Synthetic Seismograms in Viscoelastic Media: I. Theory

Thierry Bourbie
Alfonso González-Serrano

This article is a continuation of a previous paper, (Effect of Reflection Coefficients on Synthetic Seismograms, SEP-26, pp205-219), where we discussed the computation of synthetic seismograms in an elastic medium, using an f-k domain wavefield extrapolation method. In this paper we consider the case of waves propagating in a linear viscoelastic medium. Our goal is to model a marine seismogram and to study both the effect of P to S conversions and the effect of attenuation contrasts.

1. Plane waves in a linear viscoelastic medium.

For the marine seismology case, the hypothesis of representing the medium by a linear viscoelastic model is completely justified. First, the phenomena we are studying are low amplitude in the far field (1 or 2 wavelengths from the source). Under this condition linearity of the stress-strain relationship becomes a good approximation. Second, at the strain amplitudes concerned by seismic exploration ($<10^{-6}$), attenuation phenomena are not friction phenomena but fluid phenomena (Winkler *et al*, 1979), which can be modeled by viscoelasticity theory.

In linear viscoelasticity the reciprocity principle is applicable (Borcherdt, 1977, Bourbie, 1981b). This allows us to write the elastodynamic equation for linear viscoelastic media by replacing, in the usual elastodynamic equation, the real elastic moduli by complex frequency-dependent ones. This gives

$$\left[K(\omega) + \frac{\mu(\omega)}{3} \right] \nabla\vartheta + \mu(\omega)\, \nabla^2 \mathbf{u} \;=\; \rho \ddot{\mathbf{u}} \tag{1}$$

where $K(\omega)$ is the frequency-dependent complex bulk modulus, $\mu(\omega)$ is the frequency dependent shear modulus, $\vartheta = \nabla \cdot u$ is the volumetric dilation, and $\mathbf{u} = \mathbf{u}(x,y,z,t)$ is the displacement vector.

Taking for the displacement $\mathbf{u}$ time-dependence of the form $e^{i\omega t}$, equation (1) can be rewritten

$$\left[K(\omega) + \frac{\mu(\omega)}{3} \right] \nabla\vartheta + \mu(\omega)\, \nabla^2 \mathbf{u} \;=\; -\rho\omega^2 \mathbf{u} \qquad (1b)$$

In this equation $\mathbf{u} = \mathbf{u}(x,y,z)$.

This equation can be transformed using the Helmholtz form for the displacement $\mathbf{u}$ as function of potentials Φ and Ψ

$$\mathbf{u} \;=\; \nabla\Phi + \nabla\times\Psi \qquad (2)$$

$$\nabla\cdot\Psi \;=\; 0$$

then (1b) separates

$$\nabla^2 \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} + \begin{bmatrix} k_P^2\Phi \\ k_S^2\Psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3)$$

where

$$k_P^2 \;=\; \frac{\rho\omega^2}{K(\omega) + \frac{4}{3}\mu(\omega)}$$

$$k_S^2 \;=\; \frac{\rho\omega^2}{\mu(\omega)}$$

that are complex-valued quantities.

The general plane wave solution of equations of type (3) is

$$\Phi \;=\; \Phi_0\, e^{i(\omega t - \mathbf{K}\cdot\mathbf{r})} \qquad (4)$$

where $\mathbf{K}$ is a complex vector, and $\mathbf{r}$ is the position vector.

We can write $\mathbf{K}$ separating its real and imaginary parts:

$$\mathbf{K} \;=\; \mathbf{P} - i\mathbf{A} \qquad (5)$$

$\mathbf{P}$ will be called the propagating vector. $\mathbf{A}$ will be called the attenuating vector.

Rewriting equation (4) using these definitions we get

$$\Phi \;=\; \Phi_0\, e^{-\mathbf{A}\cdot\mathbf{r}}\, e^{i(\omega t - \mathbf{P}\mathbf{r})} \qquad (6)$$

In general, vectors **P** and **A** are not parallel, and the wave is said to be *inhomogeneous*. For the other case, where the angle between **P** and **A** is zero, $\gamma = (\mathbf{P},\mathbf{A}) = 0$, the wave is said to be *homogeneous*.

From (3), (4) and (5) we also get

$$\mathbf{K}\cdot\mathbf{K} \;=\; |\mathbf{P}|^2 - |\mathbf{A}|^2 + 2i\,|\mathbf{A}|\cdot|\mathbf{P}|\,\cos\gamma \;=\; \frac{\rho\omega^2}{M} = \frac{\rho\omega^2}{|M|^2}[\,M_R - i\,M_I\,] \qquad (7)$$

with

$M = M(\omega)$	complex modulus of the wave under consideration:
	$M = K + \frac{4}{3}\mu$ for a P wave
	$M = \mu$ for an S wave
M_R	Real part of M
M_I	Imaginary part of M
$\lvert M\rvert$	Modulus of M

In $2D$ (6) gives

$$\Phi \;=\; \Phi_0\, e^{-(A_x x + A_z z)}\, e^{i\,[\,\omega t\, -\,(P_x x\, +\, P_z z)\,]}$$

so (7) can be written as

$$(P_x - iA_x)^2 + (P_z - iA_z)^2 \;=\; \frac{\rho\omega^2}{M} \qquad (8a)$$

or

$$k^2 + k_z^2 \;=\; \frac{\rho\omega^2}{M} \qquad (8b)$$

2. Green's function and Wavefield Extrapolators in linear viscoelastic medium

Since the reciprocity principle can be applied in the situation we are interested, we are going to use it to derive the operators we need to extrapolate the wavefield.

2a. Green's function for the acoustic wave equation

We have shown in SEP-26 that for a Dirac source located at the origin and on the surface of an elastic liquid, the Green's function for the downgoing wave is given in the f-k domain by:

$$G_{DW}(k_x,\omega,z) = i\pi \frac{e^{-ik_z z}}{k_z} \tag{9a}$$

with

$$k_z = \frac{\omega}{v}\left[1 - \left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} \tag{9b}$$

The Fourier transform convention is

$$f(x,z,t) = \frac{1}{(2\pi)^3}\int\int\int F(k_x,k_z,\omega)\, e^{-ik_x x - ik_z z + i\omega t}\, dk_x\, dk_z\, d\omega$$

In this expression the ratio vk_x/ω is related to the angle of propagation of the wave ϑ by

$$\sin\vartheta = vk_x/\omega \tag{10}$$

The reciprocity principle gives us for a viscoelastic medium

$$G_{DW}(k_x,\omega,z) = i\pi \frac{e^{-k_z z}}{k_z} \tag{11a}$$

with

$$k_z = \frac{\omega}{v}\, p.v. \left[1 - \left(\frac{vk}{\omega}\right)^2\right]^{1/2} \tag{11b}$$

$p.v.$ meaning principal value of the square-root,
i.e. the square-root which has a positive real-part

v is a complex velocity given by $v^2 = \frac{M}{\rho}$, $v_R \geq 0$ (11c)

k is a complex number $k = k_x + i\,\mathrm{Im}(k) = P_x - iA_x$ (11d)
[cf. (8a), (8b)].

Note: To simplify notations, the fact that we are taking the principal value of every complex square-root will be understood in all following equations.

In our synthetic seismogram, the incident wave will be homogeneous. In this case, using (7), (11d) becomes

$$k = (|\mathbf{P}| - i|\mathbf{A}|)\sin\vartheta = \omega\sqrt{\rho/M}\,\sin\vartheta \qquad (12)$$

when ϑ is the angle defined in figure(1).

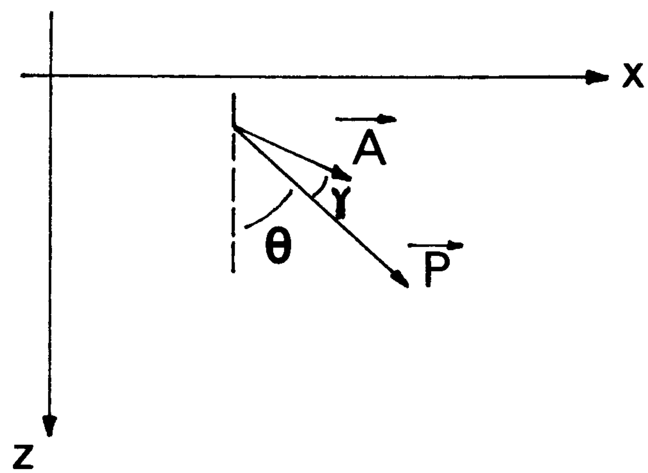

FIG. 1.

From figure(1), we can find an equivalent to (10),

$$\sin\vartheta = P_x / |\mathbf{P}|$$

For an homogeneous wave, and by using (7) we obtain

$$\sin\vartheta = \frac{v_R^2 + v_I^2}{v_R}\frac{k_x}{\omega} \qquad (13)$$

with $v = \sqrt{M/\rho}$, or

$$v_R = \left[\frac{|M| + M_R}{2\rho}\right]^{1/2}$$

$$v_I = \left[\frac{|M| - M_R}{2\rho}\right]^{1/2}$$

2b. Wave field extrapolator for the acoustic wave equation

It is well known that in an elastic medium the wavefield extrapolator for the acoustic wave equation in the f-k domain is

$$\Theta(\omega,k_x,z) = e^{-ik_z z}$$

This relation can be written for a viscoelastic medium, as done in paragraph 2.a, and so we have

$$\Theta(\omega,k_x,z) = e^{-ik_z z}$$

with

$$k_z = \frac{\omega}{v}\left[1 - \left(\frac{vk}{\omega}\right)^2\right]^{1/2}$$

$$v = \sqrt{M/\rho}$$

$$k = k_x + i\,\mathrm{Im}k = \frac{\omega}{v}\sin\vartheta \text{ , or}$$

$$\sin\vartheta = \frac{v_R^2 + v_I^2}{v_R}\frac{k_x}{\omega} \text{ , for an homogeneous wave.}$$

Remark: Boundary conditions will imply, as is shown in the next paragraph, a conservation of the x-coordinate of the wave vector **K**. This will make us able to use equations obtained in the first medium for other media. In particular, in the case of an homogeneous incident wave, we will apply equations (12) and (13) everywhere.

3. Reflections and Transmissions

In all the following equations, time dependence has been omitted, and it is understood to be of the form $e^{i\omega t}$.

Figure(2) shows the two media we are dealing with, and the notation employed. Only the propagation vector has been represented. All variables associated with the lower medium will be primed. In the first medium, subscript 1 refers to the amplitude of the incident wave, while subscript 2 refers to the reflected wave.

In all the following derivations, we are going to use potentials. As function of potentials, we have the following system of equations.

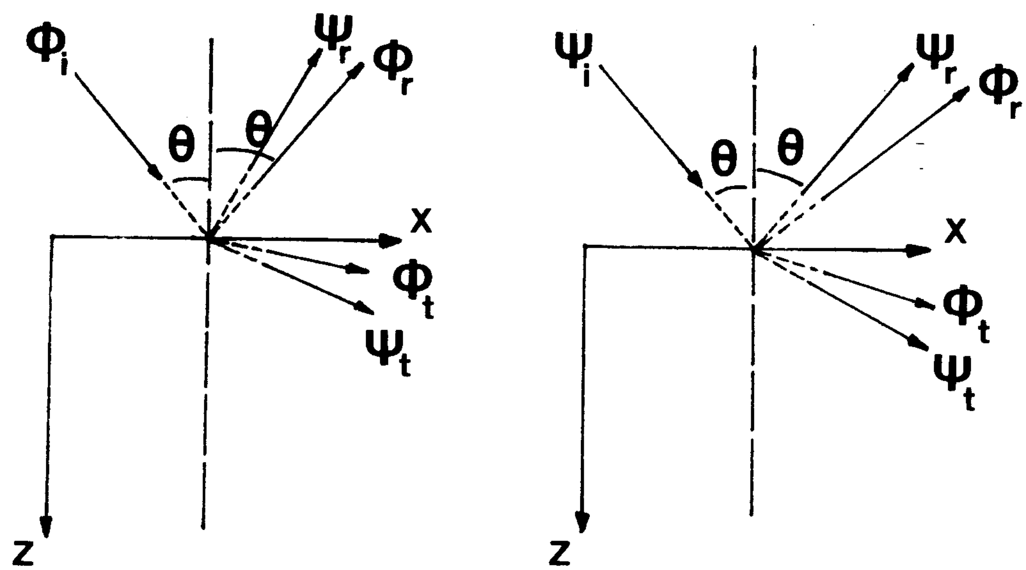

FIG. 2. P wave incident S wave incident

$$\Phi = \Phi_i + \Phi_r = A_1 e^{-i(kx + dz)} + A_2 e^{-i(kx - dz)}$$
$$\Psi = \Psi_i + \Psi_r = B_1 e^{-i(kx + fz)} + B_2 e^{-i(kx - fz)}$$
$$\Phi' = \Phi_t = A' e^{-i(k'x + d'z)}$$
$$\Psi' = \Psi_t = B' e^{-i(k'x + f'z)}$$

with

$$k^2 + d^2 = \frac{\rho\omega^2}{\lambda + 2\mu} \quad \mathrm{Re}(d) \geq 0$$
$$k^2 + f^2 = \frac{\rho\omega^2}{\mu} \quad \mathrm{Re}(f) \geq 0$$
$$k'^2 + d'^2 = \frac{\rho'\omega^2}{\lambda' + 2\mu'} \quad \mathrm{Re}(d') \geq 0$$
$$k'^2 + f'^2 = \frac{\rho'\omega^2}{\mu'} \quad \mathrm{Re}(f') \geq 0$$

The displacements and stresses for the $2D$ case are given by:

Displacements:

$$u_x = \frac{\partial\Phi}{\partial x} - \frac{\partial\Psi}{\partial z}$$

$$u_z = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x}$$

Stresses:

$$\sigma_{zz} = \lambda \frac{\partial^2 \Phi}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 \Phi}{\partial z^2} + 2\mu \frac{\partial^2 \Psi}{\partial x \partial z}$$

$$\sigma_{zx} = \mu \left[2 \frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial x^2} \right]$$

Three cases are to be examined separately.

3a. Liquid-solid interface.

The incident wave is traveling in a liquid medium, this implies $\mu = 0$, $B_1 = B_2 = 0$. Continuity of displacements and stresses gives

$$u_z = u'_z$$
$$0 = \sigma'_{zx}$$
$$\sigma_{zz} = \sigma'_{zz}$$

This implies $k = k'$, which can be considered as a generalized Snell's law.

$$\frac{A_2}{A_1} = R = \frac{\rho' d[(f'^2 - k^2)^2 + 4k^2 d' f'] - \rho d'(f'^2 + k^2)^2}{\rho' d[(f'^2 - k^2)^2 + 4k^2 d' f'] + \rho d'(f'^2 + k^2)^2}$$

$$\frac{A'}{A_1} = T_P = \frac{2\rho d(f'^4 - k^4)}{\rho' d[(f'^2 - k^2)^2 + 4k^2 d' f'] + \rho d'(f'^2 + k^2)^2}$$

$$\frac{B'}{A_1} = T_S = \frac{4\rho d d' k(f'^2 + k^2)}{\rho' d[(f'^2 - k^2)^2 + 4k^2 d' f'] + \rho d'(f'^2 + k^2)^2}$$

3b. Solid-Liquid interface.

The incident wave is now traveling in a solid medium, and the transmitted wave is traveling in a liquid medium, therefore we have $\mu' = 0$, $B' = 0$. The continuity equations are now

$$u_z = u'_z$$
$$\sigma_{zx} = 0$$
$$\sigma_{zz} = \sigma'_{zz}$$

This implies $k = k'$, as in case 3a. Now two incident waves are possible:

1. P-wave incident: $B_1 = 0$, we get

$$\frac{A_2}{A_1} = R_{PP} = \frac{\rho' d(f^2+k^2)^2 + 4k^2 dd' f\rho - \rho d'(f^2-k^2)^2}{\rho' d(f^2+k^2)^2 + 4k^2 dd' f\rho + \rho d'(f^2-k^2)^2}$$

$$\frac{B_2}{A_1} = R_{PS} = \frac{4k\rho dd'(f^2-k^2)}{\rho' d(f^2+k^2)^2 + 4k^2 dd' f\rho + \rho d'(f^2-k^2)^2}$$

$$\frac{A'}{A_1} = T_{PP} = \frac{2\rho d(f^4-k^4)}{\rho' d(f^2+k^2)^2 + 4k^2 dd' f\rho + \rho d'(f^2-k^2)^2}$$

2. S-wave incident: $A_1 = 0$, then we get

$$\frac{B_2}{B_1} = R_{SS} = \frac{4k^2 dd' f\rho - \rho d'(f^2-k^2)^2 - \rho' d(f^2+k^2)^2}{\rho' d(f^2+k^2)^2 + \rho d'(f^2-k^2)^2 + 4k^2 dd' f\rho}$$

$$\frac{A_2}{B_1} = R_{SP} = \frac{-4kd' f\rho(f^2-k^2)}{\rho' d(f^2+k^2)^2 + \rho d'(f^2-k^2)^2 + 4k^2 dd' f\rho}$$

$$\frac{A'}{B_1} = T_{SP} = \frac{4kdf\rho(f^2+k^2)}{\rho' d(f^2+k^2)^2 + \rho d'(f^2-k^2)^2 + 4k^2 dd' f\rho}$$

3c. Solid-Solid interface.

Two cases are possible: we can have either a P or an S incident wave. In both cases the equations of continuity are

$$\begin{aligned} u_x &= u'_x \\ u_z &= u'_z \\ \sigma_{zx} &= \sigma'_{zx} \\ \sigma_{zz} &= \sigma'_{zz} \end{aligned}$$

b1: P-wave incident: $B_1 = 0$, if we define

$$\begin{aligned} R_{PP} &= A_2/A_1 \\ R_{PS} &= B_2/A_1 \\ T_{PP} &= A'/A_1 \\ T_{PS} &= B'/A_1 \end{aligned}$$

They are solutions of the linear system

$$\mathbf{Ap} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} k & f & -k & f' \\ -d & k & -d' & -k \\ -2kd\mu & -(f^2-k^2)\mu & -2kd'\mu' & (f'^2-k)\mu' \\ (f^2-k^2)\mu & -2kf\mu & -\mu'(f'^2-k^2) & -2kf'\mu' \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} R_{PP} \\ R_{PS} \\ T_{PP} \\ T_{PS} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -k \\ -d \\ -2kd\mu \\ -\mu(f^2-k^2) \end{bmatrix}$$

b2: S-wave incident: $A_1 = 0$, the following definitions

$$\begin{aligned} R_{SP} &= A_2/B_1 \\ R_{SS} &= B_2/B_1 \\ T_{SP} &= A'/B_1 \\ T_{SS} &= B'/B_1 \end{aligned}$$

are solutions of the linear system

$$\mathbf{As} = \mathbf{c}$$

where

$$\mathbf{s} = \begin{bmatrix} R_{SP} \\ R_{SS} \\ T_{SP} \\ T_{SS} \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} f \\ -k \\ \mu(f^2-k^2) \\ -2kf\mu \end{bmatrix}$$

4. Attenuation model

In an internal report of the *Stanford Rock Physics Project*, (Bourbie, 1981a,c), it has been shown that the effect of attenuation on reflections was not, to a first order approximation, a function of the viscoelastic model chosen. In the same report it was also shown (following equations derived by Borcherdt, 1977) that the attenuation function in two dimensions was a slowly varying function of the angle γ between the attenuation and the propagation vectors. Under these conditions, we are confident on using a *constant* $-Q$ model (Kjartansson, 1980) to represent viscoelastic behavior. This model will enable us to have the right order of variations modeled in the synthetic sesimograms.

For review purposes, the constant-Q model is characterized by a frequency free representation of the quality factor Q, the model is the following:

The viscoelastic modulus is given by

$$M(\omega) = M_0 \left\{ \frac{i\omega}{\omega_0} \right\}^{2\gamma} = M_0 \left| \frac{\omega}{\omega_0} \right|^{2\gamma} e^{i\pi\gamma sgn\,\omega}$$

implying a phase velocity of

$$C_\varphi(\omega) = C_0 \left| \frac{\omega}{\omega_0} \right|^{\gamma}$$

In all these, ω_0 is a reference frequency at which M_0 and C_0 have been measured. γ is given by

$$\gamma = \frac{1}{\pi} \tan^{-1}(1/Q)$$

5. Synthetic Seismogram

Our goal in this paper is to model a marine seismogram. For this purpose, we will assume a flat-layered earth model. Our experiment will be to put a source at the sea surface, as in our SEP-26 article. To simplify the mathematical writing, we shall use a 3 layered earth. The generalization for more layers follows immediately.

The notations and the experimental setting are described in figure(3).

Five primary waves are recorded at the sea-surface. They are the following, where the letters refer to the type of wave in each path: *PP*, *PPPP*, *PPSP*, *PSPP*, *PSSP*.

To find the expression for each of these waves in the f-k domain, we are going to consider them as potentials. It is much easier to do so because potentials satisfy the acoustic wave equation, for which wave extrapolators are well known. On the other hand, if we were dealing with displacements, propagation would have to be done with the elastodynamic wave equation, for which we do not have wave extrapolators in the f-k domain. Thus we have:

PP:

$$i\pi R \,\frac{\exp[-i(2k_{zP_0}z_0)]}{k_{zP_0}} \cdot S(k_x,\omega)$$

PPPP:

$$i\pi T_P R_{12PP} T_{10PP} \,\frac{\exp[-i(2k_{zP_0}z_0 + 2k_{zP_1}z_1)]}{k_{zP_0}} \cdot S(k_x,\omega)$$

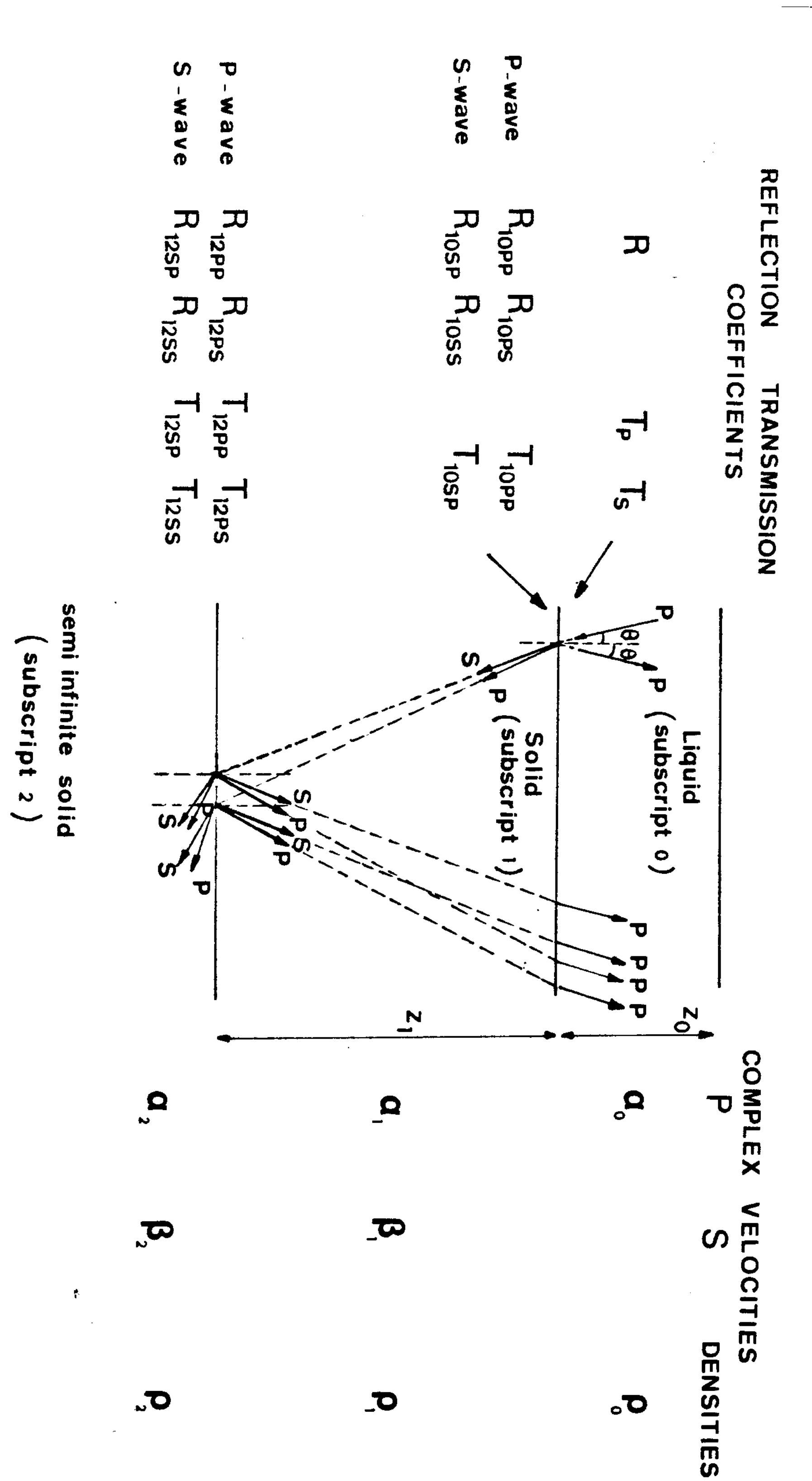
REFLECTION TRANSMISSION
COEFFICIENTS
R
T_P T_S
P-wave R_10PP R_10PS T_10PP
S-wave R_10SP R_10SS T_10SP
P-wave R_12PP R_12PS T_12PP T_12PS
S-wave R_12SP R_12SS T_12SP T_12SS
Liquid (subscript 0)
Solid (subscript 1)
semi infinite solid (subscript 2)
z_0
z_1
COMPLEX VELOCITIES
P
S
DENSITIES
α_0 ρ_0
α_1 β_1 ρ_1
α_2 β_2 ρ_2

FIG. 3.

PSSP:

$$i\pi T_S R_{12SS} T_{10SP} \frac{\exp[-i(2k_{zP_0}z_0 + 2k_{zS_1}z_1)]}{k_{zP_0}} \cdot S(k_x,\omega)$$

PPSP:

$$i\pi T_P R_{12PS} T_{10SP} \frac{\exp[-i(2k_{zP_0}z_0 + k_{zP_1}z_1 + k_{zS_1}z_1)]}{k_{zP_0}} \cdot S(k_x,\omega)$$

PSPP:

$$i\pi T_S R_{12SP} T_{10PP} \frac{\exp[-i(2k_{zP_0}z_0 + k_{zP_1}z_1 + k_{zS_1}z_1)]}{k_{zP_0}} \cdot S(k_x,\omega)$$

In these expressions

$$\begin{aligned} S(k_x,\omega) &= 2D \text{ Fourier Transform of the source function} \\ k_{zP_0} &= \frac{\omega}{\alpha_0}\left[1 - \left(\frac{\alpha_0 k}{\omega}\right)^2\right]^{1/2} \\ k_{zP_1} &= \frac{\omega}{\alpha_1}\left[1 - \left(\frac{\alpha_1 k}{\omega}\right)^2\right]^{1/2} \\ k_{zS_1} &= \frac{\omega}{\beta_1}\left[1 - \left(\frac{\beta_1 k}{\omega}\right)^2\right]^{1/2} \end{aligned}$$

with $k = \frac{\omega}{\alpha_0}\sin\vartheta$, and $\sin\vartheta = \frac{\alpha_{0R}^2 + \alpha_{0I}^2}{\alpha_{0R}}\frac{k_x}{\omega}$.

For primary waves, the synthetic seismogram in the f-k domain is the sum of the five upper expressions.

If we want to introduce multiple reflections or pegleg multiples, it is straightforward. Some examples are given in figure(4). The procedure can easily be generalized. The synthetic seismogram itself is obtained by taking an inverse $2D$ Fourier Transform of the expressions chosen in the f-k domain.

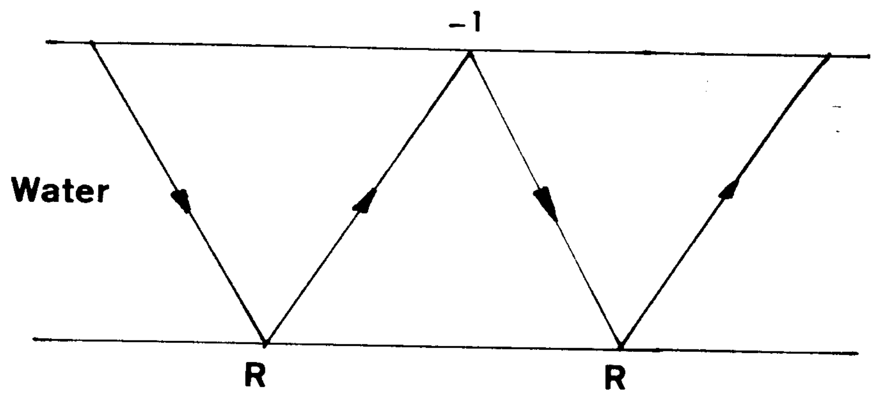

FIG. 4a. Water-Bottom multiple. Multiply the expression for *PP* by the factor: $-Re^{-2ik_{zP_0}z_0}$

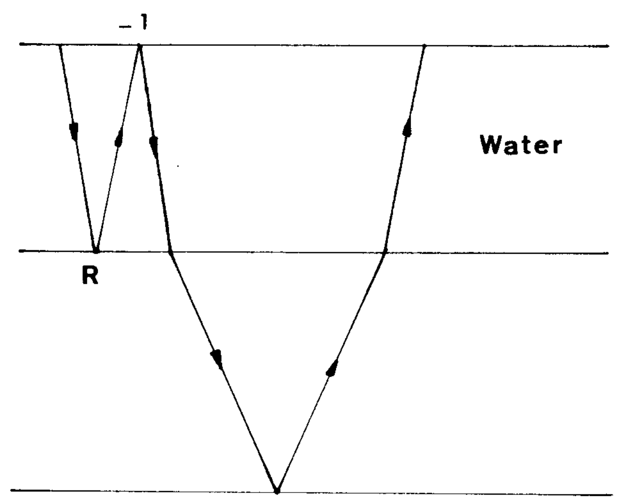

FIG. 4b. Peg-leg multiple. Multiply the expressions for *PP PPSPP PSPP PSSP* by the factor: $-Re^{-2ik_{zP_0}z_0}$

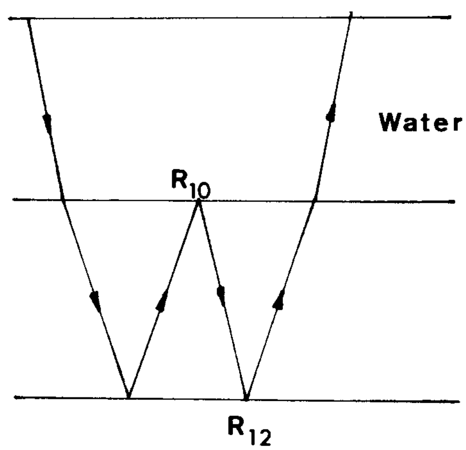

FIG. 4c. Intrabed multiples. If we take $PPPP$ as the main wave, we can have as intrabed multiples:

path	computation
$PP(PP)PP$	$PPPP \cdot R_{12PP} R_{10PP}\, e^{-i(2k_{zP_1})z_1}$
$PP(SP)PP$	$PPPP \cdot R_{12PS} R_{10SP}\, e^{-i(2k_{zP_1} + k_{zS_1})\, z_1}$
$PP(PS)PP$	$PPPP \cdot R_{12SP} R_{10PS}\, e^{-i(2k_{zP_1} + k_{zS_1})\, z_1}$
$PP(SS)PP$	$PPPP \cdot R_{12SS} R_{10SS}\, e^{-i(2k_{zS_1})\, z_1}$

REFERENCES

Ewing W.M., W.S. Jardetzky, F. Press, 1957; Elastic Waves in Layered Media. McGraw-Hill Book Co, New York.

Morse P.M., H. Feshbach, 1953; Methods of Theoretical Physics. McGraw-Hill Book Co, New York.

Pilant, W.L., 1979; Elastic waves in the earth. Elsevier, The Netherlands.

Winkler, K., 1979; The effects of pore fluids and frictional sliding on seismic attenuation. Stanford Rock Physics Project, 6.

Winkler, K., A. Nur and M. Gladwin, 1979; Friction and seismic attenuation in rocks. Nature, 277: 525-531.

Borcherdt, R.D., 1977; Reflection and Refraction of type-II S waves in elastic and anelastic medium. Bull. Seis. Soc. of Am., 67: 43-67.

Kjartansson, E., 1979; Constant-Q, Wave propagation and attenuation, Jr Geophys Res (84): 3737-3748.

Bourbie, T., 1981a; The effect of attenuation on reflection coefficients. (1 dimensional case). Stanford Rock Physics Project, 10: 153-172.

Bourbie, T., 1981b; Reflection and Transmission in attenuating media. Stanford Rock Physics Project, 10: 173-194.

Bourbie, T., 1981c; Constant Q model in two dimensions?. Stanford Rock Physics Project, 10: 195-207.

Bourbie, T., A. González, 1981; Effect of Reflection Coefficients on Synthetic Seismograms. Stanford Exploration Project, 26: 205-230.