Ray Tracing and Lagrangians

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In the previous segment on ray tracing we found that for "smooth", inhomogeneous media the ray tracing equations took the form

$$\frac{dx_j}{d\tau} = p_j V^2 \tag{2-1}$$

and

$$\frac{dp_j}{d\tau} = - \frac{\partial}{\partial x_j} ln \ V \tag{2-2}$$

where x_j are the spatial coordinates, p_j are the slownesses, V is the velocity of the propagating wave and τ is the accumulated ray travel time. An interesting result may be obtained by differentiating (2-1) with respect to τ to get

$$\frac{d}{d\tau}\left[\frac{1}{V^2} \frac{dx_j}{d\tau}\right] = \frac{dp_j}{d\tau} = - \frac{\partial}{\partial x_j} ln \ V \ , \quad j = 1,2,3 \ . \tag{2-3}$$

Equation (2-3) is a second order differential equation which can be derived from a variational principle, as shown below.

Consider the Fermat functional

$$\int_a^b \frac{dS}{V} \tag{2-4}$$

where dS is an incremental arc length, V is the velocity, and a and b mark the beginning and end of a piece of arc. The functional, of course, is the travel time, and the particular arc which minimizes the travel time is the ray from a to b . We can rewrite (2-4) by introducing

$$G = \frac{1}{V}\left[\frac{dx_j}{d\tau} \frac{dx_j}{d\tau}\right]^{1/2} \quad j = 1,2,3$$

where $V = V(x_j)$ and the sum over j is understood. The quantity

$$\left[\frac{dx_j}{d\tau}\frac{dx_j}{d\tau}\right]^{1/2}\cdot d\tau = dS \qquad (2\text{-}5)$$

so (2-4) can be rewritten as

$$\int_a^b \frac{dS}{V} = \int_a^b G\left[x_j, \frac{dx_j}{d\tau}, \tau\right] d\tau \qquad (2\text{-}6)$$

The particular path which minimizes the Fermat functional is obtained from the calculus of variations and is given by

$$\frac{d}{d\tau}\frac{\partial G}{\partial \dot{x}_j} - \frac{\partial G}{\partial x_j} = 0 \qquad j = 1, 2, 3 \qquad (2\text{-}7)$$

Now

$$-\frac{\partial G}{\partial x_j} = -\left[\dot{x}_k\dot{x}_k\right]^{1/2}\frac{\partial}{\partial x_j}\frac{1}{V} \qquad (2\text{-}8)$$

and

$$\frac{\partial G}{\partial \dot{x}_j} = -\frac{1}{V}\frac{\partial}{\partial x_j}\left[\dot{x}_k\dot{x}_k\right]^{1/2} = \frac{1}{V}\frac{\dot{x}_j}{[\dot{x}_k\dot{x}_k]^{1/2}} \qquad (2\text{-}9)$$

where $\dot{x}_j = dx_j/d\tau$, and again summation convention over the double indices is understood. By noting that $\left[\dot{x}_k\ \dot{x}_k\right]^{1/2}$ is simply V, the magnitude of the ray velocity, which is the same as the "material" velocity, (2-9) becomes

$$\frac{\partial G}{\partial \dot{x}_j} = \frac{1}{V^2}\frac{dx_j}{d\tau} \qquad (2\text{-}10)$$

Inserting (2-10) and (2-8) into (2-7), we obtain

$$\frac{d}{d\tau}\left[\frac{1}{V^2}\frac{dx_j}{d\tau}\right] = -\frac{\partial}{\partial x_j}\ln V\ , \qquad (2\text{-}11)$$

which is identical to (2-3). Thus, we have shown that the extremals of the travel time functional are simply the rays, which we obtained using local conservation principles in Section I.

In analogy to classical mechanics, G can be identified as L, the Lagrangian. The coordinates x_j are the equivalents of the generalized coordinates q_j. The generalized momenta are obtained by differentiation, i.e.

$$m_j = \frac{\partial L}{\partial \dot{x}_j} = \frac{\dot{x}_j}{V^2}$$

where m_j are the generalized momenta, and of course are the slownesses which we previously denoted by p_j .

The above identification of geometrical optics with mechanics provides us with a powerful theoretical tool for many ray investigations. For example, given the foregoing construction of the Lagrangian and generalized momenta, we can immediately write down a Hamiltonian, which is given by

$$H = m_j \dot{x}_j - G \tag{2-12}$$

or,

$$H = V^2 p_j p_j - V(p_j p_j)^{1/2} \tag{2-13}$$

With the Hamiltonian given above, we obtain the ray tracing equations exactly as before, with

$$\frac{dx_j}{d\tau} = \frac{\partial H}{\partial p_j} \text{ and } \frac{dp_j}{d\tau} = -\frac{\partial H}{\partial x_j} \quad j = 1, 2, 3$$