The Wave Equation in Ray-Centered Coordinates

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The previous three lectures provide a basis for this lecture. It is, of course, now no problem to put our wave equation in the ray-centered coordinate system. The general form of the Laplacian is

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} \left[\frac{h_2 h_3}{h_1} \frac{\partial u}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[\frac{h_1 h_3}{h_2} \frac{\partial u}{\partial x_2} \right] + \frac{\partial}{\partial x_3} \left[\frac{h_1 h_2}{h_3} \frac{\partial u}{\partial x_3} \right] \right\} \qquad (4\text{-}1)$$

In our case $h_1 = h$, $h_2 = 1$, and $h_3 = 1$. Also x_1 is identified with S , x_2 with q_1 and x_3 with q_2 . Use of (4-1) in the acoustic equation results in

$$\frac{1}{h} \left\{ \frac{\partial}{\partial S} \left(\frac{1}{h} \frac{\partial u}{\partial S} \right) + \frac{\partial}{\partial q_1} \left(h \frac{\partial u}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(h \frac{\partial u}{\partial q_2} \right) \right\} = \frac{1}{V^2} \frac{\partial^2 u}{\partial t^2} \qquad (4\text{-}2)$$

Since we eventually will be concerned with the parabolic approximation to (4-2), we would first like to remove the traveling wave contribution to (4-2). Also, we will explicitly introduce the frequency ω , which will serve as a large parameter. This will allow us to concentrate on rays near the central ray under consideration. For that ray, the traveling wave solution is simply given by

$$\exp\left\{ -i\omega \left(t - \int_{S_0}^{S} \frac{dS}{V_0} \right) \right\} \qquad (4\text{-}3)$$

where $V_0 = V_0(S)$ only and S_0 denotes the starting point of our ray. The new variable to be used in (4-2) will be defined as

$$u\left(S, q_1, q_2, t\right) = \exp\left\{ -i\omega \left(t - \int_{S_0}^{S} \frac{dS}{V_0} \right) \right\} U(S, q_1, q_2) \qquad (4\text{-}4)$$

Inserting (4-4) into (4-2) results in

$$\omega^2 h \left[\frac{1}{V^2} - \frac{1}{h^2 V_0^2} \right] U + \omega \left[\frac{-i}{h V_0^2} \frac{\partial V_0}{\partial S} U + \frac{2i}{h V_0} \frac{\partial U}{\partial S} + \frac{\partial}{\partial S} \left(\frac{1}{h} \right) \frac{i}{V_0} U \right]$$

$$+ \frac{\partial h}{\partial q_1} \frac{\partial U}{\partial q_1} + \frac{\partial h}{\partial q_2} \frac{\partial U}{\partial q_2} + h \left[\frac{\partial^2 U}{\partial q_1^2} + \frac{\partial^2 U}{\partial q_2^2} \right] + \frac{\partial}{\partial S} \left(\frac{1}{h} \right) \frac{\partial U}{\partial S} + \frac{1}{h} \frac{\partial^2 U}{\partial S^2} = 0 \quad (4\text{-}5)$$

Now in order to utilize ω as a large parameter in (4-5), we want to do a coordinate scaling. This means that we are making a paraxial approximation along the central ray. From experience, we know that in Cartesian coordinates, the simplest form of the parabolic equation is

$$\frac{\partial^2 U}{\partial Z \partial t} = \frac{V}{2} \frac{\partial^2 U}{\partial X^2} \quad (4\text{-}6)$$

If we Fourier transform (4-6) over time, then the following scaling is suggested:

$$X' = x\sqrt{\omega}$$

For such a scaling, the X' coordinate behaves like the Z coordinate. Application of such a scaling to (4-5) results in

$$\omega^2 h \left[\frac{1}{V^2} - \frac{1}{h^2 V_0^2} \right] U + \omega \left[\frac{-i}{h V_0^2} \frac{\partial V_0}{\partial S} U + \frac{2i}{h V_0} \frac{\partial U}{\partial S} + \frac{\partial}{\partial S} \left(\frac{1}{h} \right) \frac{i}{V_0} U + h \frac{\partial^2 U}{\partial \nu_1^2} + h \frac{\partial^2 U}{\partial \nu_2^2} \right]$$

$$+ \omega \left[\frac{\partial h}{\partial \nu_1} \frac{\partial U}{\partial \nu_1} + \frac{\partial h}{\partial \nu_2} \frac{\partial U}{\partial \nu_2} \right] + \frac{\partial}{\partial S} \left(\frac{1}{h} \right) \frac{\partial U}{\partial S} + \frac{1}{h} \frac{\partial^2 U}{\partial S^2} = 0 \quad (4\text{-}7)$$

where $\nu_1 = \sqrt{\omega} q_1$ and $\nu_2 = \sqrt{\omega} q_2$

Now we would like to solve (4-7) as ω tends to infinity, a ray-type approximation. We will only keep terms whose coefficients vary as ω^β, where $\beta \geq 1$. Also, we shall use the following approximations:

1) $h \approx 1$;

2) $\frac{\partial}{\partial s}(\frac{1}{h}) = 0$;

3) A Taylor's expansion of the first coefficient of U in (4-7).

Only the last approximation is somewhat involved, and it rests on the fact that

$$h = 1 + \frac{1}{V} \left[\frac{\partial V}{\partial q_1} q_1 + \frac{\partial V}{\partial q_2} q_2 \right] \quad (4\text{-}8)$$

where the derivatives are evaluated along the central ray. For h along the central ray, we replace V, which depends on all three coordinates, S, q_1 and q_2 by V_0 which depends only on S . Then,

$$hV_o = V_o + \frac{\partial V}{\partial q_1} q_1 + \frac{\partial V}{\partial q_2} q_2 \qquad (4\text{-}9)$$

Similarly, we can express the velocity V as a Taylor's expansion about the central ray as follows.

$$V(S, q_1, q_2) = V_o + \frac{\partial V}{\partial q_1} q_1 + \frac{\partial V}{\partial q_2} q_2 + \frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q} = hV_o + \frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q} \qquad (4\text{-}10)$$

where

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} \frac{\partial^2 V}{\partial q_1^2} & \frac{\partial^2 V}{\partial q_1 \partial q_2} \\ \frac{\partial^2 V}{\partial q_2 \partial q_1} & \frac{\partial^2 V}{\partial q_2^2} \end{pmatrix}$$

Use of (4-9) and (4-10) and replacement of q_1 and q_2 by ν_1 and ν_2 yields (after some algebra) that

$$\omega^2 h \left[\frac{1}{V^2} - \frac{1}{h^2 V_o^2} \right] \approx - \frac{\omega}{h^2 V_o^3} \nu^T \mathbf{V} \nu \qquad (4\text{-}11)$$

Inserting (4-11) and the foregoing approximations for h and its derivatives into (4-7), and then dropping the $\partial^2 / \partial S^2$ term results in the following parabolic equation

$$\frac{2i}{V_o}\frac{\partial U}{\partial S} + \frac{\partial^2 U}{\partial \nu_1^2} + \frac{\partial^2 U}{\partial \nu_2^2} - \left(\frac{\nu^T \mathbf{V} \nu}{V_o^3} + \frac{i}{V_o^2}\frac{\partial V_o}{\partial S} \right) U = 0 \qquad (4\text{-}12)$$

where ν is a column vector with first component ν_1 and second component ν_2.

The complete solution to our wave equation in the ray centered coordinate system, with the parabolic approximation, is given by

$$u(S, q_1, q_2\, t) = \exp \left[-i\omega \left(t - \int_{S_o}^{S} \frac{dS}{V_o} \right) \right] U(S, \nu_1, \nu_2)$$

where U is a solution of (4-12). It is clear, from the foregoing discussion, that the ray-centered coordinate system is simple, elegant and simplifies much algebra. This coordinate system is also a very powerful tool for theoretical investigations, such as the derivation of the parabolic equation in a three dimensional inhomogeneous elastic medium.

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