The Gaussian Beam in Energy Variables

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Is it hard to do a good job of computing Gaussian beams? It seems inevitable that approximations get made, both in theory and in numerical analysis. It is best when small errors don't grow, bad when errors grow rapidly to disasterous proportions, and worse when they grow more slowly and remain unrecognized. We have a lot of experience with the stability of wave extrapolation PDE's but none with the ODE's of ray tracing. With PDE's we have learned that the way to "bullet proof" the calculations is to use energy variables instead of the physical pressure variable. With the idea that energy variables might be helpful in the calculation of Gaussian beams, the Gaussian beam problem will be reformulated in terms of energy variables.

Given the differential equation

$$\frac{d\Psi}{dz} = -R\,\Psi \tag{1}$$

one may easily show that

$$\frac{d}{dz}\Psi^*\Psi = -\Psi^*(R + R^*)\Psi \leq 0 \tag{2}$$

We will interpret Ψ as an energy flux variable. We demand that $R + R^*$ be non-negative definate. Then (2) may be interpreted as a demand that the energy carried by the beam must decrease along the beam. In PDE work we sometimes regard Ψ as a column vector, with elements of the vector denoting successive locations along the x-axis. Other times we regard Ψ as a function of the Fourier variable k_x. In either case $\Psi^*\Psi$ is regarded as an integral over the x-axis. For explicit functions of x as with Gaussian beams, equation (2) has an implied integration over x .

The wave flux extrapolation equation is

$$\frac{d}{dz}\Psi = i\omega\frac{1}{\sqrt{v}}\sqrt{1-\frac{v}{-i\omega}\frac{\partial^2}{\partial x^2}\frac{v}{-i\omega}}\,\frac{1}{\sqrt{v}}\Psi \tag{3}$$

With the usual 15 degree approximation we get

$$\frac{d}{dz}\Psi = i\omega\frac{1}{\sqrt{v}}\left[1-\frac{1}{2}\frac{v}{-i\omega}\frac{\partial^2}{\partial x^2}\frac{v}{-i\omega}\right]\frac{1}{\sqrt{v}}\Psi \tag{4}$$

Replacing $-i\omega$ by $-i\omega+\varepsilon$ one sees that (4) is of the required disipative form (2).

$$\frac{d}{dz}\Psi = \frac{i\omega}{v}\Psi + \frac{\sqrt{v}}{-i\omega 2}\frac{\partial^2}{\partial x^2}\sqrt{v}\ \Psi \tag{5}$$

$$\frac{d}{dz}\Psi = \frac{i\omega}{v}\Psi + \frac{\sqrt{v}}{-i\omega 2}\frac{\partial}{\partial x}\left[(\sqrt{v}\,)_x\Psi + \sqrt{v}\,\Psi_x\right] \tag{6}$$

$$\frac{d}{dz}\Psi = \frac{i\omega}{v}\Psi + \frac{\sqrt{v}}{-i\omega 2}\left[(\sqrt{v}\,)_{xx}\Psi + 2(\sqrt{v}\,)_x\Psi_x + \sqrt{v}\,\Psi_{xx}\right] \tag{7}$$

Now we recall the Gaussian beam trial solution

$$\Psi(x,z) = \Psi_0 \exp[\ -i\omega t - \frac{1}{2}Mx^2 + i\omega\int_0^z\frac{dz}{v(z)} - \int_0^z a(z)\,dz\ ] \tag{8}$$

and take some partial derivatives

$$\Psi_x = (\frac{1}{2}Mx^2)_x\ \Psi = (Mx)\ \Psi \tag{9}$$

$$\Psi_{xx} = [(Mx)^2 + (Mx)_x]\ \Psi = (M^2x^2+M)\ \Psi \tag{10}$$

$$\Psi_z = [i\omega\int_0^z\frac{dz}{v(z)} - \int_0^z a(z)\ dz - M(z)x^2]_z\ \Psi \tag{11}$$

$$\Psi_z = [\frac{i\omega}{v(z)} - a(z) - x^2M_z]\ \Psi \tag{12}$$

We get a happy feeling when we note that we don't need Ψ_{zz} and all its attendent complications and approximations. This is because we don't plan to substitute into the scalar wave equation. We will substitute into the 15-degree flux extrapolation equation (7). First the z-derivatives.

$$(\frac{i\omega}{v} - a - x^2M_z)\ \Psi = \frac{i\omega}{v}\Psi + \frac{\sqrt{v}}{-i\omega 2}\left[(\sqrt{v}\,)_{xx}\Psi + 2(\sqrt{v}\,)_x\Psi_x + \sqrt{v}\,\Psi_{xx}\right] \tag{13}$$

$$(-a - x^2M_z)\ \Psi = \frac{\sqrt{v}}{-i\omega 2}\left[(\sqrt{v}\,)_{xx}\Psi + 2(\sqrt{v}\,)_x\Psi_x + \sqrt{v}\,\Psi_{xx}\right] \tag{14}$$

Now for the x-derivatives. We plan a power series expansion for everything, keeping all terms up to x^2. We will have a lens like medium where the beam stays columnated along the z-axis.

$$\sqrt{v} = b_0 + b_2 x^2 \tag{15a}$$

$$(\sqrt{v})_x = 2b_2 x \tag{15b}$$

$$(\sqrt{v})_{xx} = 2b_2 \tag{15c}$$

$$v = b_0^2 + 2b_0 b_2 x^2 \tag{15d}$$

Completing the substitution of (9), (10), and (15) into (14)

$$-i\omega 2(-a - x^2 M_z) = (b_0+b_2x^2)[2b_2 + 4b_2x^2 M + (b_0+b_2x^2)(M+M^2x^2)] \tag{16}$$

We find for the coefficient of x^0 that

$$2i\omega\, a = 2b_0 b_2 + b_0^2 M \tag{17}$$

We find for the coefficient of x^2 that

$$2i\omega \frac{dM}{dz} = 2b_2^2 + 6b_0 b_2 M + b_0^2 M^2 \tag{18}$$

In order to make things a bit more readable, we'll return to the definitions

$$v = b_0^2 \qquad v_{xx} = 4b_0 b_2 \qquad \frac{v_{xx}^2}{v} = 16 b_2^2 \tag{19}$$

We have finally deduced that $a(z)$ and $M(z)$ must satisfy

$$2i\omega\, a = \frac{v_{xx}}{2} + v\, M \tag{20}$$

$$2i\omega \frac{dM}{dz} = \frac{v_{xx}^2}{8v} + \frac{3}{2} v_{xx} M + v\, M^2 \tag{21}$$

I guess the linear equations are easier for numerical analysis than the Ricatti equations because they are reversible. Let us represent the Ricatti equation (21) in matrix form. Recall the equivalence of

$$\frac{d}{dz}\frac{p}{q} = b + (a-d)\frac{p}{q} - c\frac{p^2}{q^2} \tag{22}$$

$$\frac{d}{dz}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{23}$$

Thus (21) is equivalent to (24)

$$\frac{d}{dz}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2i\omega} \begin{bmatrix} \frac{3}{4} v_{xx} & \frac{v_{xx}^2}{8v} \\ v & -\frac{3}{4} v_{xx} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{24}$$

Now is the time for an erudite discussion of equation (24) explaining why it is more stable and computable than the form Cerveny uses. At the moment, I don't even know if it is as good ! This deserves further analysis. Perhaps I must note that the physics falls apart if the velocity ever becomes negative inside the beam, that is, if

$$v_{xx} \;>\; -v\;|M|$$