Looking at Wave Equations in Amplitude - Phase Coordinates

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Introduction

Many of the characteristic properties of various wave equations become clearer if the equations are expressed in amplitude-phase coordinates. These coordinates are instructive and make a good device for making clear what the one-way equations really do.

Amplitude-Phase Coordinates

For example, take the one-way wave equation (in retarded coordinates):

$$\frac{\partial Q}{\partial z} = -i\frac{\omega}{v}(1 - \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}})Q.$$

Substitute in $Ae^{i\phi}$ for Q. Dividing through by the factor $e^{i\phi}$ which appears in every term, you get

$$\frac{\partial A}{\partial z} + i\frac{\partial \phi}{\partial z}A = -i\frac{\omega}{v}(1 - \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}})A.$$

Now we get to the interesting part. Since A and ϕ are both real, we can pull this equation apart into its imaginary and real components, which must each be independently and simultaneously satisfied. Doing this (assuming for the moment that the square root is real), and again dividing through by factors appearing in every term, we get:

$$\frac{\partial A}{\partial z} = 0 \qquad \text{Real part}$$

and

$$\frac{\partial \phi}{\partial z} = -\frac{\omega}{v}(1 - \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}}). \qquad \text{Imaginary part}$$

Both of these terms are interesting. The first states the well known fact that diffraction is an all-pass filter. Diffraction only produces phase shifts, no attenuation. The second term is a bit harder to understand. For a plane wave travelling at an angle θ to straight down, $\frac{vk_x}{\omega}$ is $\sin\theta$. Viewing our wave field as a sum of plane waves, we can rewrite the phase shifting equation as

$$\frac{\partial \phi}{\partial z} = -\frac{\omega}{v}(1 - \sqrt{1 - \sin^2\theta}).$$

Using the most basic trigonometric identity in existence, this becomes

$$\frac{\partial \phi}{\partial z} = -\frac{\omega}{v}(1 - |\cos\theta|).$$

Discarding the absolute value brackets would change this to the two-way wave equation. Notice that the $\cos\theta$ is simply taking the downgoing component of the wave. Since we are using retarded coordinates, which "ride along" with a wave going straight down ($\theta = 0$), there is no phase shifting at all in this case.

The most important thing to notice about this equation is that for a given plane wave component, it is a simple phase shift of the form *constant* times ω. From the shift theorem of fourier transforms, this is seen to just be equivalent to a time shift proportional to the constant. Thus for a plane wave component, a depth extrapolation is equivalent to a time delay. In plain English, it takes a while for the plane wave passing you to be seen further down. The $\cos\theta$ accounts for the fact that it takes longer if the wave is using some of its velocity to move to the side as well. Only the downgoing component counts. The v term accounts for the velocity of the wave; a slower wave takes longer to travel the same distance. Viewed this way, our knowledge of how waves physically propagate is shown to be very simply represented in the wave equation.

Earlier we passed over the possibility that the square root was imaginary; the resulting equations applied only to real waves. If the root were instead imaginary (evanescent waves), the equation's real and imaginary parts would instead turn out to be:

$$\frac{\partial A}{\partial z} = -\frac{\omega}{v}(\sqrt{\frac{v^2 k_x^2}{\omega^2} - 1})\ A \qquad \text{Real Part}$$

and

$$\frac{\partial \phi}{\partial z} = -\frac{\omega}{v}. \qquad \text{Imaginary Part}$$

These two equations spell out the familiar properties of evanescent waves quite clearly. The amplitude equation describes a wave decaying exponentially with depth. From Snell's law, as v changes, the direction of propagation θ changes so that the quantity $\frac{k_x}{\omega} = \frac{\sin\theta}{v}$ is a constant over the life of any plane wave. Thus, regarding $\frac{k_x}{\omega}$ as a constant, we can see how the attenuation varies with ω and v.

For a given wave, higher frequencies decay more quickly than lower ones. This conforms to our expectation that there should be a constant attenuation per cycle. For any wave, as we increase v eventually the wave will cross the boundary into evanescent behavior. As v increases past this point, the wave decays more and more rapidly with depth. However, the rate of decay is not a linear function of v but rather increases toward a finite limit as v grows large.

The phase equation is less complicated. It merely shows that the wave propagates downward as fast as a real wave would that was going straight down.

We now turn our attention Muir's approximate one-way equations in amplitude-phase coordinates.

Paraxial Equations in Amplitude-Phase Coordinates

As we would expect, Muir's approximations again lead us to equations of the form

$$\frac{\partial A}{\partial z} = 0$$

and

$$\frac{\partial \phi}{\partial z} = -\frac{\omega}{v} F(\theta).$$

We can now summarize in a table $F(\theta)$ for various wave equations:

$F(\theta)$ for Various Wave Equations	
Two-Way	$1 - \cos\theta$
5°	0
15°	$\frac{\sin^2\theta}{2}$
45°	$\frac{\sin^2\theta}{2 - \frac{\sin^2\theta}{2}}$
One-Way	$1 - \lvert\cos\theta\rvert$

Viewed in this way, Muir's expansion is attempting to approximate $\cos x$ using $\sin x$. $\cos x$ is even, whereas $\sin x$ is odd, so the sine is squared to make it even. In the limit, the fit is perfect over the range $-90°$ to $+90°$. Outside of this range, $\sin^2 x$ can not hope to fit $\cos x$; but this is fine, because for a one-way equation we really wanted to fit $|\cos x|$ anyway!

The so-called 5°, 15°, 45°, etc..., equations are called such because in some sort of vague qualitative way this is the maximum angle of wave propagation they effectively handle. However, when they are dissected numerically they never seem to deserve their names. $F(\theta)$ is yet another parameter describing their behavior. It would seem that a polar plot would be appropriate:

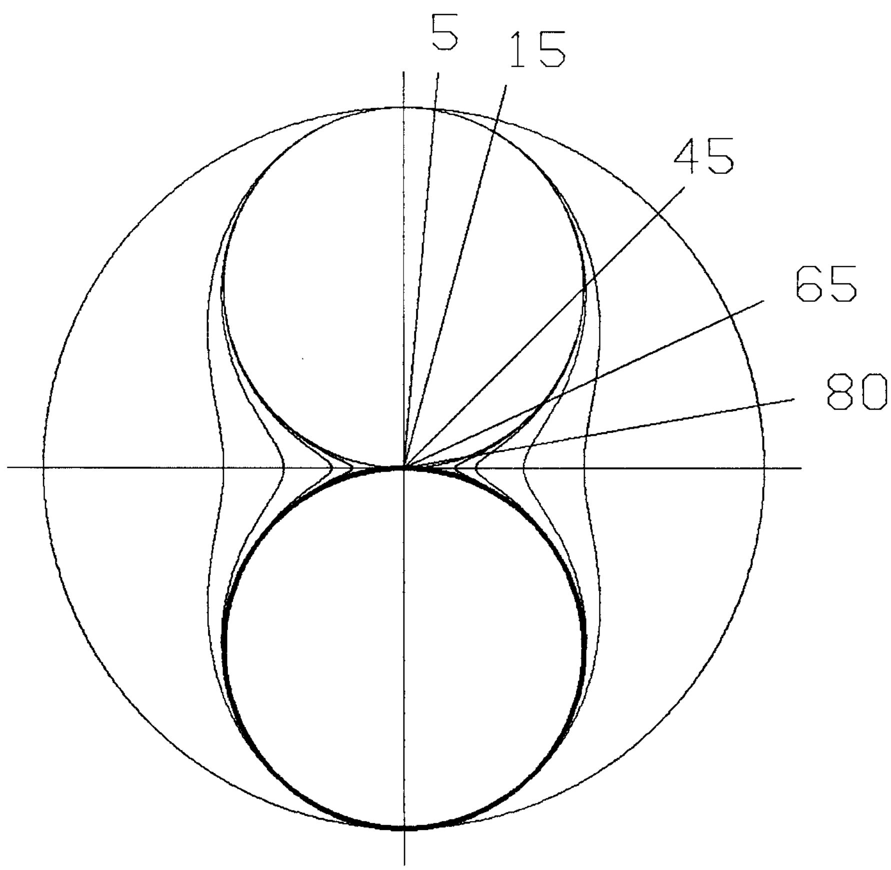

FIG. 1. A plot of $-F(\theta)$ for the two-way wave equation, the one-way wave equation, and Muir's 5°, 15°, 45°, 65°, and 80° approximations to the one-way equation. θ is zero pointing down.

Retarded coordinates produce a bizarre looking graph, so we have drawn the plot for regular coordinates. This is done by merely subtracting 1 from the values of $F(\theta)$ given in the table. Then, to make it positive, we have plotted $-F$.

The two-way wave equation is represented by the bold circle in the lower half of the graph. As θ sweeps through 360°, this circle is traversed twice. The one-way wave equation graphs as two circles, the lower bold one and its mirror image above it. Muir's various approximations give the various outer curves. The 5° equation plots as the outer circle. It is a very poor approximation to the one-way equation for any but very small angles. The 15° equation plots as a slightly pinched ellipse. Continuing inward, we see the graph for the 45°, 65°, and 80° equations. Lines with slopes of 5, 15, 45, 65, and 80 degrees have been plotted to show the angles to which we should expect the various approximate equations to still be a good approximation to the one-way equation. Surprisingly, except for the 15°, the various equations seem to deserve their names. Perhaps this means that errors in $F(\theta)$ are the determining factor as to whether an equation is considered "good" at an angle θ by a person's eye? This still wouldn't explain why the 15° equation is a special case, though. Perhaps the fact that it is an odd order approximation has something to do with this, since all of the others are even. This is supported by the fact that the next odd order approximation, the 50° equation, (not shown), also seems to be seriously underrated in a similar way.

If F is a natural parameter to approximate, it may make sense to try some other expansion of F. Since we are looking at circular functions, a Fourier series seems like a reasonable sort of thing to try. On the interval $-\pi$ to π, this would entail an expansion of $\cos \frac{\theta}{2}$ in terms of $\cos \theta$. This would then have to be converted into an expansion in $\sin^2 \theta$. This expansion, however, would not fit F well for $\theta = 0$, which should be where the fit is best. An additional constraint could be added to force this. Whereas a Taylor's expansion extrapolates a function based on local behavior near a point, such an expansion would approximate the function globally. This could be the subject of more research in the future.

Dip Filtering Using a Modified 45° Equation

In the 45° equation,

$$\frac{\partial Q}{\partial z} - \frac{k_x^2 v^2}{4\omega^2} \frac{\partial Q}{\partial z} = \frac{-ivk_x^2}{2\omega} Q,$$

substitute in for the ω on the right hand side of the equation ω_*, where ω_* will be allowed to be complex. Change variables to $Q = Ae^{i\phi}$, as before. Then we get

$$\frac{\partial A}{\partial z} + iA\frac{\partial \phi}{\partial z} - \frac{k_x^2 v^2}{4\omega^2} \frac{\partial A}{\partial z} - iA\frac{k_x^2 v^2}{4\omega^2} \frac{\partial \phi}{\partial z} = -iA\frac{vk_x^2}{2\omega_*}.$$

The real part of this gives

$$\frac{\partial A}{\partial z} = -\mathrm{Im}(\omega_*) A \frac{v k_x^2}{2\|\omega_*\|^2 (1 - \frac{k_x^2 v^2}{4\omega^2})}.$$

If $\|\omega_*\|^2 \approx \omega^2$, then this represents dip filtering.

The imaginary part gives (after factoring out an A from all terms)

$$(1 - \frac{k_x^2 v^2}{4\omega^2}) \frac{\partial \phi}{\partial z} = -\frac{v k_x^2 \mathrm{Re}(\omega_*)}{2\|\omega_*\|^2}.$$

For there to be no phase errors, we require that

$$\frac{\mathrm{Re}(\omega_*)}{\|\omega_*\|^2} = \frac{1}{\omega}.$$

$\mathrm{Im}(\omega_*)$ is fixed by how much attenuation you want. So we must pick $\mathrm{Re}(\omega_*)$ to suit. It is easily found that we want

$$\mathrm{Re}(\omega_*) = \frac{\omega + \sqrt{\omega^2 - 4\mathrm{Im}(\omega_*)^2}}{2}.$$

There is a limit to how large the imaginary part of ω_* can be for this equation to still have a solution. Unfortunately, to produce significant dip filtering requires the imaginary part of ω_* to be outside the acceptable range. If the problem is just ignored and $\mathrm{Re}(\omega_*)$ is left equal to ω, significant phase errors result. An example of this is shown in figure 2. On the left is the impulse response of the original 45° equation. On the right is the impulse response of the "dip filtering" 45° equation. The units are not important here. What is important is that the vertical and horizontal scales are the same and that in the right hand plot significant amounts of energy can be seen to have been neither attenuated away nor migrated properly.

The solution would seem to be to not do dip filtering in this way. A dip-filtering equation could be run alternately with the 45° equation, for example. One promising candidate is

$$\frac{\partial Q}{\partial z} = -\epsilon(\frac{k_x v}{\omega} - \sin\theta_0)^2 Q.$$

Writing this equation in amplitude-phase coordinates shows that this equation should select for waves propagating at an angle θ_0 to the vertical. Implementing this requires solving a pentadiagonal system of matrices, however, unless $\theta_0 = 0$. This is a significant increase in time over using the modified 45° equation alone.

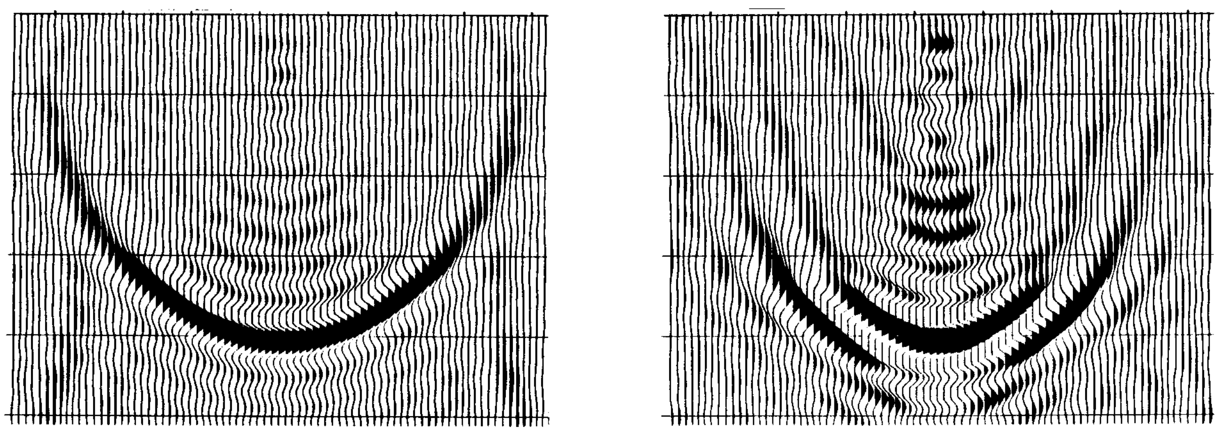

FIG. 2. Impulse responses for the 45° equation (left) and the modified 45° dip-filtering equation (right). The semicircular impulse response has been distorted into an ellipse, and there is also a great deal of "ringing" as well.

Conclusion

Amplitude - phase coordinates are a more natural way of writing wave equations, making their important properties show very clearly. They also are a good way to design new equations as well.

Acknowledgements

I wish to thank Yves Dezard for his encouragement, help, and patience, and especially for not smoking.

U.S. News survey ranks Stanford as top university

By a narrow margin, America's college presidents more often include Stanford than Harvard among the nation's best universities for undergraduate education, a new survey shows.

But Harvard continues to enroll a higher proportion of students offered admission as freshmen than does Stanford, which has ranked second nationally in this respect for several years among major private universities.

The survey was conducted by *U.S. News and World Report* magazine, drawing responses from 662 of 1,308 four-year college and university presidents. Each was asked to pick the five best undergraduate schools from a list of institutions most similar to their own in enrollment and range of programs.

Among national universities, Stanford was included in the top five by 48.8 percent of the respondents, with Harvard mentioned by 47.6 percent.

Yale followed with 37.8 percent, Princeton with 28.0 percent, and the University of California with 24.4 percent, the highest of any public institution.

Annual surveys by Stanford have shown that roughly one third of those offered admission but not enrolling as freshmen attend Harvard, Yale, Princeton, or MIT, which ranked 10th in the survey. Berkeley is the most popular choice among those admitted to Stanford who attend public universities.

All six of these institutions ranked very highly in a recent national comparison of graduate faculties by their peers across the country.

Berkeley, Stanford, and Harvard had the largest number of academic departments among the top 10 in their respective fields. MIT, Yale, and Princeton did well in the smaller number of disciplines in which they offer graduate programs.

Stanford makes no distinction between faculty who teach graduates and undergraduates. University President Donald Kennedy has frequently noted that strong research and teaching often are found in the same individual faculty members.

In 1957 a national survey by the University of Pennsylvania ranked Stanford 13th in overall quality of its graduate programs. By 1966, the University was among the top six, and in 1969 among the top three.

In its Nov. 28 issue, *U.S. News and World Report* noted that "David Riesman, emeritus professor of social science at Harvard, is among educators who were not surprised by the strong (undergraduate) showing of Stanford. . . .

"Calling the university 'a meteor in the business,' Riesman says a combination of excellent leadership, attractive climate, and the addition of many respected faculty members from Eastern colleges, including Harvard, have bolstered its reputation as a first-rate undergraduate school. 'It has risen spectacularly.' "

Several years ago, Riesman told a Stanford reporter that the University should stop trying so hard, that it "really wasn't the Avis of the West."

"That's easy to say when you're Hertz," the Stanford staffer replied.