# The Simplest Gaussian Beam

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The finiteness of the amplitude of any Gaussian beam has important implications for ray tracing programs. In traditional ray tracing the energy density in a wave is inverse to the separation of nearby rays. During focusing, rays cross and incorrectly predict infinite energy density. Phase shift at the focus is not modeled. On the other hand, Gaussian beams propagate smoothly and correctly through a focus. Vlastislav Červený has shown that the computation of Gaussian beams is no more difficult than ray tracing. In fact it is even easier because it is much more stable. He has convincingly argued that ray amplitude and phase calculations should be done by Gaussian beams as an improvement to traditional methods. This paper is written to introduce students to the the Gaussian beam concept without demanding additional skills in curvilinear coordinates as are also required for ray tracing.

A mathematical expression for vertically downgoing plane waves is

$$P(x,z) \ = \ P_0 \exp(-i\omega t + i\frac{\omega}{v} z)$$

It will be little more difficult to let the velocity vary with $z$. Clutter can be avoided by introducing the definition *levocity* $= \lambda(z) = 1/v(z)$.

$$P(x,z) \ = \ P_0 \exp[-i\omega t + i\omega \int_0^z \lambda(z) \, dz \ ] \qquad (1)$$

Let us change this expression so that the amplitude drops off slowly as a Gaussian function of $x$.

$$P(x,z) \ = \ P_0 \, exp(-i\omega t + i\omega \int_0^z \lambda(z) \, dz - \frac{1}{2} M x^2) \qquad (2)$$

If $M$ is very small then the wavefield becomes very similar to a plane wave. Since (2) is nearly equal to a plane wave it should nearly satisfy the wave equation. But there are several discrepancies, and these are the things of interest. First of all, waves always eventually spread out so we should indicate that the beam width $1/\sqrt{M}$ is a function of $z$ and seek to find the function. We should expect that $M(z)$ will not only have a positive real part but that it will develope an imaginary part representing phase curvature, *i.e.* on a *very* large scale our broad wave looks like a directional point source. As the beam spreads out, the amplitude should also decrease and it may develope a phase shift. A convenient expression of these phenomena is given by some complex, non-zero $a(z)$ in the Gaussian beam trial solution

$$P(x,z) = P_0 \exp[ -i\omega t - \frac{1}{2} M x^2 + i\omega \int_0^z \lambda(z)\, dz - \int_0^z a(z)\, dz ] \tag{3}$$

Marcuse (p.236) asserts that a function such as (3) will not exactly satisfy the wave equation, but, with suitably determined $M(z)$ and $a(z)$ it does exactly satisfy the parabolic wave equation. We will find the equations which determine $M(z)$ and $a(z)$. We will need to assert that the beam is narrow, that is

$$\omega\lambda(z) \gg x^2 \left| \frac{\partial M}{\partial z} \right| \qquad (all\ z) \tag{A1}$$

$$\omega\lambda(z) \gg |a(z)| \qquad (all\ z) \tag{A2}$$

The stronger the inequality the narrower is the angular bandwidth of the beam. After we find determining equations for $M(z)$ and $a(z)$ we will want to study their behavior to see whether the assumptions (A1) and (A2) are satisfied for all $z$ as well as initially. Let us form the derivatives of (3) which will be required for substitution into the wave equation.

$$P_x = (\frac{1}{2} M x^2)_x \, P = (Mx)\, P$$

$$P_{xx} = [(Mx)^2 + (Mx)_x]\, P = (M^2 x^2 + M)\, P \tag{4}$$

$$P_z = [i\omega \int_0^z \lambda(z)\, dz - \int_0^z a(z)\, dz - M(z) x^2]_z \, P$$

$$P_z = [i\omega\lambda(z) - a(z) - x^2 M_z]\, P$$

$$P_{zz} = [\, (i\omega\lambda(z) - a(z) - x^2 M_z)^2 + (i\omega\lambda(z) - a(z) - x^2 M_z)_z \,]\, P \tag{5}$$

Before inserting into the wave equation we discuss some simplifications to (5). On the left we simplify by neglect of the squares of the small quantities in (A1) and (A2). On the right we make a paraxial type approximation, that is, we drop the $a_z$ and the $M_{zz}$ terms to eliminate the backward going wave. Justification is by the asserted smallness of the perturbation from a plane wave. Thus

$$P_{zz} = [-\omega^2\lambda^2 - 2i\omega\lambda\,(a+x^2M_z) + i\omega\lambda_z]\,P \tag{6}$$

Inserting (4) and (6) into the Helmholtz wave equation $P_{xx}+P_{zz} = -P\omega^2\lambda^2$ we get

$$0 = x^2M^2 + M - 2i\omega\lambda\,(a+x^2\frac{dM}{dz}) + i\omega\lambda_z \tag{7}$$

There are two unknowns, $M(z)$ and $a(z)$, and only one equation (7). But the equation should be satisfied for all possible values of $x$ so it is really an infinite number of equations. Here is where we are saved by our judicious choice of a trial solution. We don't need to ask $M$ or $a$ to have $x$−dependence because we can get (7) to be satisfied for all $x$ if we note that (7) is a two-coefficient polynomial in $x$. We get the coefficient of $x^2$ to vanish by our choice of $M(z)$ and the coefficient of $x^0$ to vanish by our choice of $a(z)$. Thus

$$\frac{dM}{dz} = \frac{M^2}{2i\omega\lambda(z)} \tag{8}$$

$$2\lambda(z)a(z) = \frac{1}{i\omega}M(z) + \lambda_z \tag{9}$$

Next we should show that the solutions of (8) and (9) are well behaved. That is, we would like to be able to assure that our initial assumptions (A1) and (A2) remain valid for all $z$. Furthermore we will probably want to be assured that the real parts of $M(z)$ and $a(z)$ are positive.

**Ricatti Equation**

An important goal is to find an expression for the total energy flux down the beam and its $z$ derivative. Physically one expects the energy carried by the beam to be invarient of the location along the beam. Mathematically we don't know if this will be strictly the case, since we have made some assumptions, (A1) and (A2). As a practical matter, we could be in deep trouble if the energy in the beam grows. I haven't yet achieved this goal, but let us review some of the traditional wisdom about energy flux.

Equation (8) is the non-linear Ricatti equation. Such equations are often associated with impedances and thus have positive real parts, so we should be in luck. The main thing about Ricatti equations is that given

$$\frac{d}{dz}\begin{bmatrix}p\\q\end{bmatrix} = \begin{bmatrix}ab\\cd\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix} \tag{10}$$

then by direct substitution

$$\frac{d}{dz}\frac{p}{q} = \frac{1}{q}\frac{dp}{dz} - \frac{p}{q^2}\frac{dq}{dz}$$

$$\frac{d}{dz}\frac{p}{q} = \frac{1}{q}(ap+bq) - \frac{p}{q^2}(cp+dq)$$

$$\frac{d}{dz}\frac{p}{q} = b + (a-d)\frac{p}{q} - c\frac{p^2}{q^2} \tag{11}$$

Associate $M$ with $p/q$. Comparing (11) to (8) we see the applicable definition of $b$ and $c$. Apparently we can take $a=d=0$. Červený introduces (10) and explains that it has a formal similarity to the ray tracing equations. At the moment we are more interested in stability of our perturbation analysis.

In many problems $\text{Re}\, pq^*$ is the energy flux. I don't know if it is that here since I am familiar with equations which control the wavefield itself. This is the first time I have seen an equation controling the logarithm of a wave. Very interesting ! Anyway, it is easy to establish positivity by

$$\text{Re}\,\frac{p}{q} = \frac{\text{Re}\,pq^*}{qq^*} \tag{12}$$

By direct substitution we have

$$\frac{d}{dz}pq^* = p\,\frac{dq^*}{dz} + \frac{dp}{dz}q^* = c^*p^*p + b\,q^*q + (d^*+a)pq^* \tag{13}$$

Since $a=b=d=0$ and $c = 1/-2i\omega$ is purely imaginary, then $\text{Re}\,pq^*$ is a constant function of $z$.

### Lens Like Media

The ray would bend if we were to include a first lateral derivative of the velocity. A bending ray requires more difficult analytical procedures, so we'll ignore it for now. An interesting case is when the first derivative vanishes, but the second derivative does not. In this case the beam can focus. This is a difficult case for traditional ray theory, but it is an ideal case for Gaussian beams. Take the velocity to be given by

$$\lambda(x,z) = \lambda + \frac{x^2}{2}\lambda_{xx} \tag{14}$$

$$\lambda(x,z)^2 \;=\; \lambda^2 + x^2 \lambda \lambda_{xx} \tag{15}$$

Looking back to the derivation of equation (7) we see that the velocity of the medium enters the Helmholtz equation only from the term on the right hand side. Including the $x^2$ term in (15) just gives an additional $(-i\omega)^2 \lambda\lambda_{xx} x^2$ term on the right hand side of (7). This gives $b = -i\omega\lambda\lambda_{xx}/2$ to be added onto (8). Results may be summarized by

$$\frac{d}{dz}\begin{bmatrix} p \\ q \end{bmatrix} \;=\; \begin{bmatrix} 0 & \frac{-i\omega}{2}\lambda\lambda_{xx} \\ \frac{1}{-i\omega 2} & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{16}$$

Returning to equation (13) we see that $\partial_z(pq^*)$ is purely imaginary in lens-like media as well as homogeneous media. Thus Re $M$ = Re $(p/q)$ can change its value, but never its sign. That is lucky because the answer would be entirely meaningless if the sign changed. We may also want to do the numerical analysis to ensure the positivity.

It seems to me that the causality properties are messed up because look what happens to (13) when $-i\omega$ is replaced by $-i\omega + \varepsilon$. The sign of the real part cannot be guaranteed because the sign of $\lambda_{xx}$ is arbitrary. That seems to be just what lenses do.

## REFERENCES

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SPURT
NO GOOD WILL EVER COME OF THAT STUFF!