Incorporating reflection and transmission coefficients into one-way finite difference equations

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Wave equation migration can be formulated as the generalized inverse to wave equation modeling. This is most easily seen by assuming that the medium has a constant velocity and examining the form of the migration operator in the wavenumber-frequency domain. Migration remaps the input domain from (ω, k_x) to (k_z, k_x) by the transformation $k_z^2 = \omega^2/v^2 - k_x^2$. The only region of the (ω, k_x) domain where the inverse cannot be handled properly is the evanescent region, $k_x^2 > \omega^2/v^2$: the solution in this region is exponentially increasing. However this data in the evanescent domain is handled, by use of an exponential decay or by a zero truncation, the result is that migration is modified from an exact inversion process to a generalized inversion process. Evanescent events as a consequence are ignored in the generalized inversion.

The term *inversion*, as it has been used up to now, has meant the effort taken to invert the wave equation modeling operation. The modeling operator is

$$d(x,\omega) = \int_0^\infty dz \; e^{i\mathbf{A}z} \, \mathbf{u}_0(z) \tag{1}$$

where **A** is the square root operator in x,

$$\mathbf{A} = \left[\frac{\omega^2}{v^2} + \partial_{xx} \right]^{1/2} \tag{2}$$

and $\mathbf{u}_0$ is a function (vector) in x. We can incorporate appropriate boundary conditions into **A** at the edge of the x grid; absorbing boundaries are a good choice (Clayton and Engquist, 1980). Equation (1) is limited to modeling zero-offset sections; a similar modeler may be written down for the more general geometry of non-zero offset.

There are many obvious limitations to equation (1) as a modeler of zero-offset seismic data; not the least of which is a disregard for reflection and transmission coefficients. These effects are taken into account in the full (two-way) wave equation, but are completely ignored in the one-way formulation. For example, the form of the downward continuation operator in (1), $\exp i\mathbf{A}z$, guarantees a transmission coefficient of unity for propagating waves. If anything is to improve upon migration as a generalized inversion procedure, it obviously must be based on a modeling scheme more accurate than that of equation (1) (Mora, 1984).

With this need in mind, we shall attempt to account for the proper reflection and transmission coefficients in a one-way operator. We shall assume that the medium is isotropic, acoustic, two-dimensional, and of constant density. The background velocity of the medium can be a function of x and z. Discontinuities in velocity can be present in z (and x).

First, the differential equation that leads to the modeling operator in equation (1) is the inhomogeneous one-way wave equation

$$\frac{\partial \mathbf{u}}{\partial z} = -i\mathbf{A}\mathbf{u} \tag{3}$$

in which **A** is a function of $v(x)$ (equation (2)). This equation is a "perfect" modeler as long as the velocity v is independent of z:

$$\begin{aligned}\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{v^2}\right]\mathbf{u} &= \left[\frac{\partial^2}{\partial z^2} + \mathbf{A}^2\right]\mathbf{u} \\ &= \left[\frac{\partial}{\partial z} - i\mathbf{A}\right]\left[\frac{\partial}{\partial z} + i\mathbf{A}\right]\mathbf{u} + i\left[\frac{\partial}{\partial z}\mathbf{A} - \mathbf{A}\frac{\partial}{\partial z}\right]\mathbf{u}\end{aligned} \tag{4}$$

If $\frac{\partial \mathbf{A}}{\partial z} \to 0$, the commutator (the second term in (4)) vanishes. The **u** that satisfies the homogeneous part of equation (3) also satisfies the "full" wave equation $(\nabla^2 + \omega^2/v^2)\mathbf{u} = 0$, as long as operator **A** is independent of z.

At the discretization stage of the implementation of equation (3), the velocity model can be assumed to be piecewise constant on a rectangular grid in the (x,z) plane. See figure 1. The one-way operator (3) can be used to downward continue through each layer of thickness Δz, backscattered energy not being a concern. When a level is reached where the velocity jumps in z, reflections and transmissions may, however, become significant. At this level, boundary conditions applicable to the full wave equation must be applied.

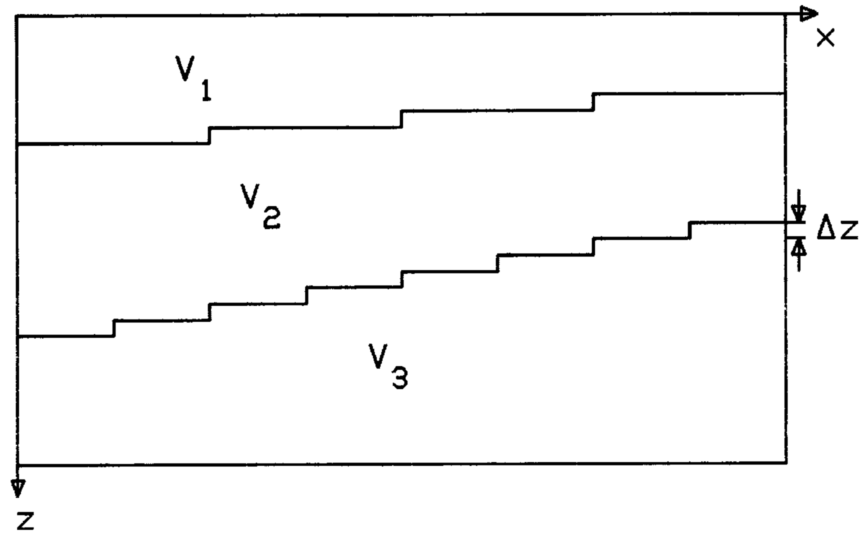

FIG. 1. A piecewise-constant velocity $v(x,z)$.

In an acoustic case the wavefield **u** is a pressure field. The boundary conditions it must satisfy are

$$\mathbf{u}_i + \mathbf{u}_r = \mathbf{u}_t \tag{5}$$

$$\frac{\partial \mathbf{u}_i}{\partial z} + \frac{\partial \mathbf{u}_r}{\partial z} = \frac{\partial \mathbf{u}_t}{\partial z} \tag{6}$$

where

$\mathbf{u}_i$ = downgoing incident wavefield

$\mathbf{u}_r$ = upgoing reflected or backscattered wavefield

$\mathbf{u}_t$ = downgoing transmitted or forward scattered wavefield

Because $\mathbf{u}_r$ must satisfy an upgoing one-way equation (equation (3)),

$$\frac{\partial \mathbf{u}_r}{\partial z} = +i\mathbf{A}_1\mathbf{u}_r \tag{7}$$

and by combining equations (6) and (7),

$$\mathbf{A}_1\mathbf{u}_i - \mathbf{A}_1\mathbf{u}_r = \mathbf{A}_2\mathbf{u}_t \tag{8}$$

or, from equation (5),

$$(\mathbf{A}_1 + \mathbf{A}_2)\mathbf{u}_r = (\mathbf{A}_1 - \mathbf{A}_2)\mathbf{u}_i \tag{9}$$

	Velocity	A	Wave Equation
Upper Medium 1	$v_1(x)$	$\mathbf{A}_1 = \sqrt{\omega^2/v_1^2 + \partial_{xx}}$	$\frac{\partial \mathbf{u}}{\partial z} = -i\mathbf{A}_1\mathbf{u}$
Lower Medium 2	$v_2(x)$	$\mathbf{A}_2 = \sqrt{\omega^2/v_2^2 + \partial_{xx}}$	$\frac{\partial \mathbf{u}}{\partial z} = -i\mathbf{A}_2\mathbf{u}$

FIG. 2. Appropriate operators for an upper and lower medium.

where the subscripts 1 and 2 refer respectively to the upper and lower medium (figure 2). Just as the continuation equations (3) and (7) may be approximated by various methods, so may equation (9). In ideal conditions, a backscattered wavefield $\mathbf{u}_r$ arises from the illuminating wavefield $\mathbf{u}_i$ upon encountering the velocity contrasts $v_1(x)$ and $v_2(x)$. If the velocity is independent of x, equation (9) yields the standard plane-wave reflection coefficients; **A** may then be diagonalized by a Fourier transform, and its eigenvalues (without special side boundary conditions) are given by

$$\lambda_A = \left[\frac{\omega^2}{v^2} - k_x^2 \right]^{1/2}$$

so that

$$u_r(k_x) = \frac{\lambda_{A1} - \lambda_{A2}}{\lambda_{A1} + \lambda_{A2}} u_i(k_x)$$

As the wavefield is propagated downward, backscattered energy may be saved for a subsequent upward continuation sweep, and previously-saved backscattered energy may be added in. In this way any order of multiple can be generated, as in a standard reflectivity approach to layered earth modeling (Kennett, 1983).

For pre-critical waves, a suitable approximation to the square root operator **A** in (9) is the 45-degree approximation

$$\tilde{\mathbf{A}} = \frac{\omega}{v} \frac{1 + \frac{3}{4}\frac{v^2\partial_{xx}}{\omega^2}}{1 + \frac{1}{4}\frac{v^2\partial_{xx}}{\omega^2}} \tag{10}$$

Performing a standard finite-difference of equation (9) in x, with the above 45-degree approximation for **A**, yields a pentadiagonal system of equations to solve at each new level in z; this system is barely more expensive to solve than the finite differencing of equations (3) or (7). However, one shortcoming of this approximation of **A** is its mishandling of evanescent energy: such an approximation is expected to give valid reflection coefficients only for pre-critical angles. Post-critical reflection strengths must be handled by some other method; it is not clear at the moment what this method may be.

Other approaches to the solutions of equation (9) are possible. If the geometry of the velocity model is simple, an eigenvalue decomposition of **A** may be attempted; this process would yield the normal modes of (9), which can be independently propagated. Or, an averaged **A** may replace $\mathbf{A}_1 + \mathbf{A}_2$ if velocity contrasts happen to be small. In any case, by making a discrete approximation to equation (9), backscattered energy with the proper relative amplitudes may be computed with about as much effort as that required by a standard reflectivity approach to modeling.

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