Stability of finite-differencing DMO

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ABSTRACT

Centered and non-centered finite-differencing DMO methods for offset extrapolation in the (k,t) and (x,t) domains are reviewed and their stability is analyzed. The centered methods that I will describe are unconditionally stable, but the non-centered method is at most only conditionally stable.

INTRODUCTION

The differential equation

$$\frac{\partial^2 P}{\partial t\, \partial h} = \frac{h}{t} \frac{\partial^2 P}{\partial x^2} \quad , \tag{1}$$

was shown by Bolondi et. al. (1982) to approximate offset continuation. $P(h,t,x)$ is the wave field as a function of the time t, the mid-point x, and the half-offset h. Equation (1) can be used to extrapolate in the offset direction to produce the zero-offset section from any given constant-offset section (after normal move-out has been applied). Finite-differencing programs were developed to apply equation (1), in the (x,t) domain (Salvador and Savelli, 1982) and in the (k,t) domain (Ronen, 1983a). The accuracy of the differential equation (1) was shown to be reasonable below the mute (Ronen, 1983b).

To simplify the differential equation (1), one can describe the data P as a function of the square of the offset; using

$$\frac{\partial}{\partial h} = 2h \frac{\partial}{\partial (h^2)} \quad ,$$

we can write equation (1) as

$$\frac{\partial^2 P}{\partial t\, \partial (h^2)} = \frac{1}{2t} \frac{\partial^2 P}{\partial x^2} \quad . \tag{2}$$

FINITE DIFFERENCING IN THE (k,t) DOMAIN

Fourier-transforming over x we have

$$\frac{\partial^2}{\partial x^2} \supset -k^2 \ .$$

and equation (2) is rewritten

$$\frac{\partial^2 P}{\partial t \, \partial(h^2)} = -\frac{k^2}{2t} P \ . \tag{3}$$

Centered finite-differencing of equation (3) gives

$$\frac{1}{\Delta t \, \Delta h^2} \left(P_{h_{j+1}}^{t+\Delta t} - P_{h_{j+1}}^{t} - P_{h_j}^{t+\Delta t} + P_{h_j}^{t} \right) =$$

$$-\frac{k^2}{2t+\Delta t} \frac{1}{4} \left(P_{h_{j+1}}^{t+\Delta t} + P_{h_{j+1}}^{t} + P_{h_j}^{t+\Delta t} + P_{h_j}^{t} \right) , \tag{4}$$

in which a uniform offset square is used for extrapolation:

$$h_{j+1}^2 = h_j^2 + \Delta h^2 \ .$$

Defining

$$\alpha_t = \frac{\Delta h^2 k^2}{8(t/\Delta t + 1/2)} \ , \tag{5}$$

we obtain from equation (4)

$$\left(1 + \alpha_t \right) P_{h_j}^{t} - \left(1 - \alpha_t \right) P_{h_j}^{t+\Delta t} = \left(1 - \alpha_t \right) P_{h_{j+1}}^{t} - \left(1 + \alpha_t \right) P_{h_{j+1}}^{t+\Delta t} \tag{6}$$

The common-offset section at the offset h_j, $P_{h_j}^t$, can be continued from the section at a further offset $P_{h_{j+1}}^t$, recursively in time, by the difference equation

$$P_{h_j}^{t} = \beta_t \left(P_{h_{j+1}}^{t} + P_{h_j}^{t+\Delta t} \right) - P_{h_{j+1}}^{t+\Delta t} \tag{7}$$

with

$$\beta_t = \frac{1 - \alpha_t}{1 + \alpha_t} \tag{8}$$

The offset continuation by equation (7) is repeated until the zero-offset is reached.

To show stability of equation (7), we first consider each temporal Fourier component separately; for each component $e^{i\omega t}$, we have

$$P_{h_j}^{t+\Delta t} = e^{i\omega \Delta t} P_{h_j}^{t} \tag{9}$$

Substituting equation (9) in equation (7), we find that

$$P_{h_j}^t = \frac{\beta_t - e^{i\omega\Delta t}}{1 - \beta_t e^{i\omega\Delta t}} P_{h_{j+1}}^t \tag{10}$$

The von Neumann condition for stability (described in many numerical analysis textbooks, for example Ferziger, 1981) is unconditionally satisfied because

$$\left| \frac{\beta_t - e^{i\omega\Delta t}}{1 - \beta_t e^{i\omega\Delta t}} \right| = 1 \ , \tag{11}$$

for all ω and any β_t; therefore, every component is stable. Stability of the sum of all the Fourier components follows from the stability of each component by linearity of the Fourier-transform.

EXPLICIT METHOD IN THE (x, t) DOMAIN

Salvador and Savelli (1982) presented an explicit finite-differencing method in the (x, t) domain, by non centered finite-differencing of equation (1) one obtains:

$$\frac{1}{\Delta h \, \Delta t} \left(P_{h+\Delta h}^{t+\Delta t, x} - P_{h+\Delta h}^{t,x} - P_h^{t+\Delta t, x} + P_h^{t,x} \right) = \tag{12}$$

$$\frac{h}{t} \frac{1}{(\Delta x)^2} \left(P_{h+\Delta h}^{t, x+\Delta x} - 2P_{h+\Delta h}^{t,x} + P_{h+\Delta h}^{t, x-\Delta x} \right)$$

and from that, the non-centered offset-extrapolation:

$$P_h^{t,x} = P_h^{t+\Delta t, x} + P_{h+\Delta h}^{t,x} - P_{h+\Delta h}^{t+\Delta t, x} + \xi \left(P_{h+\Delta h}^{t, x+\Delta x} - 2P_{h+\Delta h}^{t,x} + P_{h+\Delta h}^{t, x-\Delta x} \right) \tag{13}$$

in which ξ is defined as

$$\xi = \frac{h \, \Delta h \, \Delta t}{t \, (\Delta x)^2} \ .$$

Using the same stability analysis as in the previous section, we now have

$$P^{t, x+\Delta x} = e^{ik\Delta x} P^{t,x} \tag{14}$$

$$P^{t+\Delta t, x} = e^{i\omega\Delta t} P^{t,x}$$

Using equations (14) in (13) we obtain

$$P_h^{t,x} = \left(1 - 2\xi \frac{1 - \cos k \, \Delta x}{1 - e^{i\omega\Delta t}} \right) P_{h+\Delta h}^{t,x} \tag{15}$$

The offset continuation of equation (13) is unstable because the right side of equation (15) is unbounded for $\omega = 0$. This instability may be avoided if the low temporal frequency is removed at every step; for example by removing the evanescent field. In this case stability is conditional; it depends on the absolute value of

$$R = 1 - 2\xi \frac{1 - \cos k\,\Delta x}{1 - e^{\,i\omega\Delta t}}$$

The condition for stability is thus

$$\left| R \right|^2 = 1 + 2\xi^2 \frac{(1 - \cos k\,\Delta x)^2}{1 - \cos\omega\Delta t} - 2\xi\,(1 - \cos k\,\Delta x) \leq 1 \quad ,$$

or, for all k and ω

$$0 < \xi \leq \frac{1 - \cos\omega\Delta t}{1 - \cos k\,\Delta x} \quad .$$

This condition cannot be satisfied for $\omega = 0$. If the low temporal frequencies are filtered out to remove the evanescent field, so that the remaining frequencies are $\omega > vk$, for some non-zero velocity v, then stability can be achieved under the following mute condition:

$$\frac{h}{t} \leq \frac{(\Delta x)^2}{\Delta h\,\Delta t} \frac{1 - \cos(vk\,\Delta t)}{1 - \cos(k\,\Delta x)} \tag{16}$$

IMPLICIT METHOD IN THE (x, t) DOMAIN

Centered finite-differencing of equation (2) gives

$$P_{h_{j+1}}^{t+\Delta t,x} - P_{h_{j+1}}^{t,x} - P_{h_j}^{t+\Delta t,x} + P_{h_j}^{t,x} = \tag{17}$$

$$\gamma_t \Big(P_{h_{j+1}}^{t,x+\Delta x} - 2P_{h_{j+1}}^{t,x} + P_{h_{j+1}}^{t,x-\Delta x}$$

$$P_{h_{j+1}}^{t+\Delta t,x+\Delta x} - 2P_{h_{j+1}}^{t+\Delta t,x} + P_{h_{j+1}}^{t+\Delta t,x-\Delta x}$$

$$P_{h_j}^{t,x+\Delta x} - 2P_{h_j}^{t,x} + P_{h_j}^{t,x-\Delta x}$$

$$P_{h_j}^{t+\Delta t,x+\Delta x} - 2P_{h_j}^{t+\Delta t,x} + P_{h_j}^{t+\Delta t,x-\Delta x} \Big)$$

in which γ_t is defined by

$$\gamma_t = \frac{\Delta h^2}{(\Delta x)^2 8(t/\Delta t + 1/2)} \tag{18}$$

Equation (17) is unconditionally stable because if we use equations (14) in equation (17), we obtain

$$P_{h_j}^{t,x} = \frac{\hat{\beta}_t - e^{\,i\omega\Delta t}}{1 - \hat{\beta}_t\, e^{\,i\omega\Delta t}} P_{h_{j+1}}^{t,x} \tag{19}$$

where

$$\hat{\beta}_t = \frac{1 - 2\gamma_t (1-\cos k\,\Delta x)}{1 + 2\gamma_t (1-\cos k\,\Delta x)} \tag{20}$$

And it is stable because

$$\left| \frac{\hat{\beta}_t - e^{i\omega\Delta t}}{1 - \hat{\beta}_t\, e^{i\omega\Delta t}} \right| = 1 \quad .$$

CONCLUSIONS

Stability is unconditional for centered finite-differencing offset continuation in the (x,t) and in the (k,t) domains. For a non-centered (x,t) method, there is a time dependent condition for stability,

$$\frac{h}{t} \leq \frac{(\Delta x)^2}{\Delta h\, \Delta t} \frac{1 - \cos(vk\,\Delta t)}{1 - \cos(k\,\Delta x)}$$

and the evanescent field, $\omega/k < v$, should be removed at every step.

REFERENCES

Bolondi G., Loinger E., and Rocca F., 1982, Offset continuation of seismic sections, Geophysical Prospecting 30, p. 813-828.

Ferziger J.,H., 1981, Numerical Methods for Engineering Application, John Wiley & Sons, p. 141-142.

Ronen J., 1983a, Finite-differencing DMO, SEP-35, p. 133-144.

Ronen J., 1983b, DMO applied, SEP-37, p. 81-90.

Salvador L. and Savelli S., 1982, Offset continuation for seismic stacking, Geophysical prospecting 30, p. 829-849.

Output of the Gubkin Institute's Soviet ES-1035 computer, which uses the Soviet operating system 'OS'. RNP is the Soviet abbreviation for CDR.

```
//RB        JOB MSGLEVEL=1
//STARTING EXEC RB,D='CHUCK(J)'
XXRB    PROC  U=SYSDA,DP='SYS1.PROCLIB',D=,L=,RG=64K,BLD=80,VP=,UP=,          *
XX BL=3360,UW=SYSDA,VW=,C=8,PR=01,TM=010,SO=050,SMO=050,RGS=050,CM=1,        *
XX BLP=0,ML=20,CL=A,S=500,SM=500,V=USER02
XXIEFPROC EXEC PGM=IEFIRC,REGION=&RG,DPRTY=(14,15),                           *
IEF653I SUBSTITUTION JCL - PGM=IEFIRC,REGION=64K,DPRTY=(14,15),
XX PARM='&C.&PR.&TM.&SO.&SMO.249&RGS.&CM.&BLP.SYSDA    E000&ML.&CL.'
IEF653I SUBSTITUTION JCL - PARM='801013050050249050105YSDA    E00020A'
XXIEFRDER DD    UNIT=&U,VOL=SER=&V,LABEL=&L,DSN=&D,DISP=SHR,                   *
IEF653I SUBSTITUTION JCL - UNIT=SYSDA,VOL=SER=USER02,LABEL=,DSN=CHUCK(J),DISP=SHR,
XX DCB=(BLKSIZE=&BL,LRECL=80,RECFM=FB,BUFL=&BL,)
IEF653I SUBSTITUTION JCL - DCB=(BLKSIZE=3360,LRECL=80,RECFM=FB,BUFL=3360)
XXIEFPDSI DD    DSN=&DP,UNIT=&UP,VOL=SER=&VP,DISP=SHR
IEF653I SUBSTITUTION JCL - DSN=SYS1.PROCLIB,UNIT=,VOL=SER=,DISP=SHR
XXIEFDATA DD    UNIT=&UW,VOL=SER=&VW,SPACE=(80,(&S,&SM),RLSE,CONTIG),          *
IEF653I SUBSTITUTION JCL - UNIT=SYSDA,VOL=SER=,SPACE=(80,(500,500),RLSE,CONTIG),
XX DCB=(RECFM=FB,DSORG=PS,LRECL=80,BLKSIZE=&BLD,BUFL=&BLD,)
IEF653I SUBSTITUTION JCL - DCB=(RECFM=FB,DSORG=PS,LRECL=80,BLKSIZE=80,BUFL=80)

//JOHN  JOB (SOT,N,'19',82,00,115,19),'BAPOB E 6',PRTY=13                  00000001
//       EXEC GSDS,IL=SL,OL=SL,V=USER02,GRG=195K,WP=24,WS=0,NU=1              00000002
//SYSIN DD * GENERATED STATEMENT

************************************************************************************************
**  SEISMIC DIGITAL SYSTEM SDS 3.2     ****  EC 1035  MEMORY-  512K OO/MVT ****  DATE  28/05/1984   TIME   17/45  **
**  CGE MNP SSSR MOSKVA  TEL 1926364   ****  RELEASE - 6.1  NUCLEUS - 156K **** NARODNOGO OPOLCHENIJA 40 - 3 - 404 **
************************************************************************************************
**  CLIENT GEOPHYSIQUE                 ****        CREW    82/09        ****    AREA MINXIGP                **
************************************************************************************************

                RNP

SS001CI                               J O B   P R E P A R A T I O N
************************************************************************************************
SS002CI                                     INPUT CARDS LISTING

CARD
REF.
------------------------------------------------------------------------------------------------
0001      OPT=(RAD=06,MSG=32)                                                    00000003
0002      #LPRN=0,0                                                              00000004
0003       PROF=RNP.DP15                                                         00000005
0004       INTX=100,3000,3000                                                    00000006
0005      *REPAR=0,H,17,DP15                                                     00000007
0006      #DIMAG=3000,03000,30,0,4000,2,350,2000,5000                            00000008
0007       PROF=.DP05,0,,PROUT0                                                  00000009
0008      *REPAR=1,3,5000                                                        00000010
0009      *XZAVO=1,,03,,1000,01,0,02,2000,03,3400,04,5400,05,6400                00000011
0010      *REPAR=1,925,1,7,6,8,5                                                 00000012
0011      *SIGNAL=2,CONV,7,3.010,13,-3,-2,5,9,15,18,21,18,15,8,5,-2,-8          00000013
0012      *PRINTER=1,HTR,14,1,200,'(15)20*'                                     00000014
0013      #END                                                                   00000015
------------------------------------------------------------------------------------------------
SS005CI                                     INPUT CARDS LISTING - END

SS007CI                                     DIAGNOSTIC
                                            ----------

SS035JI   ***LEADER(MAIN)***NO ERROR IN SYNTAX AND CARD ORDER

SS145CI    PREPARATION OPTIONS IN EFFECT: MSG=3,BLK= 3520,MDS=   20,RAD=  6,REC= 210,PRV=P,OMIT=  140,TEST= NO
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SS270CI                                     PROCESS FLOW CHART
```