0 = \frac{1 + \frac{jw\,\Delta x}{2v_1}}{1 - \frac{jw\,\Delta x}{2v_1}} \Psi_1 \tag{67}$$

A similar result can be found for the left side boundary. This relation can be used in the case of the downward continuation of a downgoing wave if a negative frequency w is used instead of a positive frequency. According to the stability criterion given by the equation (64), the absorbing side boundary condition given by the equation (67) is unconditionally stable.

In the Figure 1, this mixed technique of using absorbing side boundary conditions is tested for the 65-degree extrapolation operator. The comparison with the results given by the zero-slope condition demonstrates that the boundaries become *transparent* when one uses the mixed technique. Physically, the boundaries are one-way transparent because the energy can go outside the grid but can't come back inside the grid.

There is still a problem one encounters in the implementation of this data-dependent boundary condition. It comes from the computation of the coefficients p_1 and p_{nx} themselves. When the denominator in the equation (47) is very small, we don't use this equation to determine the coefficient p_1 but instead use the zero-slope condition or the condition given by the equation (67).

We compute the maximum of the energy of the wavefield on a trace of the stacked section at the beginning of the migration program and if the denominator in equation (47) is less than 0.001, we don't apply the Burg prediction filter but instead the equation (67) or the zero-slope condition.

Finally, we have related the eigenvalues of the diffraction operator to the absorbing side boundary coefficients. The stability of the extrapolation scheme requires that the energy going outside the grid by one side of it is greater than the energy entering in the grid by the other vertical boundary. When this condition is not satisfied, we use instead the zero-slope or B1 absorbing conditions, which are unconditionally stable.

This analysis gives a criterion to guarantee the stability of the extrapolation scheme, when the absorbing condition can be implemented in the corners of the matrix T; i.e., in the case of z-independent absorbing side boundary conditions. This criterion can not be applied to the B2 and B3 absorbing conditions because they involve derivatives with respect to z.

NUMERICAL IMPLEMENTATION

We emphasize in that part the implementation of the 45- and 65-degree extrapolation operators, which are the operators of interest for our seismic data. We test the performance of the 45-degree operator on a synthetic zero-offset section and a field data stacked section.

Most of our stacked seismic data has dipping steep beds less than 60 degrees and therefore, the 45-degree operator will migrate accurately these zero-offset sections. The loss of precision (Brysk, 1983) comes from the sparse sampling of the midpoint axis.

The 65-degree operator has been derived for the pre-stack migration of profiles. In accordance with Jacobs (1983), offset angles in a shot profile can be high, even if the dip of the reflector is small. In that case, the shot profiles (and eventually the receiver profiles) are downward continue with this wide-angle operator.

We will compare the computational cost and the accuracy of phase-shift plus interpolation, full scalar wave equation and finite difference operators in the space-frequency domain.

A method which saves 50% of the computational cost for both the migration by the full scalar wave equation, and the migration by finite difference, is briefly presented. It also saves 50% computational cost of the pre-stack migration of profiles.

Synthetic examples

The synthetic zero-offset section used in this paper to test the 45-degree operator has been generated by the program syns83 (Cerveny and Psencik, 1983). The velocity model has both severe lateral and vertical velocity variations as shown in Figure 6.

The zero-offset section shown in Figure 7 has been generated by point sources and has no converted waves. The 3D geometrical spreading and the attenuation due to the reflection coefficients have been included. The reflectors below the dome have small amplitudes because of strong reflections on the first flat bed.

The synthetic section has 500 common depth points, the time sampling is equal to 5 ms, the intertrace Δx to 20 meters and the frequency bandwidth extends from 5 Hz to 75 Hz. An important point is that the modeling program syns83 does not generate diffractions at the corners of the reflectors.

The parameter Δz uses for the extrapolation is equal to 5 meters. The depth migrated section, extrapolated with the 45 degree operator, is shown in Figure 8. As expected, the absence of diffractions in the modeling program, gives rise to migration artifacts on the corner points of the synthetic section.

The edges of the migrated salt dome have moved laterally to their correct subsurface location. The lateral displacement of time patterns by the depth migration procedure, is an important feature of the downward extrapolation in media with laterally varying velocity. Here, this displacement is approximatively equal to 1 kilometer.

Again, this property of time patterns to move laterally toward their correct subsurface location can avoid wrong structure interpretations and enables the interpretators to better determine the subsurface location of stratigraphic or structural traps.

The migrated reflectors below the dome and the first reflector have a weak amplitude and are hardly distinguishable. It is because a too slight dip-filter has been used to attenuate the evanescent waves, and therefore, these waves create migration artifacts which weakens the accuracy of the migration. An other point is that we have depth migrated a section where no equalization has been applied.

We then change the dip-filtering parameters (γ_1=.7 as before, but γ_2=.01 instead of 0.05) and depth-migrated the zero-offset synthetic section after having equalized by a *Tpow* time gain function. The synthetic section is now shown in Figure 9 while the depth migrated section is shown in Figure 10.

First, a stronger dip-filtering gives a better migrated section because the artifacts coming from, either a sparse lateral sampling, or the evanescents waves, are attenuated. Second, the *Tpow* migrated section better shows that the reflectors below the dome and the edges of the dome itself appear now quite clearly. Finally, no migration artifacts come from the boundaries which are transparent to the energy going outside the grid.

The field data example is a stacked section of a profile recorded in Eastern-Nevada. This profile has been shot in the Spring-valley Basin. The velocity model is shown in Figure 11; it presents both severe lateral and vertical velocity variations.

The stacked section which is 7.5 kilometers long, has 500 common depth-points, a time sampling of 4ms and the trace interval is equal to 15.2 meters. This zero-offset section is shown in Figure 12. It shows an dissymmetric basin, fault dips (normal faults) to the East approximatively equal to 40 degrees, diffraction patterns and pull-up due to the velocity of the structure.

The downward extrapolation has been achieved with the 45-degree extrapolation operator, a Δz step equal to 10 meters and a stronger dip-filtering than one used to migrate the synthetic section. The depth migrated section is shown in Figure 13. The West side diffraction patterns of the stacked section have been collapsed along the fault dipping to the East side. The dissymmetry of the Basin is more clear on the depth migrated section. The pull-ups have also been properly migrated to their subsurface location. Also, the continuity of the reflector dipping to the East and starting below the CDP 350 was not obvious.

The velocities model given by the conventional processing are the RMS velocities. It means that we have a velocity model in the time-midpoint coordinates while we need the interval velocities in space-midpoint coordinates to do the depth migration.

This paper does not address the important problem of the determination of these velocities. Briefly, the velocity model shown in Figure 11 was first estimated by the use of the Dix formula. It maps RMS velocities in depth-midpoint velocities. We then used Stolt migration method to collapse the diffractions of the stacked section, and therefore improve the accuracy of the velocity model.

A METHOD TO HALVE COMPUTATION TIME

In this part, we do not address the trick, that is to increase the depth extrapolation parameter Δz with the depth (this is done by using the property of the velocity to increase with the depth), but instead a more fundamental feature of the depth migration in the frequency domain.

In accordance with Schultz et al (1980), if you downward continue in depth the stacked section up to a certain depth, and then come back in the time domain by the use of a Fourier transform on each trace, you get a time section which has the same number of time samples than the zero-offset at the earth surface. But here, only a part of the data is relevant.

If the depth migration has reached a depth such as for example only 512 time samples are relevant out of 1024; ones comes back in the frequency domain by performing it only on these samples, and then restarts the depth migration by downward continuing only 2 times less frequency.

In practice, this trick is used two times and save approximatively 50% of the whole computational cost.

Finally, the numerical analysis of Dubrulle (1983) shows that the 45-degree extrapolation operator has the same computational cost than the phase-shift method. The trick above makes still our extrapolation operators more attractive.

The migration by the full wave equation (Kosloff and Baysal, 1983) solves a Runge-Kutta algorithm at each z-step extrapolation and for each frequency. In order to stabilize the scheme, they do also two Fourier transforms in the lateral direction, operation which is quite consuming and makes their method less attractive.

In accordance with Larner et al (1977), the relative merits of these three methods (and the others) are no more the current debate. The most important issue is the proper treatment of velocity. It determines the precision (as defined by Brysk, 1983) of the migration result itself.

DERIVATION OF THE CONTINUED FRACTION EXPANSION FROM THE SCALAR WAVE EQUATION

The paraxial equation

Claerbout (1976) extracts the 15-degree equation from the scalar wave equation by transforming this equation in the retarted coordinate frame and canceling the second derivative with respect to the depth.

In his coordinate frame, $z' = z$ and $t' = t + \frac{z}{v}$, the scalar wave equation becomes:

$$\frac{v}{2}\frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial z'\,\partial t'} + \frac{v}{2}\frac{\partial^2}{\partial x^2} = 0 \tag{68}$$

Canceling the term $\frac{\partial^2}{\partial z'^2}$, Claerbout found the scalar wave paraxial equation:

$$\frac{\partial^2}{\partial z'\,\partial t'} + \frac{v}{2}\frac{\partial^2}{\partial x^2} = 0 \tag{69}$$

In the frequency domain, this equation fits the first order of the focusing operator expansion first introduced in reflection seismology by Muir. This expansion is generated by the following recursion:

$$F_{k+1}(S) = \frac{S}{2 + F_k(S)} \;\; ; \;\; F_0(S) = 0 \;\; ; \;\; k \geq 0 \;\; ; \;\; S = \frac{v^2}{w^2}\frac{\partial^2}{\partial x^2} \tag{70}$$

while the focusing equation is:

$$\frac{\partial}{\partial z} = \frac{jw}{v}\left[-1+\sqrt{1+\frac{v^2}{w^2}\frac{\partial^2}{\partial x^2}}\right] \tag{71}$$

The analog of the paraxial equation in the frequency domain is to solve two partial differential equations, which are the phase-shift equation and the 15-degree focusing equation:

$$\frac{\partial}{\partial z} = \frac{jw}{v} \tag{72a}$$

$$\frac{\partial}{\partial z} = \frac{jw}{v}\frac{S}{2} \tag{72b}$$

The analog of the equation (72b) in the time domain is the paraxial equation and the analog of the equation (72a) is the phase-shift equation given by:

$$\frac{\partial}{\partial z} = \frac{1}{v}\frac{\partial}{\partial t} \tag{73}$$

When the velocity has only depth variations, the Claerbout coordinate transform solves implicitly the equation (73) and therefore the remaining work is only to solve the 15-degree equation.

The 45-degree approximation of the square-root first given by Claerbout (1976) in the frequency domain was derived in the time domain by Stolt (1978) in three steps. First, he multiplies the equation (68) by $\frac{\partial}{\partial z'}$, then cancels the term $\frac{\partial^3}{\partial z'^2}$ and uses the equation (68) to substitute the term $\frac{\partial^2}{\partial z'^2}$ to get finally the 45-degree equation.

$$\frac{\partial^3}{\partial z'\,\partial t'^2} - \frac{v^2}{4}\frac{\partial^3}{\partial z'\,\partial x^2} + \frac{v}{2}\frac{\partial^3}{\partial x^2 \partial t'} = 0 \tag{74}$$

As for the 15-degree equation, this equation can be found by a Fourier inverse transform of the second order of the Muir expansion. Finally, This suggests that the Muir expansion can be derived from the scalar wave equation itself.

Derivation of the continued fraction expansion

We suppose that the focusing operator approximation is given by the recursive relation (equation [70]), up to the order n. At this order, the equation to solve is:

$$\frac{\partial}{\partial z} = \frac{jw}{v}F_n(S) \tag{75}$$

We multiply each side of the above equation by $\frac{\partial}{\partial z}$ and compute the term $\frac{\partial^2}{\partial z^2}$ by its value given by the equation (68) in the frequency domain, i.e.:

$$\frac{\partial^2}{\partial z^2} = -\frac{2jw}{v}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial x^2} \tag{76}$$

We then substitute the second order derivative with respect to z in the equation (75), by its value given by the equation (76), and it yields:

$$\left(-\frac{2jw}{v} - \frac{jw}{v} F_n(S)\right)\frac{\partial}{\partial z} = \frac{\partial^2}{\partial x^2} \tag{77}$$

We finally substitute the second order derivative with respect to the parameter x by its value $\frac{w^2}{v^2}S$, and we get:

$$\boxed{\frac{\partial}{\partial z} = \frac{jw}{v}\frac{S}{2+F_n(S)}} \tag{78}$$

which is nothing but the recursive relation given in the equation (3). Therefore, the implicit expansion first used by Claerbout (1976), then Stolt (1978), Berkhout (1980), and Ma (1980) to derive one-way wave extrapolation operators in the time domain is exactly the same as the explicit expansion of the focusing operator in the frequency domain.

Therefore, the strategy to derive extrapolation operators in the time domain is first to build these operators in the frequency domain with the explicit expansion and then come back in the time domain with an inverse Fourier transform.

One-way wave extrapolation operators in the time-space domain

The extrapolation operators in the frequency domain are split in 45-degree operators. The odd orders of the recursive relation given by the equation (70) are never used because their computational cost in the frequency domain is the same than the next even order.

In the time domain, the strategy is to split the extrapolation operator in 15-degree operators as shown by Ma (1980). These operators involve only the first derivatives of the wavefield with respect to the time and the depth. Therefore, a double Crank-Nicholson transform with respect to the time and the depth can be used to stabilize the scheme and improve the accuracy.

We already know how to split a finite matrix periodic fraction expansion in 45-degree operators. The partial fraction decomposition of this finite expansion is first calculated (see the paragraph "Splitting in 45-degree type extrapolation operators") as a sum of 45-degree type approximations of the square-root. Then the extrapolation operator is split in 45-degree type extrapolation operators.

The procedure to build 15-degree extrapolation operators in the time domain is still the same. The partial fraction decomposition of the approximant $F_n(S)$ of the continued fraction expansion (equation [70]) is given by:

$$F_n(S) = c_n\, S + \sum_{k=1}^{k=\frac{n}{2}} \frac{a_k\, S}{b_n + S} \tag{79}$$

Here, $\frac{n}{2}$ stands for the integer part of this number. For even orders, say $2n$, the above relation shows that the $2n^{th}$ approximant is the sum of n 45-degree type approximants (the coefficient c_{2n} is equal to zero). For odd orders, say $2n+1$, the corresponding approximant is the sum of n 45-degree type approximants plus one 15-degree type approximant.

Therefore, the decomposition of each approximant in 15-degree type approximations is achieved by expanding each 45-degree type approximation as the sum of two partial fractions as following:

$$\frac{jw}{v}\,\frac{a\, S}{b + S} = \frac{a\,\frac{v}{2\sqrt{b}}\,\frac{\partial^2}{\partial x^2}}{v\,\frac{\partial}{\partial x} - jw\sqrt{b}} - \frac{a\,\frac{v}{2\sqrt{b}}\,\frac{\partial^2}{\partial x^2}}{v\,\frac{\partial}{\partial x} + jw\sqrt{b}} \tag{80}$$

The partial differential equation associated with each 45-degree approximation is:

$$\frac{\partial}{\partial z} = \frac{jw}{v}\,\frac{a\, S}{b + S} \tag{81}$$

We then split this PDE into two PDEs by using the equation (80), and we get:

$$\frac{\partial}{\partial z} = \frac{a\,\frac{v}{2\sqrt{b}}\,\frac{\partial^2}{\partial x^2}}{v\,\frac{\partial}{\partial x} - jw\sqrt{b}} \tag{82a}$$

$$\frac{\partial}{\partial z} = -\frac{a\,\frac{v}{2\sqrt{b}}\,\frac{\partial^2}{\partial x^2}}{v\,\frac{\partial}{\partial x} + jw\sqrt{b}} \tag{82b}$$

Then, coming back in the time domain by an inverse Fourier transform we find the two

15-degree type equations which are:

$$\boxed{\frac{\partial^2}{\partial z\,\partial t} \pm \frac{v}{\sqrt{b}}\,\frac{\partial^2}{\partial z\,\partial x} + \frac{av}{2b}\,\frac{\partial^2}{\partial x^2} = 0} \qquad (83)$$

Implementation of the extrapolation operators in the time-space domain

The first advantage of the equivalence of the Muir partial fraction (explicit) expansion and the implicit expansion has been so far to derive extrapolation operators in the time domain by the simple procedure we have described previously.

The second advantage is to modify the explicit expansion, still in the frequency-space domain in order to improve the accuracy of this expansion. Following Jacobs (1982), we modify the recursive relation given in the equation (70) by introducing a parameter γ as:

$$F_{k+1}(S) = \frac{S}{2 + F_k(S)} \;;\; F_1(S) = \frac{v^2}{\gamma\, w^2}\,\frac{\partial^2}{\partial x^2} \;;\; k \geq 1 \qquad (84)$$

where the matrix operator S has the same definition as before and the parameter γ is a real positive number. In a way similar to what we did to improve the accuracy of the extrapolation operators in the frequency domain, the parameter γ enables us to build more accurate approximations of the square root.

Figure 14 shows the group velocity parametric curves for the 15- (Figure 14a and 14b) and 55-degree extrapolation operators (Figure 14c and 14d). Choosing γ equal to .92 transforms the 15-degree approximation in a 20-degree accurate one. The optimal parameter γ for the 55-degree approximations is equal to .7 and the third approximant of the recursive relation (84) becomes a 65-degree accurate operator.

In practice, it means that the implementation of the parameter γ transforms the three first orders of the recursive relation (84) in wider-angle operators and that is realized at no computational cost.

The operators of interest in the time domain are the three first approximants of the recursive relation (84). The trick used to improve the accuracy of these approximants has no computational cost. The implementation of these operators is to solve alternatively, at each time-step 15-degree type extrapolation operators. The numerical implementation of these operators has been described extensively by Claerbout (1976) and Ma

(1981, 1983).

We assume that the velocity has only vertical velocity variations. Therefore, Claerbout's coordinate transform solves implicitly the phase-shift equation.

As noticed by Claerbout (1983), the solution of the phase-shift equation, for a medium with laterally velocity variations, doubles the computational cost of the paraxial wave solution. In that case, the finite difference method in the space-frequency domain is used because the phase-shift equation in that domain is solved analytically.

CONCLUSION

The depth extrapolation of the wavefield by finite difference of the wavefield in the frequency-space domain is achieved either by the 45- or the 65-degree extrapolation operators. We have derived these operators by the use of a periodic continued fraction expansion.

They are WKBJ accurate in the 1D vertical case and produce the correct amplitudes for both smooth and discontinuous velocity variations. Splitting of this one in 45-degree type extrapolation operators (at no loss of accuracy) leads to a unique code to build wide-angle extrapolation operators.

These operators can use detailed velocity information and are built to handle both complex structures and severe velocity variations. A causal dip-filter has been implemented inside these operators at no extra computational cost. It enables us to remove the evanescent waves, dip-filter high dips and improve the accuracy of the extrapolation operators.

The stability analysis gives a condition sufficient to guarantee the stability of the extrapolation scheme. Data-dependent absorbing side boundary conditions have been implemented inside the operators at no extra computational cost. This is equivalent to one-way *transparent* boundaries.

We have shown how to save 50% of the computational cost for the migration by the full scalar wave equation and the migration by finite difference. Our method, which takes advantage of the low cost of the fast Fourier transform, also saves 50% computational cost of the pre-stack migration of profiles.

The extrapolation operators can be used for the pre-stack migration processing both to downward continue the shot profiles and the receiver profiles. We have shown that our finite difference method compares favorably with the phase-shift plus interpolation method.

We have demonstrated the equivalence of the implicit expansion of the scalar wave and the continued fraction expansion of the square root to derive one-way wave extrapolation operators.

The wide-angle extrapolation operators do not separate in the 3D case and therefore, one must use the phase-shift plus interpolation method to downward continue in the wavefield depth.

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