Chapter III: Dip-Moveout and Prestack Migration

3.1. Prestack Migration Reviewed

An intuitive understanding of dip-moveout (DMO) by Fourier transform requires only the ray-theoretical analysis of chapter I. Reflection seismology, however, has progressed far beyond the geometrical analysis of raypaths. In particular, migration algorithms, though originally based on ray theory, are now firmly rooted in the scalar wave equation. While ray theory remains useful in understanding migration with respect to reflection times, wave theory is necessary to properly migrate reflection seismograms as a recorded wavefield, a wavefield that contains amplitude as well as timing (or phase) information.

The purpose of this chapter is to put the ray-theoretical DMO process derived in chapter I on a stronger, wave-theoretical foundation. A key assumption made toward this purpose is that prestack migration is the wave-theoretical process that should, ideally, be used to compute a subsurface image from reflection seismograms. Therefore, a review of this important process is given below. Because prestack migration has been well discussed by others, the following review is brief and is constrained to the purpose of deriving the DMO transformation. For more complete discussions, I particularly recommend those by Claerbout (1976,1983), Stolt (1978), and Yilmaz and Claerbout (1980).

The first step in migration is to extrapolate the results of a seismic experiment conducted on the earth's surface to obtain results for seismic experiments (not actually conducted) in the subsurface. The results of seismic experiments are, of course, seismograms, which will be denoted by a function $p(t,s,r,z)$ of recording time t, source location s, receiver location r, and experiment depth z. The seismic experiment actually performed is here assumed to be that in which sources and receivers are placed along a line on the earth's surface, $z = 0$. Therefore, the first step in migration, the extrapolation step, is to compute $p(t,s,r,z)$ from the recorded $p(t,s,r,z=0)$.

Given the extrapolated seismic data, the subsurface image is taken to be that portion of $p(t,s,r,z)$ for which time and source-receiver offset both equal zero. Recorded seismograms $p(t,s,r,z=0)$ are usually sorted into CMP gathers $p(t,h,y,z=0)$, where h denotes half-offset and y denotes source-receiver midpoint defined by

$$h \equiv \frac{r-s}{2} \quad , \quad y \equiv \frac{r+s}{2} .$$

The arguments of the function p, either (s,r) or (h,y), will be used to imply the appropriate interpretation of that function. For example, $p(t=0,s=y,r=y,z)$ is the subsurface image extracted from $p(t,s,r,z)$, the extrapolated data in source-receiver coordinates; and $p(t=0,h=0,y,z)$ is the same image extracted from $p(t,h,y,z)$, the extrapolated data in offset-midpoint coordinates.

Extrapolation of surface-recorded data is the key to imaging the subsurface. To compute $p(t,s,r,z)$ from $p(t,s,r,z=0)$, one must know how the seismic wavefield changes with depth z. Equivalently, one needs the partial derivative of p with respect to z; and this derivative is supplied by the scalar wave equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} ,$$

where x is a horizontal coordinate (such as s or r) along the survey line, and v is the wave velocity that, throughout this chapter, is assumed to be constant. By including only two spatial dimensions (x and z) in the wave equation, one also assumes that the seismic wavefield is constant with respect to the third dimension perpendicular to the (x,z)-plane. This latter assumption is presently made only for simplicity; and in the last section, the results of this chapter are generalized to three-dimensional seismic wavefields.

To use the wave equation in extrapolating $p(t,s,r,z=0)$, one must interpret the horizontal coordinate x in that equation as representing either the receiver location r or the source location s. Actually, both interpretations are used, one after the other. The details were given by Claerbout (1976) and Stolt (1978), and will not be repeated here; but one of the many results in Stolt's paper was that prestack migration may be performed in offset-midpoint coordinates using the following equations:

$$p(\omega,k_h,k_y,z=0) \;=\; \int dt\, e^{i\omega t} \int dh\, e^{-ik_h h} \int dy\, e^{-ik_y y}\, p(t,h,y,z=0) \;, \quad (3.1a)$$

$$p(\omega,k_h,k_y,z) \;=\; e^{ik_z(\omega,k_h,k_y)z}\, p(\omega,k_h,k_y,z=0) \;, \quad (3.1b)$$

where

$$k_z(\omega,k_h,k_y) \;\equiv\; -\frac{\omega}{v}\left\{\left[1 - \frac{v^2}{4\omega^2}(k_y + k_h)^2\right]^{1/2} + \left[1 - \frac{v^2}{4\omega^2}(k_y - k_h)^2\right]^{1/2}\right\} , \quad (3.1c)$$

and

$$p(t=0,h=0,y,z) \;=\; \frac{1}{(2\pi)^3}\int d\omega \int dk_h \int dk_y\, e^{ik_y y}\, p(\omega,k_h,k_y,z) \;. \quad (3.1d)$$

Equation (3.1a) denotes a three-dimensional Fourier transform, where the appropriate interpretation of the function p is again specified by its arguments. Equations (3.1b) and (3.1c) perform the required wavefield extrapolation in the frequency-wavenumber domain. And equation (3.1d) is an inverse three-dimensional Fourier transform, evaluated for all y but for only $t=0$ and $h=0$, thereby yielding the subsurface image $p(t=0,h=0,y,z)$. Together, these equations represent an algorithm for performing what is often called *migration before stack* or, equivalently, *prestack migration.* These terms are somewhat misleading because they imply that stacking (integrating over offset) will be performed after migration. In fact, no stacking is necessary or even conceivable after migration via equations (3.1). Nevertheless, for lack of a better term, *prestack migration* will hereafter denote the process defined by equations (3.1).

For several reasons, prestack migration is seldom used. In practice, a subsurface image is more often computed through the following processing sequence: (1) normal-moveout (NMO) correction, (2) CMP stacking, and (3) poststack migration. One of the most important reasons for subdividing the imaging process stems from the need, in practice, to estimate velocities. Initial (and often final) estimates of velocities are typically obtained by repeated application of the NMO and stacking processes for different velocities. Although one might imagine a similar repeated application of the prestack migration equations (3.1) for different velocities, the computational cost of this procedure is high and, in most cases, unwarranted. Furthermore, as noted by Yilmaz and Claerbout (1980, p. 1754), "an unmigrated CMP stack section helps the interpreter a great deal in resolving spurious events on a migrated section due to inaccurate velocities." Unlike the typical processing sequence, prestack migration in one step via equations (3.1) does not provide the CMP stack as an intermediate product.

Unfortunately, the practical and time-honored sequence of NMO, stack, and poststack migration, hereafter referred to as *conventional CMP processing*, is not equivalent to prestack migration. This fact was first noted by Judson et al (1978) and later discussed in detail by Yilmaz and Claerbout (1980). In the latter paper, the authors noted that conventional CMP processing yields an accurate subsurface image only if the source-receiver offset or the dip of reflectors is zero. Because neither offset nor dip is typically zero, conventional CMP processing is rarely accurate. One might also argue that prestack migration via equations (3.1) is hardly accurate, particularly with regard to the assumption that velocity is constant; but, in fact, equations (3.1) may be easily generalized to cope with depth-variable velocity, and finite-difference algorithms derived from these equations extend their usefulness to regions with lateral velocity variations (Claerbout, 1976, 1983; Schultz and Sherwood, 1980). In any case, none of the assumptions made in prestack migration are as restrictive as the conventional assumptions of zero offset or zero dip.

To obviate the choice between practical but inaccurate conventional CMP processing and accurate but impractical prestack migration, a process was developed that could be inserted into the conventional CMP processing sequence to make it better approximate prestack migration. This process has been variously called *DEVILISH* (Judson et al, 1978), *prestack partial migration* (Yilmaz and Claerbout, 1980), and *dip-moveout* (Bolondi et al, 1982). Although I will use the latter term (or its acronym, DMO), the reader should remember that DEVILISH, prestack partial migration, and DMO are different names for the same process. Algorithms for applying DMO may differ, however, just as algorithms for performing migration may differ.

The remainder of this chapter is devoted to the derivation of an algorithm for performing DMO by Fourier transform. Beginning with the assumption that equations (3.1) yield a theoretically correct subsurface image, the DMO algorithm is derived by showing that each step in the conventional CMP processing sequence is precisely represented in these equations. The process remaining after successively eliminating the NMO, stacking, and poststack migration processes from equations (3.1) must then represent the difference between conventional CMP processing and prestack migration. DMO is defined to be the leftover process. In other words, DMO added to conventional CMP processing yields *exactly* the same subsurface image as prestack migration via equations (3.1).

3.2. Prestack Migration Dissected

The first step in expressing prestack migration as a cascade of CMP processes will be to find the conventional poststack migration represented in equations (3.1). For simplicity, equations (3.1b) and (3.1d) should be combined, dropping the irrelevant 2π scaling factors and the inverse Fourier transform over k_y to obtain:

$$p(t=0,h=0,k_y,z) = \int d\omega \int dk_h \, e^{ik_z(\omega,k_h,k_y)z} \, p(\omega,k_h,k_y,z=0) \ , \qquad (3.2)$$

which, together with the definition of $k_z(\omega,k_h,k_y)$ in equation (3.1c), will be taken to represent prestack migration. As shown by a lengthy algebraic proof in Appendix 3.A, $k_z(\omega,k_h,k_y)$ may be equivalently written as

$$k_z(\omega_0,k_y) \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2k_y^2}{4\omega_0^2}\right]^{1/2} \ , \qquad (3.3a)$$

where ω_0 is defined implicitly by

$$\omega(\omega_0,k_h,k_y) \equiv \omega_0\left[1 + \frac{v^2k_h^2}{4\omega_0^2}\left[1 - \frac{v^2k_y^2}{4\omega_0^2}\right]^{-1}\right]^{1/2} \qquad (3.3b)$$

$$= \omega_0\left[1 + \frac{k_h^2}{k_z^2(\omega_0,k_y)}\right]^{1/2} .$$

That the two definitions of k_z are equivalent is hardly obvious, but the reader should at least verify the special case of $k_h = 0$.

Equations (3.3) may be used to change the integration over ω in the prestack migration equation (3.2) to an integration over ω_0, yielding

$$p(t=0,h=0,k_y,z) = \int dk_h \int d\omega_0 \left[\frac{d\omega}{d\omega_0}\right] e^{ik_z(\omega_0,k_y)z} \, p[\omega(\omega_0,k_h,k_y),k_h,k_y,z=0] \qquad (3.4)$$

$$= \int d\omega_0 \, e^{ik_z(\omega_0,k_y)z} \int dk_h \left[\frac{d\omega}{d\omega_0}\right] p[\omega(\omega_0,k_h,k_y),k_h,k_y,z=0]$$

$$= \int d\omega_0 \, e^{ik_z(\omega_0,k_y)z} \, p_s(\omega_0,k_y) \ ,$$

where the last equality follows from defining

$$p_s(\omega_0,k_y) \ \equiv \ \int dk_h \left| \frac{d\omega}{d\omega_0} \right| p[\omega(\omega_0,k_h,k_y),k_h,k_y,z=0] \ . \qquad (3.5)$$

The Jacobian of the change of variable is shown in Appendix 3.B to be

$$\frac{d\omega}{d\omega_0} \ = \ \frac{\omega_0}{\omega(\omega_0,k_h,k_y)} \left[1 - \frac{k_h^2 k_y^2}{k_z^4(\omega_0,k_y)} \right] \ . \qquad (3.6)$$

Readers familiar with frequency-wavenumber domain methods of migration should recognize equation (3.4), with the definition of $k_z(\omega_0,k_y)$ given by equation (3.3a), as representing poststack migration (e.g., Stolt, 1978; Gazdag, 1978). In conventional CMP processing, $p_s(\omega_0,k_y)$ would be the Fourier transform of a CMP stack $p_s(t_0,y)$. The subscript on ω_0 and t_0 seems appropriate because poststack migration is based on the assumption that the CMP stack well approximates a zero-offset section. Think of t_0 as zero-offset time, and of ω_0 as zero-offset frequency. Equation (3.4) yields the subsurface image, a function of k_y and z, by extracting (integrating over ω_0) the $t_0 = 0$ portion of an extrapolated version of $p_s(\omega_0,k_y)$.

The first step in dissecting prestack migration is completed. Prestack migration is just the familiar poststack migration of a function $p_s(t_0,y)$. But what is $p_s(t_0,y)$? Conventional CMP processing would assume that $p_s(t_0,y)$ is the stack, an integral over offset, of NMO-corrected seismograms; but the definition of $p_s(\omega_0,k_y)$ in equation (3.5) contains only an integration over offset-wavenumber k_h. Equation (3.5) may be easily rewritten, however, to include the conventional CMP stacking, by inverse Fourier transforming $p(\omega,k_h,k_y,z=0)$ over k_h to obtain

$$p_s(\omega_0,k_y) \ = \ \int dk_h \left| \frac{d\omega}{d\omega_0} \right| \int dh \, e^{-ik_h h} \, p[\omega(\omega_0,k_h,k_y),h,k_y,z=0] \qquad (3.7)$$

$$= \int dh \int dk_h \, e^{-ik_h h} \left| \frac{d\omega}{d\omega_0} \right| p[\omega(\omega_0,k_h,k_y),h,k_y,z=0]$$

$$= \int dh \, p_0(\omega_0,h,k_y) \ ,$$

where the last equality follows from the definition

$$p_0(\omega_0,h,k_y) \ \equiv \ \int dk_h \, e^{-ik_h h} \left| \frac{d\omega}{d\omega_0} \right| p[\omega(\omega_0,k_h,k_y),h,k_y,z=0] \ . \qquad (3.8)$$

Equations (3.7) and (3.8) complete the second step in dissecting prestack migration. Prestack migration is just the poststack migration of the CMP stack of a function $p_0(t_0,h,y)$. The next question is, of course, what is $p_0(t_0,h,y)$? In particular, how is $p_0(t_0,h,y)$ related to NMO-corrected seismograms? Once NMO has been identified, the remaining process must be the exact DMO transformation.

NMO-corrected seismograms are most easily computed from recorded seismograms in the time domain. Therefore, to find an NMO transformation in equation (3.8), rewrite that equation in terms of $p(t,h,k_y,z=0)$ as follows:

$$p_0(\omega_0,h,k_y) \ = \ \int dk_h \, e^{-ik_h h} \left| \frac{d\omega}{d\omega_0} \right| \int dt \, e^{i\omega(\omega_0,k_h,k_y)t} \, p(t,h,k_y,z=0) \qquad (3.9)$$

$$= \int dt \, p(t,h,k_y,z=0) \int dk_h \left| \frac{d\omega}{d\omega_0} \right| e^{-ik_h h \, + \, i\omega(\omega_0,k_h,k_y)t}$$

$$= \int dt \, p(t,h,k_y,z=0) \ I(t,\omega_0,h,k_y) \ ,$$

where the last equality follows from defining

$$I(t,\omega_0,h,k_y) \ \equiv \ \int dk_h \left| \frac{d\omega}{d\omega_0} \right| e^{-ik_h h \, + \, i\omega(\omega_0,k_h,k_y)t} \ . \qquad (3.10)$$

Unfortunately, the prestack process represented by equation (3.9) still does not resemble NMO correction. NMO is derived by the following change of variable from recording time t to NMO time t_n:

$$t(t_n,h) \equiv \left[t_n^2 + \frac{4h^2}{v^2} \right]^{1/2} , \tag{3.11}$$

which may be used to rewrite equation (3.9) as

$$p_0(\omega_0,h,k_y) = \int dt_n \left[\frac{dt}{dt_n} \right] p[t(t_n,h),h,k_y,z=0]\, I[t(t_n,h),\omega_0,h,k_y] . \tag{3.12}$$

But NMO-corrected seismograms may be defined by

$$p_n(t_n,h,y) \equiv \left[\frac{dt}{dt_n} \right] p[t(t_n,h),h,y,z=0] \tag{3.13}$$

$$= \left[1 + \frac{4h^2}{v^2 t_n^2} \right]^{-1/2} p(\sqrt{t_n^2 + 4h^2/v^2},h,y,z=0) .$$

The scaling by the Jacobian in equation (3.13), although sometimes omitted in NMO correction, ensures that data for large offsets and early times are not amplified by NMO stretch (Dunkin and Levin, 1973). Using equation (3.13), equation (3.12) becomes

$$p_0(\omega_0,h,k_y) = \int dt_n\, p_n(t_n,h,k_y)\, I(\sqrt{t_n^2 + 4h^2/v^2},\omega_0,h,k_y) . \tag{3.14}$$

The final step in dissecting prestack migration is now completed. Prestack migration is equivalent to poststack migration of the CMP stack p_s of a function p_0 which is related, through equation (3.14), to NMO-corrected seismograms p_n. Table 3.1 provides a summary of the complete processing sequence.

The definition of DMO in equations (3.10) and (3.14) (or Table 3.1) does not suggest a practical DMO algorithm unless the integral $I(t,\omega_0,h,k_y)$ is evaluated analytically. (One would certainly not want to compute this integral numerically for

Prestack Migration Dissected	
(1) NMO	$p_n(t_n,h,y) \;=\; \left[1 + \dfrac{4h^2}{v^2 t_n^2}\right]^{-1/2} p(\sqrt{t_n^2 + 4h^2/v^2},h,y,z=0)$
(2) DMO	$p_0(\omega_0,h,k_y) \;=\; \int dt_n\, p_n(t_n,h,k_y)\; I(\sqrt{t_n^2 + 4h^2/v^2},\omega_0,h,k_y)$
(3) Stack	$p_s(\omega_0,k_y) \;=\; \int dh\, p_0(\omega_0,h,k_y)$
(4) Migration	$p(t=0,h=0,k_y,z) \;=\; \int d\,\omega_0\, e^{ik_z(\omega_0,k_y)z}\, p_s(\omega_0,k_y)$
Definitions	$k_z(\omega_0,k_y) \;\equiv\; -\dfrac{2\omega_0}{v}\left[1 - \dfrac{v^2 k_y^2}{4\omega_0^2}\right]^{1/2}$ $\omega(\omega_0,k_h,k_y) \;\equiv\; \omega_0\left[1 + \dfrac{k_h^2}{k_z^2(\omega_0,k_y)}\right]^{1/2}$ $I(t,\omega_0,h,k_y) \;\equiv\; \int dk_h \left[\dfrac{d\,\omega}{d\,\omega_0}\right] e^{-ik_h h \,+\, i\omega(\omega_0,k_h,k_y)t}$

TABLE 3.1. Prestack migration rewritten to resemble conventional CMP processing. Prestack migration is usually thought of as a one-step transformation from data $p(t,h,y,z=0)$ to subsurface image $p(t=0,h=0,y,z)$. The functions p_n, p_0, and p_s represent intermediate outputs of an equivalent, but more conventional, four-step processing sequence. DMO, however, is typically omitted in conventional CMP processing; i.e., p_0 and p_n are typically assumed to represent the same function. Note that a Fourier transform over midpoint y is implied between (1) NMO and (2) DMO, and that an inverse Fourier transform over k_y would be performed after (4) migration.

each t, ω_0, h, and k_y!) An exact evaluation of the integral is given in the next section, along with an asymptotic (high-frequency) approximation that makes implementation of DMO via equation (3.14) quite practical.

3.3. The Dip-Moveout Transformation

The details in evaluating the integral $I(t,\omega_0,h,k_y)$ are provided in Appendix 3.C; the result is that

$$I(t,\omega_0,h,k_y) \;=\; \pm\frac{i\pi|k_z|}{B^2}\left[\left(1-\frac{4h^2}{v^2t^2}\right)H_0^{\binom{1}{2}}(|\omega_0|tB)\right. \tag{3.15a}$$

$$\left. +\;\frac{k_y^2\left(2-B^2\right)}{k_z^2|\omega_0|tB}H_1^{\binom{1}{2}}(|\omega_0|tB)\right] \;,\; \omega_0 \gtrless 0 \;,$$

where $H_0^{\binom{1}{2}}$ and $H_1^{\binom{1}{2}}$ denote zero and first-order Hankel functions of the first and second kinds, B is defined by

$$B \;=\; B(t,\omega_0,h,k_y) \;\equiv\; \left(1-\frac{k_z^2h^2}{\omega_0^2t^2}\right)^{1/2} \;, \tag{3.15b}$$

and k_z is defined as in the previous section by

$$k_z \;=\; k_z(\omega_0,k_y) \;\equiv\; -\frac{2\omega_0}{v}\left(1-\frac{v^2k_y^2}{4\omega_0^2}\right)^{1/2} \;. \tag{3.3a}$$

Equations (3.14) and (3.15) define the DMO process that, when applied after NMO correction but before CMP stacking and poststack migration, yields *exactly* the same subsurface image as prestack migration. In practice, one may prefer to use an asymptotic approximation to the kernel $I(t,\omega_0,h,k_y)$ defined in equations (3.15), if only because Hankel functions are not as readily available as complex exponentials in typical computing environments. Furthermore, an asymptotic approximation of equations (3.15) enables one to check these equations against the ray-theoretical DMO equations given in chapter I.

The asymptotic ($x \to \infty$) approximations of the Hankel functions are

$$H_\nu^{\binom{1}{2}}(x) \;\approx\; \left(\frac{2}{\pi x}\right)^{1/2}\left[e^{\pm i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)} \;+\; O\left(\frac{1}{x}\right)\right]$$

(see Abramowitz and Stegun, 1965). The high-frequency approximation to $I(t,\omega_0,h,k_y)$ may be obtained by letting $|\omega_0|t \to \infty$ while keeping h/t and k_y/k_z (and, hence, B) fixed. The ratio k_y/k_z equals the tangent of the emergence angle of waves recorded with zero source-receiver offset; and, for constant velocity, this angle equals the dip of a subsurface reflector. For any given emergence angle or dip, note that dropping the second term in I (the H_1 term) is consistent with keeping only the leading term in the approximation of H_0. Therefore, the high-frequency approximation of I is

$$I(t,\omega_0,h,k_y) \approx \left[\frac{2\pi k_z^2}{|\omega_0|tB^5}\right]^{1/2} \left[1 - \frac{4h^2}{v^2t^2}\right] e^{i\omega_0 tB + i\,sgn(\omega_0)\frac{\pi}{4}} .$$

Incidentally, this approximation can also be obtained by the method of stationary phase applied to the integral of equation (3.10).

Inserting the high-frequency approximation of I into the DMO transformation equation (3.14) yields

$$p_0(\omega_0,h,k_y) \approx \int dt_n\, p_n(t_n,h,k_y) \left[\frac{2\pi k_z^2}{|\omega_0|t_n A^5}\right]^{1/2} e^{i\omega_0 t_n A + i\,sgn(\omega_0)\frac{\pi}{4}} , \quad (3.16a)$$

where

$$A = A(t_n,\omega_0,h,k_y) \equiv \left[1 + \frac{k_y^2h^2}{\omega_0^2t_n^2}\right]^{1/2} . \quad (3.16b)$$

In chapter I, DMO was defined by

$$p_0(\omega_0,h,k_y) = \int dt_n\, p_n(t_n,h,k_y)\, A^{-1}\, e^{i\omega_0 t_n A} , \quad (3.17)$$

where A is defined as in equation (3.16b). [See equations (1.12).] The differences between DMO in this chapter and DMO in chapter I lie primarily in amplitude factors. Because the phases in equations (3.16a) and (3.17) are identical, except for a constant $\pi/4$ phase shift, a DMO algorithm based on either of them would have the same effect on reflection times. In other words, the ray-theoretical DMO process of chapter I is consistent, with regard to reflection times, with the wave-theoretical prestack migration process.

One may, of course, use the exact representation of $I(t,\omega_0,h,k_y)$ given by equations (3.15) in the DMO transformation. However, for practical applications, one can easily show that the error in using only the leading term in the asymptotic approximation is negligible for frequency-time products greater than about ten cycles. And in analyzing the approximation error in detail, one should remember that worse approximations are implied by the assumptions made in deriving the "exact" equations (3.15). Two assumptions that are particularly suspect are (1) that the wave propagation velocity is constant, and (2) that the seismic wavefield is two-dimensional. With regard to the first assumption, I have not derived an exact wave-theoretical DMO transformation for variable velocity, primarily because the change of variable employed in dissecting prestack migration is useful only for constant velocity. Approximate corrections to the DMO equations for depth-variable velocity were discussed in chapter II. With regard to the second assumption, a DMO transformation for three-dimensional wavefields is derived in the following section.

3.4. Dip-Moveout in Three Dimensions

In three-dimensional (3-D) seismic surveys, sources and receivers are not constrained to lie along a single line, but may be located anywhere in a plane on the earth's surface. The recorded seismic wavefield should then be denoted by the function $p(t,\mathbf{s},\mathbf{r},z=0)$, where **s** and **r** denote source and receiver coordinate vectors. Specifically, **s** and **r** may be defined by

$$\mathbf{s} \equiv s_1\hat{\mathbf{e}}_1 + s_2\hat{\mathbf{e}}_2 \quad , \quad \mathbf{r} \equiv r_1\hat{\mathbf{e}}_1 + r_2\hat{\mathbf{e}}_2 \ ,$$

where $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are orthogonal unit vectors in the plane of the survey. For convenience, one may assume that $\hat{\mathbf{e}}_1$ points in the "inline" direction and that $\hat{\mathbf{e}}_2$ points in the "crossline" direction.

The recorded seismic wavefield may alternatively be denoted by $p(t,\mathbf{h},\mathbf{y},z=0)$, where **h** and **y** denote half-offset and source-receiver midpoint vectors defined by

$$\mathbf{h} \equiv \frac{\mathbf{r}-\mathbf{s}}{2} \quad , \quad \mathbf{y} \equiv \frac{\mathbf{r}+\mathbf{s}}{2} .$$

Yilmaz (1979) showed that the desired subsurface image $p(t=0,\mathbf{h}=0,\mathbf{y},z)$ may be obtained through the following 3-D generalization of the two-dimensional (2-D) prestack migration equations:

$$p(\omega,\mathbf{k}_h,\mathbf{k}_y,z=0) = \int dt\, e^{i\omega t} \int d^2h\, e^{-i\mathbf{k}_h\cdot\mathbf{h}} \int d^2y\, e^{-i\mathbf{k}_y\cdot\mathbf{y}} p(t,\mathbf{h},\mathbf{y},z=0) \quad ,(3.18a)$$

$$p(\omega,\mathbf{k}_h,\mathbf{k}_y,z) = e^{ik_z(\omega,\mathbf{k}_h,\mathbf{k}_y)z}\, p(\omega,\mathbf{k}_h,\mathbf{k}_y,z=0) \quad , \qquad (3.18b)$$

where

$$k_z(\omega,\mathbf{k}_h,\mathbf{k}_y) \equiv -\frac{\omega}{v}\left\{\left[1-\frac{v^2}{4\omega^2}|\mathbf{k}_y+\mathbf{k}_h|^2\right]^{1/2}+\left[1-\frac{v^2}{4\omega^2}|\mathbf{k}_y-\mathbf{k}_h|^2\right]^{1/2}\right\} \quad (3.18c)$$

and

$$p(t=0,\mathbf{h}=0,\mathbf{y},z) = \frac{1}{(2\pi)^5}\int d\omega \int d^2k_h \int d^2k_y\, e^{i\mathbf{k}_y\cdot\mathbf{y}} p(\omega,\mathbf{k}_h,\mathbf{k}_y,z) \quad . \qquad (3.18d)$$

Equations (3.18) are quite similar to equations (3.1), but with line integrals over h and y replaced by surface integrals. Equation (3.18a) is a five-dimensional Fourier transform. Equation (3.18b) extrapolates the Fourier transformed wavefield to nonzero depths z. In the definition of $k_z(\omega_0,\mathbf{k}_y)$ in equation (3.18c), $|\mathbf{k}_y + \mathbf{k}_h|$ denotes the length of the sum of the vectors $\mathbf{k}_y$ and $\mathbf{k}_h$; and $|\mathbf{k}_y - \mathbf{k}_h|$ denotes the length of their difference. Equation (3.18d) is an inverse Fourier transform, but evaluated only for $t = 0$ and $\mathbf{h} = 0$ to yield the subsurface image $p(t=0,\mathbf{h}=0,\mathbf{y},z)$.

In section 3.2 of this chapter, 2-D prestack migration was shown to be equivalent to a cascade of four CMP processes. Three of these processes were the conventional NMO, stack, and poststack migration processes; and the fourth was the typically neglected DMO process. Although the algebra is slightly more complicated, the steps used to dissect 2-D prestack migration may also be used to dissect 3-D prestack migration. The result is summarized in Table 3.2, which may be viewed as a 3-D generalization of Table 3.1.

Comparing Tables 3.1 and 3.2, the most significant difference is in the definition of $\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)$ given by

$$k_z(\omega_0,\mathbf{k}_y) \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{1/2} \tag{3.19a}$$

and

$$\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y) \equiv \omega_0\left\{1 + \frac{v^2}{4\omega_0^2}\left[k_h^2 + \frac{(\mathbf{k}_y\cdot\mathbf{k}_h)^2}{k_z^2(\omega_0,\mathbf{k}_y)}\right]\right\}^{1/2} . \tag{3.19b}$$

These equations follow from a 3-D generalization of the algebra in Appendix 3.A, with the result that equations (3.18c) and (3.19) are equivalent representations of k_z. k_y^2 and k_h^2 in equations (3.19) denote the squared lengths of the vectors $\mathbf{k}_y$ and $\mathbf{k}_h$. Similarly, h^2 in Table 3.2 denotes the squared length of the half-offset vector $\mathbf{h}$. The dissection of 3-D prestack migration then proceeds exactly as in the 2-D case, but with line integrals over h, y, k_h, and k_y replaced by surface integrals.

Analogous to the 2-D case, practical implementation of the DMO process requires the analytical evaluation of the kernel

$$I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \int d^2k_h \left(\frac{d\omega}{d\omega_0}\right) e^{-i\mathbf{k}_h\cdot\mathbf{h} + i\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)t} .$$

3-D Prestack Migration Dissected	
(1) NMO	$p_n(t_n,\mathbf{h},\mathbf{y}) = \left[1 + \frac{4h^2}{v^2 t_n^2}\right]^{-1/2} p(\sqrt{t_n^2 + 4h^2/v^2},\mathbf{h},\mathbf{y},z=0)$
(2) DMO	$p_0(\omega_0,\mathbf{h},\mathbf{k}_y) = \int dt_n\, p_n(t_n,\mathbf{h},\mathbf{k}_y)\, I(\sqrt{t_n^2 + 4h^2/v^2},\omega_0,\mathbf{h},\mathbf{k}_y)$
(3) Stack	$p_s(\omega_0,\mathbf{k}_y) = \int d^2h\, p_0(\omega_0,\mathbf{h},\mathbf{k}_y)$
(4) Migration	$p(t=0,\mathbf{h}=0,\mathbf{k}_y,z) = \int d\omega_0\, e^{ik_z(\omega_0,\mathbf{k}_y)z}\, p_s(\omega_0,\mathbf{k}_y)$
Definitions	$k_z(\omega_0,\mathbf{k}_y) \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{1/2}$ $\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y) \equiv \omega_0\left\{1 + \frac{v^2}{4\omega_0^2}\left[k_h^2 + \frac{(\mathbf{k}_y\cdot\mathbf{k}_h)^2}{k_z^2(\omega_0,\mathbf{k}_y)}\right]\right\}^{1/2}$ $I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \int d^2k_h \left[\frac{d\omega}{d\omega_0}\right] e^{-i\mathbf{k}_h\cdot\mathbf{h} + i\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)t}$

TABLE 3.2. The 3-D generalization of Table 3.1 in section 3.2. 3-D prestack migration is usually thought of as a one-step transformation from recorded data $p(t,\mathbf{h},\mathbf{y},z=0)$ to subsurface image $p(t=0,\mathbf{h}=0,\mathbf{y},z)$. The functions p_n, p_0, and p_s represent intermediate outputs of an equivalent, but more conventional, four-step processing sequence. DMO, however, is typically omitted in conventional CMP processing; i.e., p_0 and p_n are typically assumed to represent the same function. Note that a Fourier transform over midpoint $\mathbf{y}$ is implied between (1) NMO and (2) DMO, and that an inverse Fourier transform over $\mathbf{k}_y$ would be performed after (4) migration.

Although an analytical evaluation of this two-dimensional integral should be possible, I have been unable to obtain it. I have, however, obtained an asymptotic approximation via the method of stationary phase. Leaving the details to Appendix 3.D, the stationary phase approximation of $I(t,\omega_0,\mathbf{h},\mathbf{k}_y)$ for large $|\omega_0|t$ is

$$I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \approx \frac{4\pi|k_z|}{vtB^3}\left[1 - \frac{4h^2}{v^2t^2}\right] e^{i\omega_0 tB + i\,sgn(\omega_0)\frac{\pi}{2}} ,$$

where

$$B = B(t,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \left[1 - \frac{4h^2}{v^2t^2} + \frac{(\mathbf{k}_y\cdot\mathbf{h})^2}{\omega_0^2t^2}\right]^{1/2} .$$

Substituting this approximation into the DMO equation of Table 3.2 yields

$$p_0(\omega_0,\mathbf{h},\mathbf{k}_y) \approx \int dt_n\, p_n(t_n,\mathbf{h},\mathbf{k}_y)\, \frac{4\pi|k_z|}{vt_nA^3}\, e^{i\omega_0 t_n A + i\, sgn(\omega_0)\frac{\pi}{2}} , \qquad (3.20a)$$

where

$$A = A(t_n,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \left[1 + \frac{(\mathbf{k}_y\cdot\mathbf{h})^2}{\omega_0^2t_n^2}\right]^{1/2} . \qquad (3.20b)$$

With regard to reflection times, the only significant difference between the 2-D DMO equations (3.16) and the 3-D DMO equations (3.20) is that the product $k_y h$ in the 2-D equations is generalized by the dot product $\mathbf{k}_y\cdot\mathbf{h}$ in the 3-D equations. For some 3-D seismic surveys, this difference may be ignored. For example, 3-D surveys are often performed by conducting a series of parallel 2-D surveys. In each of these 2-D surveys the half-offset vector $\mathbf{h}$ typically has a zero (or near-zero) crossline component – typically, $\mathbf{h} = h_1\hat{\mathbf{e}}_1$. The dot product in equation (3.20b) then becomes $\mathbf{k}_y\cdot\mathbf{h} = k_{y_1}h_1$, which implies that DMO may be applied to each of the 2-D lines independently. In other words, the DMO process does not depend on k_{y_2}, the cross-line component of midpoint wavenumber, when source-receiver offsets are inline. For such 3-D surveys, variation of the seismic wavefield in the crossline direction need be considered only in the poststack migration process.

3.5. Summary

Prestack migration for constant velocity may be exactly represented as a cascade of four CMP processes: NMO, DMO, stack, and poststack migration. This convenient representation of prestack migration is summarized by Table 3.1 for two-dimensional seismic wavefields, and by Table 3.2 for three-dimensional wavefields. Ideally, the 3-D equations should be used, because recorded seismic wavefields are never constant in any direction.

DMO is defined to be that process remaining after each of the more conventional CMP processes is extracted from the prestack migration equations. Equations (3.14) and (3.15), which were derived by extracting NMO, stack, and poststack migration from the prestack migration equations (3.1), represent an exact 2-D DMO transformation. The asymptotic approximation of this transformation is consistent with the ray-theoretical DMO equations of chapter I.

An exact 3-D DMO transformation, like that obtained for two dimensions, was not derived, due to difficulty in evaluating the 3-D DMO kernel $I(t,\omega_0,\mathbf{h},\mathbf{k}_y)$ in Table 3.2. However, an asymptotic approximation of this kernel was obtained via the method of stationary phase.

Appendix 3.A

The purpose of this appendix is to show that $k_z(\omega,k_h,k_y)$ defined by

$$k_z(\omega,k_h,k_y) \equiv -\frac{\omega}{v}\left\{\left[1 - \frac{v^2}{4\omega^2}(k_y + k_h)^2\right]^{1/2} + \left[1 - \frac{v^2}{4\omega^2}(k_y - k_h)^2\right]^{1/2}\right\} \qquad (3.1c)$$

is equivalent to $k_z(\omega_0,k_y)$ defined by

$$k_z(\omega_0,k_y) \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{1/2} , \qquad (3.3a)$$

where ω_0 is defined implicitly by

$$\omega(\omega_0,k_h,k_y) \equiv \omega_0\left[1 + \frac{k_h^2}{k_z^2(\omega_0,k_y)}\right]^{1/2} . \tag{3.3b}$$

First, to simplify notation, define

$$Z \equiv \frac{vk_z}{2\omega} \quad , \quad Y \equiv \frac{vk_y}{2\omega} \quad , \quad H \equiv \frac{vk_h}{2\omega} .$$

Then square both sides of equation (3.1c) to obtain

$$4Z^2 = 1 - (Y+H)^2 + 1 + (Y-H)^2 + 2\sqrt{1-(Y+H)^2}\sqrt{1-(Y-H)^2}$$

$$= 2(1 - Y^2 - H^2) + 2\sqrt{1-(Y+H)^2}\sqrt{1-(Y-H)^2} .$$

Isolate the square roots and square both sides again:

$$(1 - Y^2 - H^2 - 2Z^2)^2 = [1-(Y+H)^2][1-(Y-H)^2] .$$

Expand both sides to find a common $(1 - Y^2 - H^2)^2$ term:

$$(1 - Y^2 - H^2)^2 - 4Z^2(1 - Y^2 - H^2) + 4Z^4$$

$$= [1 - Y^2 - H^2 - 2HY][1 - Y^2 - H^2 + 2HY]$$

$$= (1 - Y^2 - H^2)^2 - 4H^2Y^2 .$$

Canceling the common term yields

$$Z^2(Z^2 + Y^2 + H^2 - 1) = -H^2Y^2$$

$$Z^2 = 1 - Y^2 - H^2 - \frac{H^2Y^2}{Z^2} .$$

Using the definitions of Z, Y, and H,

$$k_z^2 = \frac{4\omega^2}{v^2} - k_y^2 - k_h^2 - \frac{k_h^2 k_y^2}{k_z^2} \; ;$$

and, using equation (3.3a),

$$\frac{4\omega_0^2}{v^2} = \frac{4\omega^2}{v^2} - k_h^2 - \frac{k_h^2}{k_z^2}\left[\frac{4\omega_0^2}{v^2} - k_z^2\right]$$

$$= \frac{4\omega^2}{v^2} - \frac{4\omega_0^2 k_h^2}{v^2 k_z^2} \, ,$$

from which follows

$$\omega^2 = \omega_0^2\left[1 + \frac{k_h^2}{k_z^2}\right] \, ,$$

which is just the square of equation (3.3b). Therefore, the two definitions of k_z in equations (3.1c) and (3.3) are equivalent.

Appendix 3.B

The purpose of this appendix is to verify

$$\frac{d\omega}{d\omega_0} = \frac{\omega_0}{\omega(\omega_0,k_h,k_y)}\left[1 - \frac{k_h^2 k_y^2}{k_z^4(\omega_0,k_y)}\right] \, , \tag{3.6}$$

given the definitions

$$k_z(\omega_0,k_y) \equiv -\frac{2\omega_0}{v}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{1/2} \tag{3.3a}$$

$$\omega(\omega_0,k_h,k_y) \equiv \omega_0\left[1 + \frac{v^2 k_h^2}{4\omega_0^2}\left[1 - \frac{v^2 k_y^2}{4\omega_0^2}\right]^{-1}\right]^{1/2} . \tag{3.3b}$$

Square both sides of equation (3.3b) and differentiate to obtain

$$2\omega d\omega = \left[2\omega_0 - \frac{v^2k_h^2}{4}\left(\frac{v^2k_y^2}{2\omega_0^3}\right)\left(1 - \frac{v^2k_y^2}{4\omega_0^2}\right)^{-2}\right]d\omega_0 \ .$$

Then solve for $d\omega/ d\omega_0$ using equation (3.3a).

$$\frac{d\omega}{d\omega_0} = \frac{\omega_0}{\omega} - \frac{\omega_0}{\omega}\frac{v^4k_h^2k_y^2}{16\omega_0^4}\left(1 - \frac{v^2k_y^2}{4\omega_0^2}\right)^{-2}$$

$$= \frac{\omega_0}{\omega}\left(1 - \frac{k_h^2k_y^2}{k_z^4}\right) ,$$

which is equation (3.6).

Appendix 3.C

The purpose of this appendix is to evaluate the following integral

$$I(t,\omega_0,h,ky) \equiv \int_{-\infty}^{\infty} dk_h \left(\frac{d\omega}{d\omega_0}\right) e^{-ik_hh + i\omega(\omega_0,k_h,k_y)t} , \qquad (3.10)$$

where $\omega(\omega_0,k_h,k_y)$ and $d\omega/ d\omega_0$ are defined by equations (3.3) and (3.6) (see Appendix 3.B). Using these definitions, note that the desired integral is the sum of two integrals,

$$I = I_1 + I_2 \ ,$$

where

$$I_1 = \int_{-\infty}^{\infty} dk_h \left(1 + \frac{k_h^2}{k_z^2}\right)^{-1/2} e^{-ik_hh + i\omega_0t\left(1 + \frac{k_h^2}{k_z^2}\right)^{1/2}}$$

and

$$I_2 = \frac{k_y^2}{k_z^4}\frac{\partial^2 I_1}{\partial h^2} \ .$$

Therefore, the first step in evaluating I is to evaluate I_1. Change the integration variable in I_1 from k_h to α using

$$\tanh\beta = \frac{k_z h}{\omega_0 t} \quad , \quad k_h = k_z \sinh(\alpha + \beta)$$

to obtain

$$I_1 = |k_z| \int_{-\infty}^{\infty} d\alpha \, e^{i\omega_0 t [\cosh(\alpha + \beta) - \sinh(\alpha + \beta)\tanh\beta]} .$$

Then use the following identities

$$\cosh(\alpha + \beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta \ ,$$

$$\sinh(\alpha + \beta) = \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta \ ,$$

$$1 - \tanh^2\beta = \frac{1}{\cosh^2\beta} \ ,$$

to obtain

$$I_1 = |k_z| \int_{-\infty}^{\infty} d\alpha \, e^{i\omega_0 t \left[1 - \frac{k_z^2 h^2}{\omega_0^2 t^2}\right]^{1/2} \cosh\alpha} .$$

An integral representation for zero-order Hankel functions of the first and second kinds is

$$H_0^{\binom{1}{2}}(x) = \pm \frac{1}{i\pi} \int_{-\infty}^{\infty} d\alpha \, e^{\pm i x \cosh\alpha} \quad , \ x > 0$$

(e.g., Carrier et al, 1966). The first kind may be used for positive frequencies, and the second kind for negative frequencies, to express I_1 as

$$I_1 = \pm i\pi |k_z| H_0^{\binom{1}{2}}\left[|\omega_0| t \left[1 - \frac{k_z^2 h^2}{\omega_0^2 t^2}\right]^{1/2} \right] \quad , \ \omega_0 \gtrless 0 \ .$$

Using the relations

$$\frac{dH_0(x)}{dx} = -H_1(x) \quad , \quad \frac{dH_1(x)}{dx} = H_0(x) - \frac{1}{x}H_1(x)$$

(see Abramowitz and Stegun, 1965), the second integral I_2 may be easily computed from I_1; and the desired sum $I = I_1 + I_2$ is given by equations (3.15) in section 3.3.

Appendix 3.D

The method of stationary phase for multidimensional integrals is thoroughly discussed by Bleistein and Handelsman (1975). This appendix outlines the application of this method in approximating the 3-D DMO kernel:

$$I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \int d^2k_h \left[\frac{d\omega}{d\omega_0}\right] e^{-i\mathbf{k}_h\cdot\mathbf{h} + i\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)t} \quad , \qquad (3.D.1)$$

where $\omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)$ is defined by equations (3.19). The details of the approximation are tedious and will be omitted, but the following outline should assist the energetic reader in verifying that

$$I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \approx \frac{4\pi|k_z|}{vtB^3}\left[1 - \frac{4h^2}{v^2t^2}\right] e^{i\omega_0 tB + i\,sgn(\omega_0)\frac{\pi}{2}} \quad , \qquad (3.D.2a)$$

where

$$B = B(t,\omega_0,\mathbf{h},\mathbf{k}_y) \equiv \left[1 - \frac{4h^2}{v^2t^2} + \frac{(\mathbf{k}_y\cdot\mathbf{h})^2}{\omega_0^2t^2}\right]^{1/2} \quad , \qquad (3.D.2b)$$

is the desired approximation.

Let $\Phi \equiv \omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y)t - \mathbf{k}_h\cdot\mathbf{h}$ denote the phase of the integrand in equation

(3.D.1), and define the curvature matrix **C** by

$$\mathbf{C} = \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial k_{h_1}{}^2} & \dfrac{\partial^2\Phi}{\partial k_{h_1}\partial k_{h_2}} \\ \dfrac{\partial^2\Phi}{\partial k_{h_1}\partial k_{h_2}} & \dfrac{\partial^2\Phi}{\partial k_{h_2}{}^2} \end{bmatrix} .$$

Let $det(\mathbf{C})$ denote the determinant of this matrix, and let $sig(\mathbf{C})$ denote its signature, the number of positive eigenvalues minus the number of negative eigenvalues. The stationary phase approximation is then given by

$$I(t,\omega_0,\mathbf{h},\mathbf{k}_y) \approx \frac{2\pi}{|\,det(\mathbf{C})\,|^{1/2}} \left(\frac{d\,\omega}{d\,\omega_0} \right) e^{i\Phi + i\,sig(\mathbf{C})\frac{\pi}{4}} , \tag{3.D.3}$$

where the right-hand side is evaluated at that $\mathbf{k}_h$ for which

$$\nabla\Phi = 0 . \tag{3.D.4}$$

In other words, to find the stationary phase approximation, one first finds the stationary point satisfying equation (3.D.4), and then evaluates each factor in the right-hand side of equation (3.D.3) at that point.

Using equations (3.19) and equation (3.D.4), one may (after considerable effort) show that the phase Φ is stationary when

$$\mathbf{k}_h = \omega(\omega_0,\mathbf{k}_h,\mathbf{k}_y) \left[\frac{4}{v^2 t}\mathbf{h} - \frac{(\mathbf{k}_y \cdot \mathbf{h})^2}{\omega_0^2 t}\mathbf{k}_y \right] .$$

Using this result to eliminate $\mathbf{k}_h$ in equation (3.19b), one may show that ω at the stationary point is

$$\omega = \omega_0 \left[1 - \frac{4h^2}{v^2 t^2} + \frac{(\mathbf{k}_y \cdot \mathbf{h})^2}{\omega_0^2 t^2} \right]^{-1/2} .$$

The phase Φ at the stationary point may then be shown to be

$$\Phi = \omega_0 t \left[1 - \frac{4h^2}{v^2 t^2} + \frac{(\mathbf{k}_y \cdot \mathbf{h})^2}{\omega_0^2 t^2} \right]^{1/2} . \tag{3.D.5}$$

After some lengthy algebra, one may verify that the determinant of the curvature matrix **C**, evaluated at the stationary point, is

$$det(\mathbf{C}) = \frac{v^2 t^2}{4k_z^2} \left[1 - \frac{4h^2}{v^2 t^2} + \frac{(\mathbf{k}_y \cdot \mathbf{h})^2}{\omega_0^2 t^2} \right]^2 , \tag{3.D.6}$$

and that the signature is

$$sig(\mathbf{C}) = 2\, sgn(\omega_0) . \tag{3.D.7}$$

The Jacobian in equation (3.D.1) may be found as in Appendix 3.B:

$$\frac{d\omega}{d\omega_0} = \frac{\omega_0}{\omega} \left[1 - \frac{(\mathbf{k}_y \cdot \mathbf{k}_h)^2}{k_z^4(\omega_0, \mathbf{k}_y)} \right] .$$

Evaluated at the stationary point, this is

$$\frac{d\omega}{d\omega_0} = \left(1 - \frac{4h^2}{v^2 t^2} \right) \left[1 - \frac{4h^2}{v^2 t^2} + \frac{(\mathbf{k}_y \cdot \mathbf{h})^2}{\omega_0^2 t^2} \right]^{-1/2} . \tag{3.D.8}$$

Substituting equations (3.D.5) to (3.D.8) into equation (3.D.3) yields the stationary phase approximation in equations (3.D.2).