Ray Tracing Equations In Three Dimensions

Mathew J. Yedlin

The following four lectures serve as an introduction to ray tracing. They are based mainly on the material quoted in the references. The purpose of these lectures is a tutorial one, in that they serve to elucidate the basic material.

In the first lecture the equations for ray tracing in a three-dimensional inhomogeneous elastic medium are derived from a conservation principle. This principle maintains that locally the wave crest density must be conserved. The second lecture deals with ray tracing and Lagrangians. The connection between the ray tracing equations and a variational principle is demonstrated. The ray tracing equations are then derived again using Hamilton's equations. There is a slight shift in emphasis in the third lecture from physics to geometry. An elegant ray-centered coordinate system is introduced. This coordinate system relies on the differential geometry of space curves. Thus, a derivation of Frénet's formulae is included. Finally, all the previous material is applied to the wave equation. It is first written in the new coordinate system. Then a high frequency approximation is used to develop the parabolic equation, in which the extrapolation direction is along a particular ray. It is very similar to the well known equation applied in migration, and differs only in the inclusion of two additional connection terms.

One Dimensional Ray Equations

In all of what follows, the most important definition encountered is that of the phase of a traveling wave. To be more specific, we shall require the phase τ to be local. Its value could be some multiple of 2π, if the wave had advanced an integral number of crests. For the one-dimensional example, the phase will be a function of x and t. Since the phase is a smoothly varying function of x and t, we can define its local derivatives. The rate of change of τ with respect to t locally is 2π /T, where T is the wave period. Similarly, the rate of change of τ with respect to x is 2π / λ. In summary,

$$\frac{\partial \tau}{\partial t} = \frac{2\pi}{T} = \omega \qquad \frac{\partial \tau}{\partial x} = -\frac{2\pi}{\lambda} = -k \tag{1-1}$$

Notice the appearance of a minus sign in the second of the equations in (1-1). This implies that the phase decreases with x by an amount 2π, between crests a distance of λ apart. The minus sign follows since late arriving crests have a large value for τ. Regardless of the sign convention chosen for our values for the local phase derivatives, the ray tracing equations should remain invariant.

With the phase properly defined, we can write down an expression for a wave disturbance as:

$$u(x,t) = \hat{u}(x,t)e^{i\tau(x,t)}$$

where $\hat{u}$ is a slowly varying amplitude function. Near a specific (x_o, t_o), we can expand the phase, τ as a Taylor series as follows

$$\tau = \tau_o + \left.\frac{\partial \tau}{\partial x}\right|_{x_o,t_o} (x - x_o) + \left.\frac{\partial \tau}{\partial t}\right|_{x_o,t_o} (t - t_o). \tag{1-2}$$

From Eq. (1 -1),

$$\left.\frac{\partial \tau}{\partial x}\right|_{x_o,t_o} = -k_o \qquad \left.\frac{\partial \tau}{\partial t}\right|_{x_o,t_o} = \omega_o$$

Therefore, u is nearly a plane wave with a local wavenumber k_o and frequency ω_o.

The above definition of the phase and its derivatives yields a very nice result. It is derived by differentiating the local wavenumber with respect to time and the local frequency with respect to x. From (1-1), we obtain

$$\frac{\partial k}{\partial t} = -\frac{\partial^2 \tau}{\partial x \partial t} \qquad \frac{\partial \omega}{\partial x} = \frac{\partial^2 \tau}{\partial x \partial t}$$

or

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{1-3}$$

Equation (1-3) is both a conservation and continuity equation. It states that the time rate of change of phase per unit length is related to the spatial rate of change of phase per unit time. Stated otherwise, the net phase-crest density is conserved.

The conservation equation is a first order partial differential equation which may be solved by the method of characteristics. Generally, there may exist a dispersion relation or equation of state which relates ω to k . Then by the chain rule,

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial k} \frac{\partial k}{\partial x}$$

Equation (1-3) then becomes

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial k} \frac{\partial k}{\partial x} = 0. \tag{1-4}$$

Equation (1-4) can be written in simpler form, if we notice that on the set of curves defined by

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k}, \tag{1-5}$$

$$\frac{dk}{dt} = 0. \tag{1-6}$$

The curves defined by the differential equation in (1-5) are called characteristic curves, and in geometrical optics, they represent the rays. Note also that the slope of the characteristic curves equals the group velocity. Physically, this means that if you wish to observe waves with the same local wavenumber (or wavelength) , you must focus on a point traveling with the group velocity $\partial \omega / \partial k$. Finally, it is important to realize that if the dispersion relation is of the form $\omega(k,x)$, the characteristic equation (1-5) no longer represents straight lines. Therefore, we have created the phenomenon of refraction.

Three Dimensional Ray Tracing

Ray tracing in three dimensions can be readily derived by following the cues obtained in solving the one dimensional problem. Again a local phase function τ is defined such that a local frequency and local wavenumber may be assigned at each point of space and time. To do this it is required that the wavenumber k vary slowly on a length scale equivalent to a wavelength. The local wavenumber and frequency is defined as before:

$$\frac{\partial \tau(x_i,t)}{\partial t} = \omega(x_i,t) \tag{1-7}$$

and

$$\frac{\partial \tau(x_i,t)}{\partial x_i} = -k_i(x_i,t) \qquad i = 1, 2, 3 \tag{1-8}$$

Equations (1-7) and (1-8) are identical to those in Eq. (1-1), except that there are three spatial coordinates instead of one. By differentiating (1-7) with respect to x_i , (1-8) with respect to t , and adding the results, we again obtain the conservation equation,

$$\frac{\partial \omega}{\partial x_i} + \frac{\partial k_i}{\partial t} = 0. \tag{1-9}$$

We also obtain from (1-8) that

$$\frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0 \tag{1-10}$$

It is most important to note that a partial derivative with respect to x_i in (1-9) means that time and the other x_i are held fixed.

Now hypothesize a known dispersion relation of the form:

$$\omega = W(k_1, k_2, k_3, x_1, x_2, x_3) \tag{1-11}$$

With the dispersion relation defined in (1-11), we calculate $\partial\omega / \partial x_i$ which is given by

$$\frac{\partial W}{\partial x_i} + \frac{\partial W}{\partial k_j}\frac{\partial k_j}{\partial x_i} \tag{1-12}$$

Substituting into the conservation equation (1-9), we obtain

$$\frac{\partial k_i}{\partial t} + \frac{\partial W}{\partial k_j}\frac{\partial k_j}{\partial x_i} = -\frac{\partial W}{\partial x_i}. \tag{1-13}$$

From (1-10) we can rewrite (1-13) as

$$\frac{\partial k_i}{\partial t} + \frac{\partial W}{\partial k_j}\frac{\partial k_i}{\partial x_j} = -\frac{\partial W}{\partial x_i} \tag{1-14}$$

As before, we notice that (1-14) can be written as an ordinary differential equation if we make the identification that

$$\frac{dx_j}{dt} = \frac{\partial W}{\partial k_j} \tag{1-15}$$

Then equation (1-14) becomes

$$\frac{dk_i}{dt} = -\frac{\partial W}{\partial x_i} \tag{1-16}$$

provided equation (1-15) is satisfied. The coupled system, Eqs. (1-15) and (1-16) determines our rays. Equation (1-15) defines the actual ray paths. Along those ray paths, (1-16) determines how the local wavenumber changes due to the direct dependence of the dispersion relation on the spatial coordinates. This means that the wave is being refracted, and the rays are no longer straight lines, but some general space curves.

Three Dimensional Ray Tracing Equations.

We shall now focus on a particular form for W. It is well known that the dispersion relation for a three- dimensional acoustic medium is

$$\omega = v\sqrt{k_1^2 + k_2^2 + k_3^2} = W(k_1, k_2, k_3, x_1, x_2, x_3) . \tag{1-17}$$

where v , the velocity, is a function of all three spatial coordinates. Then, for the j^{th} component of the ray velocity, we obtain

$$\frac{\partial W}{\partial k_j} = \frac{vk_j}{\sqrt{k_1^2 + k_2^2 + k_3^2}} = \frac{v^2k_j}{v\sqrt{k_1^2 + k_2^2 + k_3^2}} = \frac{v^2k_j}{\omega} = p_jv^2 \tag{1-18}$$

where $p_j = \frac{k_j}{\omega}$. Thus, for the dispersion relation chosen, (1-15) becomes

$$\frac{dx_j}{d\tau} = p_jv^2 . \tag{1-19}$$

where τ is the travel time along the ray.

Similarly, we find that

$$\frac{dk_j}{d\tau} = -\frac{\partial W}{\partial x_j} = -\frac{\partial v}{\partial x_j}\sqrt{k_1^2 + k_2^2 + k_3^2} \tag{1-20}$$

Equation (1-20) can be rewritten in neater form by setting $k_j = \omega p_j$, and by noting that ω doesn't depend on time. Therefore, we obtain

$$\omega\frac{dp_j}{d\tau} = -\frac{\partial v}{\partial x_j}\frac{v\sqrt{k_1^2 + k_2^2 + k_3^2}}{v} = -\frac{\omega}{v}\frac{\partial v}{\partial x_j}$$

or

$$\frac{dp_j}{d\tau} = -\frac{\partial}{\partial x_j}ln\ v \tag{1-21}$$

Equations (1-19) and (1-21) are the ray tracing equations for our three-dimensional acoustic medium. In an elastic medium, there are two dispersion relations of the form of (1-17), with two different velocities. Each ray system (P or S) would have to be traced separately, with the appropriate interface condition applied, when a boundary is encountered.