The paradoxical elliptical reflector

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Introduction

Elsewhere in this report, Shuki Ronen gives an ingenious derivation of the phase of the dip-moveout operator. This consists of zero-offset migration of an impulse recorded at non-zero offset and then laterally stretching the resulting circle into the equal traveltime ellipse predicted by ray tracing. When he tried to ascertain how accurately his operator handled amplitudes, using ray-tube arguments I suggested, he discovered a paradox. Simply put, the dip-moveout operator, being based on source-receiver reciprocity, is symmetric but the impulse response Shuki derived from ray energy arguments is not. Shuki's result seems paradoxical because image source arguments show the reflection response of each tangent plane to the ellipse to be identical to the response of a symmetrically placed planar reflector dipping the other way. Superposing all these symmetrically placed tangents produces a symmetric ellipse, just as dip-moveout predicts. How, then, could Shuki have validly obtained reflection coefficients along the ellipse larger on receiver end of the ellipse than on the shot side?

In this paper I will tackle resolution of this paradox by first using the methods of asymptotic ray theory to verify Shuki's result for the underlying model of an elliptical reflecting barrier and then discuss why our impulse response assumptions broke down.

Asymptotic wave theory result

To obtain an impulse response we ask what reflection coefficient distribution in the earth would be manifested as a single impulse on a non-zero offset survey. The immediate answer is that it is some distribution of reflection coefficients along an ellipse with the shot at one focus and the geophone at the other. Comparing this to the result of laterally distorting a uniform circular reflector into this ellipse would, we thought, show how Shuki's method treated amplitude.

So what reflection coefficient distribution along an ellipse will give rise to an impulse in non-zero offset? Here I use the wave-theoretic methods of asymptotic ray theory (Keller and Lewis 1965) to treat this question.

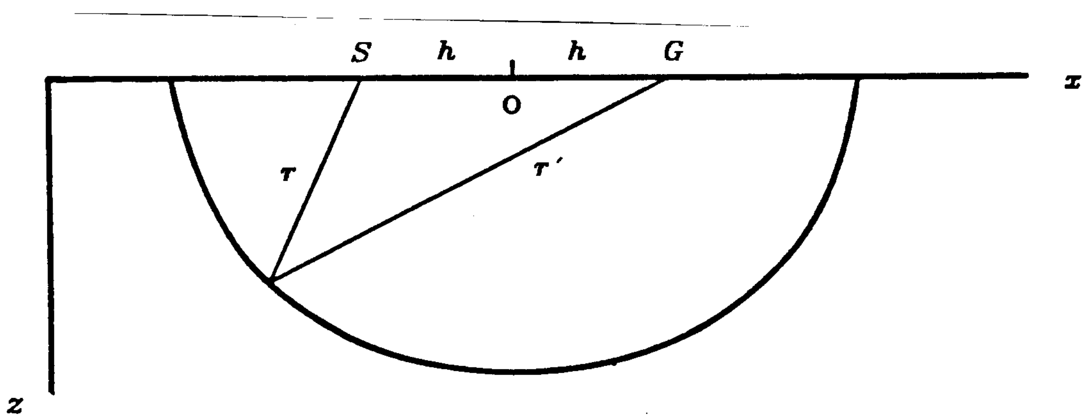

FIG. 1. An elliptical reflecting boundary surrounding a constant velocity region. The source is placed at the focus labelled S and the geophone at focus G. The distance from a boundary point to the source is denoted by r and to the geophone by r'. The origin for Cartesian coordinates is placed midway between the source and geophone.

The basic geometry I will be using is sketched in Figure 1. The ellipse, bounding a region with constant velocity, v, is characterized by equal traveltime t_o to any of the points on its surface whence is given by the equation

$$r + r' = vt_o \tag{1}$$

or, in the more familiar Cartesian form,

$$\frac{4x^2}{v^2t_o^2} + \frac{4y^2}{v^2t_o^2} + \frac{4z^2}{v^2t_o^2 - 4h^2} = 1 \quad . \tag{2}$$

Here the source is located at $[-h,0,0]$ and the geophone at $[+h,0,0]$.

We now excite the spherically symmetric source at time zero. In temporal Fourier space this produces the incident field

$$U^I = \frac{e^{ikr}}{r} \tag{3}$$

where $k = \omega / v$ and $r^2 = (x+h)^2 + y^2 + z^2$.

Similarly, after reflection, a uniform spherical wave arriving at the geophone is given by

$$U^R = \frac{A\, e^{ik(vt_o - r')}}{r'} \tag{4}$$

with $r'^2 = (x-h)^2 + y^2 + z^2$ and A an unknown amplitude.

By rotational symmetry, we need only work in the vertical $x-z$ plane. Along the ellipse, the total field $U = U^I + U^R$ will satisfy the mixed boundary condition

$$\frac{\partial U}{\partial \nu} + ik\eta U = 0 \quad , \tag{5}$$

connecting the field with its normal derivate, for some, to be determined, boundary impedance $\eta(x,k)$.

To compute normal derivatives, we can form the dot product of the normal vector with the gradient. We parametrize the ellipse using the coordinates

$$2x = vt_o \cos\varphi \quad \text{and} \quad 2z = \sqrt{v^2 t_o^2 - 4h^2}\,\sin\varphi \quad . \tag{6}$$

r and r' are given on the ellipse in terms of φ by the simple formulas

$$r = \frac{vt_o}{2} + h\cos\varphi \tag{7a}$$

and

$$r' = \frac{vt_o}{2} - h\cos\varphi \quad . \tag{7b}$$

Taking φ derivatives of (6), we have

$$\frac{dx}{d\varphi} = -\frac{vt_o}{2}\sin\varphi \quad \text{and} \quad \frac{dz}{d\varphi} = \frac{\sqrt{v^2 t_o^2 - 4h^2}}{2}\cos\varphi \tag{8}$$

whence

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dz}{d\varphi}\right)^2 = \frac{v^2 t_o^2}{4} - h^2\cos^2\varphi = rr' \quad . \tag{9}$$

Therefore the unit tangent in the vertical plane is given by

$$\mathbf{T} = (4rr')^{-1/2}\ [-vt_o\sin\varphi, 0, \sqrt{v^2t_o^2 - 4h^2}\cos\varphi] \tag{10}$$

and the unit (inward) normal by

$$\mathbf{N} = -(4rr')^{-1/2}\ [\sqrt{v^2t_o^2 - 4h^2}\cos\varphi, 0, vt_o\sin\varphi] \quad . \tag{11}$$

Naturally, for other azimuths away from the vertical plane we rotate this normal accordingly.

At the elliptical boundary, the gradients of U^I and U^R are given by

$$\nabla U^I = \left[\frac{ik}{r} - \frac{1}{r^2}\right] e^{ikr}\ \hat{\mathbf{r}} \tag{12a}$$

$$\nabla U^R = -A\left[\frac{ik}{r'} + \frac{1}{r'^2}\right] e^{ik(vt_o - r')}\ \hat{\mathbf{r}}' \tag{12b}$$

with $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ unit vectors in the indicated directions. Now

$$\hat{\mathbf{r}}\cdot\mathbf{N} = \hat{\mathbf{r}}'\cdot\mathbf{N} = -\frac{\gamma}{\sqrt{rr'}} \quad , \tag{13}$$

where $2\gamma = \sqrt{v^2t_o^2 - 4h^2}$, evidencing the equality of incident and reflected angles. Therefore the normal derivatives of the incident and reflected wavefields are given along the boundary by

$$\frac{\partial U^I}{\partial \nu} = -\frac{\gamma}{\sqrt{rr'}}\left[\frac{ik}{r} - \frac{1}{r^2}\right] e^{ikr} \tag{14a}$$

$$\frac{\partial U^R}{\partial \nu} = A\frac{\gamma}{\sqrt{rr'}}\left[\frac{ik}{r'} + \frac{1}{r'^2}\right] e^{ik(vt_o - r')} \quad . \tag{14b}$$

Substituting these expressions in equation (5) and dropping the common phase factor $\exp(ikr)$ gives the relation

$$\frac{\gamma}{\sqrt{rr'}}\left\{\frac{A\,ik}{r'} - \frac{ik}{r} + \frac{A}{r'^2} + \frac{1}{r^2}\right\} + ik\eta\left\{\frac{1}{r} + \frac{A}{r'}\right\} = 0 \quad . \tag{15}$$

Solving for η we obtain

$$\eta = \frac{\gamma}{\sqrt{rr'}}\left\{\frac{r' - Ar}{r' + Ar} - \frac{1}{ikrr'}\,\frac{r'^2 + Ar^2}{r' + Ar}\right\} \quad . \tag{16}$$

The reflection coefficient corresponding to η is given by

$$R = \frac{-\mathbf{N}\cdot\hat{\mathbf{r}} - \eta}{-\mathbf{N}\cdot\hat{\mathbf{r}} + \eta}$$

$$= \frac{\frac{\gamma}{\sqrt{rr'}} - \eta}{\frac{\gamma}{\sqrt{rr'}} + \eta}$$

$$= \frac{2ikA\,r^2r' + A\,r^2 + r'^2}{2ikrr'^2 - A\,r^2 - r'^2} \qquad . \qquad (17)$$

In the high frequency limit, $k \to \infty$, we recover Shuki's result

$$R = \frac{A\,r}{r'} \qquad . \qquad (18)$$

For low frequencies, R is approximately -1, a free surface.

This reflection coefficient is surprising because it is not symmetric, while the ellipse, the shot and geophone placement and the Green's function for the wave equation all are. Further, if we attempt to conserve energy by making the magnitude of the reflected amplitude A equal to one, we find that the reflection coefficient (18) is greater than one on the half the ellipse closest to the receiver, another dissatisfying result.

Where did we go wrong?

First, it might be argued that the simple boundary model or asymptotic analysis we used does not fully predict all the complexities of reflection at an interface between two media. However Eisner (1983) works a similar calculation for a more general scalar wave model in his discussion of reciprocity where he states "it is easily shown that an isotropic expanding spherical wave at [one focus] and a similar contracting one at [the other focus] fail to satisfy the boundary conditions on the ellipse." I, myself, obtained the high frequency reflection coefficient R of equation (18) when I reformulated the problem as an elliptical contact between two media.

Yet, as I view it, all questions of reciprocity aside, there is an essential difference between the above calculations, in which we went about solving for a coefficient model, e.g. η in equation (5), and dip-moveout. Dip-moveout is based on the linear theory of wavefield extrapolation, to which the principle of superposition applies, while coefficient inversion, which is spectral division in its simplest, high-frequency form, is nonlinear. In other words, we may superpose dipping wavefields along an elliptical path to create an elliptical wavefront but we cannot, meaningfully, superpose dipping reflectors to create an elliptical reflector*.

Estimating reflection coefficients is not all wrong, however. Dip-moveout, as Hale (1982) shows, is firmly rooted in the double square root equation whose coincident shot-receiver imaging condition, as Claerbout (1978) points out, does produce an estimate of some form of reflection coefficient. The real difference is that we derived our reflection coefficients from just a single common-offset section whereas dip-moveout obtains its reflection strengths by applying double-square root migration to the above common-offset section *with zeros assumed at all other offsets.*

This zero data assumption is not the same as a zero information assumption. It is a mathematical convenience. Indeed this author cannot imagine any subsurface reflector that could only be detected at just one offset. Hence for dip-moveout to give a physically meaningful image it must be applied to the full suite of common-offset sections with all the results stacked together. When this is done dip-moveout will also produce an asymmetric elliptical image.

Conclusions

A single-spike common-offset section, recorded over a constant velocity region, can only be produced by an asymmetric elliptical reflector. This conclusion does not contradict the fact that a symmetric ellipse results from applying dip-moveout and zero-offset migration because DMO+migration solves a different problem than that of estimating subsurface reflection coefficients from a single, common-offset survey. Shuki and I, in equating DMO+migration with inversion of a common-offset survey, solved the wrong problem.

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* John Toldi (private communication) suggests that, from the point of view of Born theory, i.e., linearized inversion, the elliptical geometry we are treating is exceptional. When Born theory does give reasonable approximations, linear superposition also approximately holds and we would be able to superpose linear dipping reflectors to build up our elliptical one. That we aren't we attribute to either the presence of the caustic (focus of rays) at the receiver location or to inapplicability of Born theory to common-offset experiments.

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