High Order Migration When V=V(x,z)

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Abstract

A good migration scheme uses rational x-derivatives, dip filtering, and dissipation. In the case of laterally invariant media a recursive scheme exists for deriving operators of successively larger order which includes all these good features. When velocity varies laterally, the recursion generates 15- and 45-degree algorithms. The recursion fails to yield computationally useful algorithms of any higher degree which includes rational x-derivatives.

It is possible to get an algorithm of one higher order than the 45-degree algorithm with a trick. This trick does not admit extensions to any other order of approximation.

Introduction

The use of rational approximations to the square root operator in migration schemes was first suggested by Francis Muir. This proposal has led to the development of a sequence of migration algorithms for use in laterally invariant media. In theory, a similar sequence of migration schemes is generated for use in media in which acoustic velocity varies in both spatial directions, but only the 45-degree equation has been coded up. A method for factoring the higher order equations which retains rational approximations to the x-derivatives is needed.

When acoustic velocity only varies with depth, the expansion of the square root operator for use in migration involves matrix operators which commute. This means that the placement of the fraction bars in the continued fraction expansion is not as crucial as it might otherwise be. Continued fraction theory leads to a recursion for generating the matrices of migration schemes of successively higher order. When acoustic velocity varies laterally the placement of fraction bars is much more constraining. A continued fraction approximation again leads to a recursive scheme for generating a sequence of migration schemes of increasing order. This time there is a problem because the derived schemes are not computationally nice for any approximants except the two corresponding to the 15- and the 45-degree equations.

The Pressure Wave Equation and Acoustic Media

One-way wave equations are the foundation of finite-difference migration procedures. In a discrete world these require the solution of a banded system of linear equations. The algebra required for obtaining the matrix coefficients is difficult. Before attempting this algebra, the proper one-way wave equation will be derived. Much attention will be devoted to the causality and symmetry properties of the operators invoked.

The starting point is the wave equation for a pressure wave field $P(x,z,t)$ in an acoustic medium. The medium is characterized by its density $\rho(x,z)$ and bulk modulus $\kappa(x,z,t)$. In the following, we will also need to consider acoustic velocity and slowness, denoted by $V(x,z,t)$ and $\Lambda(x,z,t)$, respectively. The time dependence in V, Λ, and κ will allow us to model the visco-acoustic effects of dissipative wave equations. The wave equation for P,

$$\frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{1}{\kappa} * \frac{\partial^2 P}{\partial t^2} = 0 \qquad (1)$$

$$f * g = \int_{-\infty}^{\infty} dt' \, f(t') \, g(t-t')$$

governs the propagation of acoustic waves when combined with appropriate boundary conditions. For causality, the reciprocal of κ will have to vanish for $t < 0$ so that

$$\frac{1}{\kappa} * g = \int_0^{\infty} dt' \, \frac{1}{\kappa}(t') \, g(t - t')$$

Unfortunately, insufficient information is available with which to completely specify the two-way propagation problem. The wave field $P(x,z,t)$ is known at $z = 0$, but its z-derivative there is not, so the information at the boundaries is incomplete. It is expedient, therefore, to consider one-way wave equations which support propagation in only one vertical direction. To do this, the causality of the derivatives needs to be specified and the wave equation needs to be rearranged.

Derivatives and Causality

The derivatives in equation (1), will be handled first. In calculus texts it is taught that the derivative of a function f(t) is defined when both

$$D_t^+ f(t) = \lim_{\Delta t \to 0+} \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad \text{and} \quad D_t^- f(t) = \lim_{\Delta t \to 0+} \frac{f(t) - f(t-\Delta t)}{\Delta t}$$

exist and are equal. In this case, $D_t^0 f(t) = D_t^+ f(t) = D_t^- f(t)$. This definition actually involves three kinds of derivative operators: a right-derivative like D_t^+, a left-derivative like D_t^-, and a derivative like D_t^0 which exists when both left- and right-derivatives exist and are equal.

The one-way wave equation is peculiar in that derivatives of the D_t^0 type are not relevant. The other two types of derivatives are and must be carefully specified. We will need the causal derivative operators D_x, D_z, and D_t, as well as the anti-causal operators $-D_x^H$, $-D_z^H$, and $-D_t^H$. Defining positive increments Δx, Δz, and Δt, the partial derivatives can be defined by the six equations

$$D_x f(x,z,t) = \frac{f(x,z,t) - f(x-\Delta x,z,t)}{\Delta x} \qquad -D_x^H f(x,z,t) = \frac{f(x+\Delta x,z,t) - f(x,z,t)}{\Delta x}$$

$$D_z f(x,z,t) = \frac{f(x,z,t) - f(x,z-\Delta z,t)}{\Delta z} \qquad -D_z^H f(x,z,t) = \frac{f(x,z+\Delta z,t) - f(x,z,t)}{\Delta z}$$

$$D_t f(x,z,t) = \frac{f(x,z,t) - f(x,z,t-\Delta t)}{\Delta t} \qquad -D_t^H f(x,z,t) = \frac{f(x,z,t+\Delta t) - f(x,z,t)}{\Delta t}$$

where limiting processes are performed if necessary. The H superscript denotes Hermitian conjugation. To see why this should be so, consider the case of x-differentiation in a discrete space consisting of an infinite set of evenly spaced grid points. In this space there are no boundaries to introduce anomalous values into the differencing matrices. D_x, for instance, is a matrix operator with a diagonal of $1/\Delta x$'s and a subdiagonal of $-1/\Delta x$'s. The negative of the transpose of D_x is a matrix has a diagonal of $-1/\Delta x$'s and a superdiagonal of $1/\Delta x$'s. Thus D_x^H has the form required for an anti-causal derivative.

The x-derivatives discussed above give a consistent estimate of the derivative as Δx approaches zero. No distinction is made between operations in a discrete world and operations in a continuum. They are not the only operators with well-defined causality that have this property. It will turn out to be useful to consider an x-derivative of the form

$$D_x f(x,z,t) = \frac{1}{\Delta x} B \left[I - \frac{\alpha}{1+\alpha} B \right]^{-1} f(x,z,t)$$

where B is a causal operator and α is a real and positive constant. The operator B

operates on a vector defined at regularly spaced, discrete points along the x-axis, but whose z and t dependence may be either continuous or discrete. The advantage of making a clear distinction between an operator and its representation is that the notation for continuous and discrete forms for the same operator becomes identical. The same operation in discrete and continuous physical models is denoted by the same symbol. Given this definition of B, the restriction on α will guarantee the existence of the causal inverse operator $[I-\alpha/(1+\alpha)B]^{-1}$. The new definition for D_x has the same limit as the old one when Δx approaches zero from above.

The new D_x can be used to construct a Hermitian second- derivative operator which provides a good approximation to the continuous second derivative over a wide range of wavenumbers. Forming the product $-D_x^H D_x$, simple algebra shows that

$$-D_x^H D_x f(x,z,t) \;=\; \frac{-1}{(\Delta x)^2} B^H B \left[I - \frac{\alpha}{(1+\alpha)^2} B^H B \right]^{-1} f(x,z,t)$$

where use has been made of the result that $B + B^H = B^H B = B B^H$. If we introduce an operator $T = BB^H$, whose matrix representation has 2's on its diagonal and -1's on its super- and sub-diagonals, then

$$-D_x^H D_x f(x,z,t) \;=\; \frac{-1}{(\Delta x)^2} T \left[I - \frac{\alpha}{(1+\alpha)^2} T \right]^{-1} f(x,z,t)$$

Finally, we may want to implement time differentiation in the frequency domain. If the Fourier transform of f(t) is defined by

$$F(\omega) \;=\; \int_{-\infty}^{\infty} dt \; e^{-i\omega t} f(t)$$

then causal differentiation is equivalent to multiplication by $i\omega+\varepsilon$ in the limit $\varepsilon \to 0+$. Similarly, anti-causal differentiation is equivalent to multiplication by the complex factor $i\omega-\varepsilon$.

Causal wave equations

Migration of upwards traveling waves is a process which is anti-causal in both time and depth so $-D_t^H$ and not D_t will be used for temporal differentiation. With the notation and concepts introduced in the last section the two-way wave equation for propagating backwards in time can be written as

$$-D_x^H \frac{1}{\rho} D_x P \;-\; D_z^H \frac{1}{\rho} D_z P \;-\; \frac{1}{\kappa} * \left[-D_t^H\right]^2 P \;=\; 0$$

where the asterisk denotes a convolution with respect to time. It will be convenient to work in the frequency domain so that the time domain convolution does not mess up the algebra. To avoid the introduction of an unnecessary number of new symbols, we let P stand for both the pressure wave and its Fourier transform with respect to time. Define

$$\frac{1}{K(x,z,\omega)} = \int_0^\infty dt \ \frac{1}{\kappa}(x,z,t)\ e^{-i\omega t}$$

so that the wave equation can be written

$$-D_z^H \frac{1}{\rho} D_z P - D_x^H \frac{1}{\rho} D_x P - \frac{1}{K}(i\omega-\varepsilon)^2 P = 0$$

Since κ may be frequency dependent, there is a frequency dependent phase velocity V and a frequency dependent slowness Λ, too.

With a change of the dependent variable on which the wave equation operates, the parameters of the medium can be grouped together. The extrapolation will be in the z-direction so the x- and t-derivatives are transposed to the right side of the equality.

$$-K^{1/2} D_z^H \frac{1}{\rho} D_z K^{1/2} \frac{P}{K^{1/2}} = K^{1/2} D_x^H \frac{1}{\rho} D_x K^{1/2} \frac{P}{K^{1/2}} + (i\omega-\varepsilon)^2 \ \frac{P}{K^{1/2}}$$

To get a one-way equation we need to take the square root of the operator on both sides of the equality. Using the approximation

$$-K^{1/2} D_z^H \frac{1}{\rho} D_z K^{1/2} \frac{P}{K^{1/2}} \approx V^{1/2} D_z^H V D_z^H V^{1/2} \frac{P}{K^{1/2}} = \left[V^{1/2} D_z^H V^{1/2} \right]^2 \frac{P}{K^{1/2}}$$

leads to a one-way wave equation which is as accurate as geometric optics in the z-direction and as accurate as physical optics in the x-direction. As an added and very necessary bonus, all traces of the causal derivative D_z have disappeared. Since the waves which migration handles are to be pushed downwards the z-derivative which is employed must be anti-causal. The two-way wave equation now under consideration is

$$\left[V^{1/2} D_z^H V^{1/2} \right]^2 \frac{P}{K^{1/2}} = K^{1/2} D_x^H \frac{1}{\rho} D_x K^{1/2} \frac{P}{K^{1/2}} + (i\omega-\varepsilon)^2 \frac{P}{K^{1/2}}$$

To convert this equation into a one-way wave equation take the square roots of the operators on both sides of the equality. There is an ambiguity of sign in the square root which is resolved by the choice of propagation direction. If the square root operator maps into a non-positive real quantity then for upwards traveling waves

$$-V^{1/2} D_z^H V^{1/2} \frac{P}{K^{1/2}} = \left\{ (i\omega-\varepsilon) - (i\omega-\varepsilon) + \left[(i\omega-\varepsilon)^2 + K^{1/2} D_x^H \frac{1}{\rho} D_x K^{1/2} \right]^{1/2} \right\} \frac{P}{K^{1/2}} \qquad (2)$$

Through a change of state variable it is possible to get a differential equation in normal form which can be approximately solved with the help of the Crank-Nicolson method. The desired form is $D_z^H f = Opf$ and can be obtained in two steps by grouping $V^{1/2}$ with $P/K^{1/2}$ and then pre-multiplying both sides of the equation by $\Lambda^{1/2}$. The result is the partial differential equation

$$-D_z^H \frac{V^{1/2}P}{K^{1/2}} = \Lambda^{1/2}\left\{(i\omega-\varepsilon) - (i\omega-\varepsilon) + \left[(i\omega-\varepsilon)^2 + K^{1/2}D_x^H \frac{1}{\rho} D_x K^{1/2}\right]^{1/2}\right\}\Lambda^{1/2}\frac{V^{1/2}P}{K^{1/2}}$$

The peculiar $(i\omega-\varepsilon) - (i\omega-\varepsilon)$ in the braces on the right side of the equality was placed there because the most accurate schemes do not solve the migration equation directly. Instead, the migration equation is split into two partial differential equations, a phase shift equation and a focusing equation. The two pieces are solved for alternately at each z-step. One of the equations of the split, a phase shift equation, is easily solved analytically. The other, a focusing equation, is the subject of this paper and can be written in the form

$$-D_z^H \frac{V^{1/2}P}{K^{1/2}} = \left\{-(i\omega-\varepsilon)\Lambda + \Lambda^{1/2}\left[(i\omega-\varepsilon)^2 + K^{1/2}D_x^H \frac{1}{\rho} D_x K^{1/2}\right]^{1/2}\Lambda^{1/2}\right\}\frac{V^{1/2}P}{K^{1/2}} \qquad (3)$$

This equation will be used to extrapolate the current state variable $V^{1/2}P/K^{1/2}$ from a depth z to a depth $z+\Delta z$. An equation of the form $-D_z^H(V^{1/2}P/K^{1/2}) = Op\,(V^{1/2}P/K^{1/2})$ in which Op is roughly independent of depth between z and $z+\Delta z$ has an approximate solution of the form

$$\frac{V^{1/2}P}{K^{1/2}}(x,z+\Delta z,\omega) = \exp\left[\int_z^{z+\Delta z} dz'\ Op(z')\right]\frac{V^{1/2}P}{K^{1/2}}(x,z,\omega)$$

The integral in this expression can be approximated by $\Delta z\, Op(z)$ leaving us with the approximate expression for the solution. Applying this result to differential equation (3) leads to

$$\frac{V^{1/2}P}{K^{1/2}}(x,z+\Delta z,\omega) =$$

$$\exp\left\{-\Delta z\,(i\omega-\varepsilon)\Lambda+\Delta z\,\Lambda^{1/2}\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\Lambda^{1/2}\right\}\frac{V^{1/2}P}{K^{1/2}}(x,z,\omega)$$

The exponential is defined in terms of its power series. A rational approximation which matches power series up to terms of second order (inclusive) in Δz is given by

$$exp(\Delta z\, Op) \approx (I-\frac{\Delta z}{2}Op)^{-1}(I+\frac{\Delta z}{2}Op)$$

This particular rational approximation is equivalent to the result obtained by applying

Crank-Nicolson to the z-derivative. Higher order rational approximations exist but will not be considered here. The result is a linear equation of the form

$$\frac{V^{1/2}P}{K^{1/2}}(x,z+\Delta z,\omega) = (I - \frac{\Delta z}{2}Op)^{-1}(I + \frac{\Delta z}{2}Op)\frac{V^{1/2}P}{K^{1/2}}(x,z,\omega)$$

$$Op = -(i\omega-\varepsilon)\Lambda+\Lambda^{1/2}\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\Lambda^{1/2}$$

The $\Lambda^{1/2}$'s in this equation are annoying. To get rid of them, introduce yet another state variable Q, defined in terms of the temporal Fourier transforms of the pressure and bulk modulus.

$$Q(x,z,\omega) = \Lambda^{1/2}\frac{V^{1/2}P}{K^{1/2}}(x,z,\omega) = \frac{P}{K^{1/2}} \tag{4}$$

$$Q(x,z+\Delta z,\omega) = \left\{V+\frac{\Delta z}{2}(i\omega-\varepsilon)-\frac{\Delta z}{2}\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\right\}^{-1}\cdot$$

$$\left\{V-\frac{\Delta z}{2}(i\omega-\varepsilon)+\frac{\Delta z}{2}\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\right\}Q(x,z,\omega)$$

This equation can be made dimensionless to improve its numerical properties. This involves scaling by $i\omega-\varepsilon$. There are several ways in which to do this. The way which seems to be the most desirable is to lump one $i\omega-\varepsilon$ with each $K^{1/2}$. This seems logically nice because it allows us to link numerical motivated dissipation via ε with the physically motivated dissipation buried in the frequency dependence of $K^{1/2}$. If we allow ε to be x-dependent then it will not commute with D_x, forcing the choice of the anti- causal half derivative of Q as another state variable. This is inconvenient but necessary if the dissipation parameters are to be allowed to be space variable.

$$Q'(x,z,\omega) = (i\omega-\varepsilon)^{1/2}Q(x,z,\omega) = \frac{(i\omega-\varepsilon)^{1/2}}{K^{1/2}}P \tag{5}$$

$$Q'(x,z+\Delta z,\omega) = \tag{6}$$

$$\left\{\frac{2}{\Delta z}(i\omega-\varepsilon)^{-1}V + I - \left[I + (i\omega-\varepsilon)^{-1}K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}(i\omega-\varepsilon)^{-1}\right]^{1/2}\right\}^{-1}\cdot$$

$$\left\{\frac{2}{\Delta z}(i\omega-\varepsilon)^{-1}V - I + \left[I + (i\omega-\varepsilon)^{-1}K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}(i\omega-\varepsilon)^{-1}\right]^{1/2}\right\}Q'(x,z,\omega)$$

This difference equation is the one we would like to solve. What prevents us is the cost of inverting and square rooting matrices. To get a cost-effective algorithm will require approximations that lead to linear systems of equations.

Continued Fractions for Focusing Equations

The operator in equation (6) has a numerator and denominator. Since both of them have the same structure a study of one will suffice. Consider the numerator operator, which we will now call N:

$$N = \frac{2}{\Delta z}(i\omega-\varepsilon)^{-1}V - I + \left[I + (i\omega-\varepsilon)^{-1}K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}(i\omega-\varepsilon)^{-1}\right]^{1/2}$$

The continued fraction for this function can be written in many ways because the coefficients in the expansion all commute with one another. To make the fraction more compact, the following notation is introduced:

$$G = \frac{2}{\Delta z}(i\omega-\varepsilon)^{-1}V \qquad -D_t^{-H} = (i\omega-\varepsilon)^{-1}$$

Since I do not know how to represent the operator $K^{1/2}D_x^H\rho^{-1}D_x K^{1/2}$ with a rational form for D_x, lateral variations in density will be assumed to be gentle enough so that this operator can be approximated by the Hermitian operator $VD_x^H D_x V$, where $D_x^H D_x$ is the negative of a second differentiation operator. To get a good representation for this, set

$$D_x^H D_x = d^{-1}n \qquad d = \Delta x(I - \beta T) \qquad n = \frac{1}{\Delta x}T$$

where β is a real number between 0 and 1/4, and T is a symmetric matrix with 2's on its diagonal and -1's on both its super- and sub-diagonals. With this restriction $(I - \beta T)$ will be a positive-definite operator. The operators d and n are defined so that the recurrences which appear later in this paper will be dimensionally correct. With these substitutions

$$N = G - I + \left[I + D_t^{-H}Vd^{-1}nVD_t^{-H}\right]^{1/2}$$

$$N = G + \frac{I}{2I + \frac{I}{2I + \cdots}D_t^{-H}Vd^{-1}nVD_t^{-H}}D_t^{-H}Vd^{-1}nVD_t^{-H} \tag{7}$$

Note that the $D_t^{-H}Vd^{-1}nVD_t^{-H}$'s all appear to the right of the fraction bars and that this is not an obvious necessity. Using results from Appendix A we can give a good reason for the operator placement used above. If $D_t^{-H}Vd^{-1}nVD_t^{-H}$ is singular and appears to the right of a fraction bar then one of the approximants of the continued fraction in equation (7) will not be well defined. In fact, $D_t^{-H}Vd^{-1}nVD_t^{-H}$ will be singular when V is a constant diagonal operator, ε in D_t^{-H} is equal to zero, and the corners in n are properly chosen, non-singularity cannot be guaranteed. Since we want all the approximants to equation (7) to be defined, it follows that the way in which we have written equation (7) is the only way in which to write the fraction (up to an equivalence transformation such as that discussed in Appendix A).

The denominator operator in equation (6) has a similar representation. Denoting the denominator by D, a little work will show that the only change from the previous continue fraction is a single change of sign. This fact will have far-reaching consequences which will be examined later.

$$D \;=\; G - \frac{I}{2I + \frac{I}{2I + \cdots} D_t^{-H} V d^{-1} n V D_t^{-H}} D_t^{-H} V d^{-1} n V D_t^{-H} \tag{8}$$

The problems with these representations are that they have too many fraction bars and have matrix inverses wherever d^{-1} appears. To get rid of these obstructions requires a more careful treatment of continued fractions. A discussion of continued fractions is found in Appendices A and B.

Recurrence Formulae for High Order Migration

The discussion in the appendices makes it clear that what equation (7) needs is an equivalence transformation that generates a recurrence for numerators and denominators which does not require any matrix inversions. Such a transformation should generate the 15- and 45-degree equations early in its sequence of approximants.

The 15-degree equation will be generated if the first right partial numerator is nVD_t^{-H}, the first left partial numerator is I, and the first partial denominator is equal to $2d\Lambda D_t^H$. Any equivalence transformation that yields these partial numerators and denominators pre-multiplies both the first left partial numerator and the first right partial numerator by $d\Lambda D_t^H$. The result of this transformation is a new formulation for N:

$$G + \frac{I}{2d\Lambda D_t^H + d\Lambda D_t^H \frac{I}{2I + \frac{I}{2I + \cdots} D_t^{-H} V d^{-1} n V D_t^{-H}} D_t^{-H} V d^{-1} n V D_t^{-H}} n V D_t^H$$

The 45-degree equation will be generated when the second partial denominator is equal to $2I$, the second left partial numerator is I, and the second right partial numerator is equal to nVD_t^{-H}. The first partial numerators and denominators must, of course, remain unaltered. Only equivalence transformations that pre-multiply the second right partial numerator by $d\Lambda D_t^H$ are allowable. Applying this transformation yields

$$G+\cfrac{I}{2d\Lambda D_t^H+d\Lambda D_t^H\cfrac{I}{2d\Lambda D_t^H+d\Lambda D_t^H\cfrac{I}{2I+\cdots}D_t^{-H}Vd^{-1}nVD_t^{-H}}nVD_t^H}nVD_t^H$$

The right partial numerators can now be obtained only with a transformation that involves post-multiplication of both the second and third left partial numerators by $D_t^{-H}Vd^{-1}$. The result of this equivalence transformation is the continued fraction

$$G+\cfrac{I}{2d\Lambda D_t^H+\cfrac{I}{2I+\cfrac{I}{2I+\cdots}D_t^{-H}Vd^{-1}nV^2D_t^{-2H}d^{-1}}nVD_t^{-H}}nVD_t^{-H}$$

We can also clear the third right partial numerator by pre-multiplying it by $d\Lambda D_t^H$. Continuing in this manner eventually gets us the continued fraction

$$G+\cfrac{I}{2d\Lambda D_t^H+\cfrac{I}{2I+\cfrac{I}{2I+\cfrac{I}{2I+\cdots}nV^2D_t^{-2H}d^{-1}}nV^2D_t^{-2H}d^{-1}}nVD_t^{-H}}nVD_t^{-H} \qquad (9)$$

Applying the fundamental recurrence for continued fractions to equation (9) generates a sequence of numerators A_k^N and denominators B_k^N. The superscript N is used to indicate that these numerators and denominators are those of the numerator operator in equation (6).

$$\begin{aligned}
A_{-1}^N &= I \\
A_0^N &= G \\
A_1^N &= nVD_t^{-H}A_{-1}^N + 2d\Lambda D_t^H A_0^N \\
A_2^N &= nVD_t^{-H}A_0^N + 2A_1^N \\
A_3^N &= nV^2D_t^{-2H}d^{-1}A_2^N + 2A_1^N \\
B_{-1}^N &= 0 \\
B_0^N &= I \\
B_1^N &= 2d\Lambda D_t^H B_0^N \\
B_2^N &= nVD_t^{-H}B_0^N + 2B_1^N \\
B_3^N &= nV^2D_t^{-2H}d^{-1}B_2^N + 2B_1^N
\end{aligned}$$

A similar recursion can be written for the denominator operator in equation (6). If we denote the numerators and denominators by A_k^D and B_k^D, respectively, then the only step of the recurrences that will change is the one for calculating A_1^D. The change is one of sign in one of the coefficients

$$A_1^D = -nVD_t^{-H}A_{-1}^D + 2d\Lambda D_t^H A_0^D$$

Since the recursions for B_k^N and B_k^D are identical, it is safe to conclude that $B_k^N = B_k^D$ for $k = -1,0,1,2,3,\ldots$.

Returning to equation (6), we can write the operator on the right side of the equality as

$$D^{-1}N \approx A_k^{D\,-1}B_k^D B_k^{N\,-1}A_k^N$$

using the kth approximant for N and D. But this approximation is equal to $A_k^{D\,-1}A_k^N$, so the linear equation which needs to be solved is

$$A_k^D Q'(x,z+\Delta z,\omega) = A_k^N Q'(x,z,\omega) \qquad (10)$$

Another consequence of the recursion is that it yields a messy form when $k = 3$. Though B_k^N still cancels B_k^D, the numerators contain a matrix inverse. This leads to a linear system with a large bandwidth and coefficients which are hard to evaluate. The upshot is that the recurrence we have been considering is not of any use for $k > 2$.

The recurrences will become more familiar if substitutions are made for n, d, and G. Only $k \le 2$ which are the useful terms in the recurrence will be considered. Dropping superscripts for compactness,

$$\begin{aligned} A_{-1} &= I \\ A_0 &= 2(\Delta z)^{-1} D_t^{-H} V \\ A_1 &= \pm(\Delta x)^{-1} T V D_t^{-H} A_{-1} + 2(I-\beta T)\Lambda D_t^H A_0 \\ A_2 &= (\Delta x)^{-1} T V D_t^{-H} A_0 + 2A_1 \end{aligned}$$

The 60-Degree Trick

By means of a trick it is possible to get a soluble system of linear equations for the case $k = 3$. This is done by leaving a d^{-1} in the first right partial numerator. A series of equivalence transformations and a use of the commutation relation $nd^{-1} = d^{-1}n$ leads to a continued fraction for N:

$$G + \cfrac{I}{2I + \cfrac{I}{2d\Lambda D_t^H + \cfrac{I}{2I + \cfrac{I}{2I + \cfrac{I}{2I + \cdots} n V^2 D_t^{-2H} d^{-1}} n V D_t^{-H}} n V D_t^{-H}} n V D_t^{-H}} D_t^{-H} V n d^{-1} V D_t^{-H}$$

The continued fraction for $Nd^{-1}VD_t^{-H}$ is obtained by pulling a factor of $d^{-1}VD_t^{-H}$ from the right side of this fraction.

$$GD_t^H\Lambda d+\cfrac{I}{2I+\cfrac{I}{2d\Lambda D_t^H+\cfrac{I}{2I+\cfrac{I}{2I+\cfrac{I}{2I+\cdots}\,nV^2D_t^{-2H}d^{-1}}\,nVD_t^{-H}}\,nVD_t^{-H}}\,nVD_t^{-H}}\,D_t^{-H}Vn$$

Applying the fundamental recurrence to this continued fraction yields a 15-degree and 45-degree equation representation which is computationally more expensive than the representation already considered. This is not true for the equation of next higher order. This time the recurrence for the first few A_k^N's is

$$\begin{aligned}
A_{-1}^N &= I \\
A_0^N &= GD_t^H\Lambda d \\
A_1^N &= D_t^{-H}VnA_{-1}^N + 2A_0^N \\
A_2^N &= nVD_t^{-H}A_0^N + 2d\Lambda D_t^HA_1^N \\
A_3^N &= nVD_t^{-H}A_1^N + 2A_2^N \\
A_4^N &= nV^2D_t^{-2H}d^{-1}A_2^N + 2A_3^N
\end{aligned}$$

This recursion is inverse free up to the fourth term and is, therefore, suitable for all k up to and including three. There is, again, a nearly identical recursion for A_k^D. The only alteration is a single sign change in the recursion for A_1^D:

$$A_1^N = -D_t^{-H}VnA_{-1}^D + 2A_0^D$$

The recursions for the denominators B_k^N and B_k^D need not be considered because they cancel each other out in the linear system which now needs to be solved.

$$A_3^Dd^{-1}VD_t^{-H}Q'(x,z+\Delta z,\omega) = A_3^Nd_{-1}VD_t^{-H}Q'(x,z,\omega) \tag{11}$$

Dropping superscripts for compactness and substituting for n, d, and G changes the recurrences to the useful forms:

$$\begin{aligned}
A_{-1} &= I \\
A_0 &= 2(\Delta z)^{-1}D_t^H\Lambda\Delta x(I-\beta T) \\
A_1 &= \pm D_t^{-H}V(\Delta x)^{-1}A_{-1}+2A_0 \\
A_2 &= (\Delta x)^{-1}VD_t^{-H}A_0+2\Delta x(I-\beta T)\Lambda D_t^HA_1 \\
A_3 &= (\Delta x)^{-1}VD_t^{-H}A_1+2A_2
\end{aligned}$$

The solution can be obtained without actually calculating a matrix inverse. This leads to a scheme in which three linear systems are successively solved. Introducing an auxiliary variable R, the three equations which need to be solved are

$$D_t^H \Lambda \Delta x (I - \beta T) R(x,z,\omega) = Q'(x,z,\omega)$$
$$A_3^D R(x,z+\Delta z,\omega) = A_3^N R(x,z,\omega)$$
$$Q'(x,z+\Delta z,\omega) = D_t^H \Lambda \Delta x (I - \beta T) R(x,z+\Delta z,\omega)$$

where the unknown appears on the left-hand side in all three cases. No such trick seems to work for the case in which $k = 4$.

Appendix A - Continued Fractions with Matrix Coefficients

The migration problem has generated a continued fraction with matrix operators for coefficients. The algebra of these fractions needs to be developed a bit before we proceed much further. A continued fraction generates a sequence of rational forms called approximants. For any continued fraction, we will need a quick algorithm for generating its sequence of approximants, rules for changing the coefficients of the continued fraction in such a way as to leave the sequence of approximants untouched, and an understanding of how the properties of matrices and continued fractions interact.

Given coefficients, each an N by N matrix,

$$\left\{ a_j \right\}_{j=1}^{\infty}, \quad \left\{ b_j \right\}_{j=0}^{\infty}, \quad \left\{ c_j \right\}_{j=1}^{\infty}$$

a continued fraction can be generated. The following argument will be formal, in that the existence of the necessary inverses and the convergence of the continued fraction will be assumed rather than proved. With this understood, a continued fraction which we denote F will be considered, where equal to

$$F = b_0 + a_1 \frac{I}{b_1 + a_2 \dfrac{I}{b_2 + a_3 \dfrac{I}{b_3 + \cdots} c_3} c_2} c_1$$

This continued fraction can be considered to be the resultant of a series of non-linear transformations t_p, where p varies from 0 to ∞. This sequence of transformations is defined on an N by N matrix argument w and takes the form

$$t_0(w) = b_0 + w$$

$$t_p(w) = a_p \frac{I}{b_p + w} c_p \qquad p = 1,2,3, \cdots$$

Transformations of this type can be combined via the operation of functional composition. For example, the resultant of operating on t_1 with t_0 is a transformation

$$t_0 t_1(w) = b_0 + a_1 \frac{I}{b_1 + w} c_1$$

Similarly, our continued fraction F can be considered to be the limit of a sequence of transformational compositions evaluated with some particular value of w. If F is well-defined then the choice of w will not matter.

$$F = F(w) = \lim_{p \to \infty} t_0 t_1 t_2 \cdots t_p(w)$$

A limit is not useful for computations because limits usually are infinitely expensive to compute. In migration applications it will turn out to be useful to consider intermediate terms of the limiting sequence. The most important result in this section, to be proved by induction in a separate section at the end, is that

$$F_k(w) = t_0 t_1 t_2 \cdots t_k(w) = \left[B_k + w a_k^{-1} B_{k-1}\right]^{-1} \left[A_k + w a_k^{-1} B_{k-1}\right]$$

$$A_{-1} = I \qquad A_0 = b_0 \qquad a_0 = I$$

$$B_{-1} = 0 \qquad B_0 = I$$

$$A_{k+1} = c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k \qquad k = 1,2,3,...$$

$$B_{k+1} = c_{k+1} a_k^{-1} B_{k-1} + b_{k+1} a_{k+1}^{-1} B_k \qquad k = 1,2,3,...$$

Now for some nomenclature: A_n is the nth numerator, B_n is the nth denominator, the ratio $B_n^{-1} A_n$ is the nth approximant, a_n is the nth left partial numerator, c_n is the nth right partial numerator, and b_n is the nth partial denominator. The difference from the usual continued fraction theory lies in the distinction between left and right partial numerators.

There exist an infinite number of continued fractions with the same value and the same series of approximants as F. Consider two sets of N by N matrices

$$\left\{ d_j \right\}_{j=1}^{\infty}, \qquad \left\{ e_j \right\}_{j=1}^{\infty}$$

where the d_j are all invertible. One continued fraction that has the same approximants as F can be obtained by simultaneously pre-multiplying c_1, b_1, and a_2 by an N by N matrix e_1. This pattern of matrix multiplication is used because the pattern of matrices, ignoring b_0 for the moment, is basically $a_1 M^{-1} c_1$ where M contains all of the rest of the continued fraction. Premultiplying c_1 by $e_1^{-1} e_1$ yields $a_1 M_{-1} e_1^{-1} e_1 c_1$ which is equivalent to $a_1 (e_1 M)^{-1} e_1 c_1$. If we look back at the fundamental recurrence formulae for A_k and B_k then it can be seen that this transformation leaves all approximants of the continued fraction unchanged. The result of this transformation is the continued fraction

$$b_0 + a_1 \frac{I}{e_1 b_1 + e_1 a_2 \frac{I}{b_2 + a_3 \frac{I}{b_3 + \cdots} c_3} c_2} e_1 c_1$$

A transformation on the coefficients of a continued fraction that preserves the sequence of approximants will be called an equivalence transformation. A much more general equivalence transformation of F yields the following continued fraction:

$$b_0 + a_1 d_1 \frac{I}{e_1 b_1 d_1 + e_1 a_2 d_2 \frac{I}{e_2 b_2 d_2 + e_2 a_3 d_3 \frac{I}{e_3 b_3 d_3 + \cdots} e_3 c_3 d_2} e_2 c_2 d_1} e_1 c_1$$

Appendix B - The Fundamental Recurrence for Continued Fractions

We have defined a sequence of rational transformations t_p, where p varies between 0 and ∞ and

$$t_0(w) = b_0 + w$$

$$t_p(w) = a_p \frac{I}{b_p + w} c_p$$

It would be desirable to find a recurrence formula for the functional compositions $t_0 t_1 t_2 \cdots t_k(w)$. Suppose the kth composition is of the form

$$F_k(w) = \left[B_k + w a_k^{-1} B_{k-1}\right]^{-1} \left[A_k + w a_k^{-1} A_{k-1}\right]$$

Then

$$F_{k+1}(w) = F_k t_{k+1}(w) = F_k(a_{k+1} \frac{I}{b_{k+1} + w} c_{k+1})$$

With a little algebra, paying close attention to the lack of commutativity among the various matrices, this expression can be simplified to look like

$$F_{k+1}(w) = \left[(c_{k+1} a_k^{-1} B_{k-1} + b_{k+1} a_{k+1}^{-1} B_k) + w a_{k+1}^{-1} B_k\right]^{-1} \left[(c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k) + w a_{k+1}^{-1} A_k\right]$$

$$F_{k+1}(w) = \left[B_{k+1} + w a_{k+1}^{-1} B_k\right]^{-1} \left[A_{k+1} + w a_{k+1}^{-1} A_k\right]$$

Equating coefficients yields a recurrence for both the A_k's and B_k's in terms of the partial numerators and denominators:

$$A_{k+1} = c_{k+1}a_k^{-1}A_{k-1} + b_{k+1}a_{k+1}^{-1}A_k$$
$$B_{k+1} = c_{k+1}a_k^{-1}A_{k-1} + b_{k+1}a_{k+1}^{-1}B_k$$

The necessary initializations for this recurrence need to be found. To get the starting points, consider the cases in which $k = 0$ and $k = 1$.

$$t_0(w) = b_0 + w = \left[I + 0w\right]^{-1}\left[b_0 + Iw\right]$$

$$t_1(w) = (b_1a_1^{-1} + wa_1^{-1})^{-1}(b_1a_1^{-1}b_0 + c_1 + wa_1^{-1}b_0)$$

Equating coefficients again, we find that for non-zero a_0

$$A_{-1} = a_0 \quad A_0 = b_0$$
$$B_0 = I \quad B_0 = I$$

will provide a suitable recurrence initialization. For convenience, we set a_0 and therefore A_{-1} equal to identity operators.

From the fundamental recurrence it can be seen that pre-multiplying b_{k+1} and c_{k+1} by the same non-singular matrix will not change the approximants of the continued fraction. The same can be said for post-multiplication of c_{k+1} and a_k and for post-multiplication of b_{k+1} and a_{k+1}. These facts were used implicitly in Appendix A in the discussion of equivalence transformations.

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