# Dynamic Ray Tracing Across Curved Interfaces

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## 1. Introduction

Any method of computing seismic wave fields in inhomogeneous media is not complete when it does not consider the structural interfaces in the medium. It is well known that the velocity and its spatial derivatives are not smooth functions of coordinates in the earth's interior. The discontinuities of velocity play an important role both in seismology and in seismic prospecting. They are often of rather complex shape and physical properties.

In the following, we shall try to find a way of performing the dynamic ray tracing across curved interfaces. First we shall consider just one interface. Then we shall generalize the results for any multiply reflected/transmitted waves (possibly converted).

We shall use many results from four previous papers (Červený 1981a, 1981b, 1981c, 1981d, hereafter denoted by I, II, III, IV). For simplicity, we shall refer to the equations and figures from these papers in a shortened way, e.g. Eq.(II-17), Fig. II-3, etc.

Let us again select an arbitrary ray $\Omega$ and use the ray-centered coordinates $(s,n)$ corresponding to this ray. The coordinate $s$ measures the arclength along the ray, from an arbitrary reference point, and $n$ is a length coordinate in the direction perpendicular to $\Omega$ at $s$. In this system, the travel-time field $\tau(s,n)$ in the vicinity of $\Omega$ can be computed using a simple equation,

$$\tau(s,n) \approx \tau(s,0) + \frac{1}{2} M(s) n^2 \ , \tag{1}$$

where M(s) is a solution of the *dynamic ray tracing equation*

$$\frac{dM(s)}{ds} + vM^2 + \frac{v_{,nn}}{v} = 0 \ , \tag{2}$$

(see (I-22)). Here $v$ is the velocity and $v_{,nn}$ its second derivative with respect to $n$, measured directly at $\Omega$.

An alternative form of (2) can be obtained from (2) when we put $M(s) = p(s)/q(s)$, with $p(s) = q_{,s}/v(s)$. Then the non-linear dynamic ray tracing equation (2) leads to two simple linear ordinary differential equations of first order for $p$ and $q$ ,

$$\frac{dq}{ds} = vp \ , \qquad \frac{dp}{ds} = -\frac{v_{,nn}}{v^2} q \ , \tag{3}$$

(see (I-27)).

The physical meaning of individual quantities and possible applications of dynamic ray tracing were discussed in [I-IV]. Here we shall study the problem of dynamic ray tracing along the ray $\Omega$ which crosses an interface.

We shall call the interface, across which the velocity is discontinuous, the interface of first order. Similarly, the interface across which the gradient of velocity is discontinuous (but the velocity remains continuous) will be called the interface of second order. Similarly, we can define even interfaces of higher order.

## 2. Phase matching method

Assume that the ray $\Omega$ strikes a curved interface $\Sigma$ at the point $O$ (see Fig. 1). As is well known, four new vaves are then generated at the interface, namely reflected and transmitted P and S waves.

The *phase matching* method requires that the phase function (travel time) of incident, reflected and transmitted waves are equal along $\Sigma$. We shall use this method and derive the new initial conditions for reflected/transmitted waves at $O$ for quantities $M$, $q$ and $p$ in the dynamic ray tracing equations. As the derivation would be the same for all four generated waves, we shall consider just one of them, e.g. one of the transmitted waves. There will be only some difference between transmitted and reflected waves in the signs of certain terms. In this case we shall use two signs, the upper always applies to the transmitted wave, the lower to the reflected wave.

We shall introduce the following notation: We denote the angle of incidence by $\vartheta$ , the angle of transmission (or reflection) by $\tilde{\vartheta}$ . We understand that these angles are acute angles between the tangent of the ray $\Omega$ and the normal to the interface $\Sigma$ at $O$ (see Fig. 1).
We emphasize that the angle $i$ we introduced in [II] is different from $\vartheta$ and $\tilde{\vartheta}$ , as the normal to $\Sigma$ at $O$ does not generally have the direction of the z-axis. Even if it has this direction, the angle $i$ is measured from 0 to $2\pi$ , but $\vartheta$ and $\tilde{\vartheta}$ only from 0 to $\frac{\pi}{2}$ .

In order to distinguish between the symbols related to the incident wave and to the reflected/transmitted wave at the point $O$, the following notation is used: The tilde above

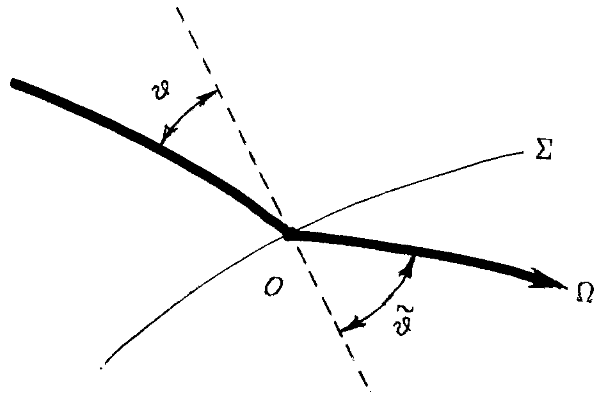


FIG. 1.

the symbol denotes the quantities pertinent to the reflected/transmitted waves, whereas the plane symbols refer to the incident wave. Thus we have, for the incident wave: $\vartheta, v, v_{,m}, v_{,l}, q, p, M$; for reflected/transmitted waves: $\tilde{\vartheta}, \tilde{v}, \tilde{v}_{,m}, \tilde{v}_{,l}, \tilde{q}, \tilde{p}, \tilde{M}$. Here $v_{,m}$ and $v_{,l}$ are the derivatives of velocity in the direction along the tangent to the ray at $O$ and in the direction perpendicular to it, respectively (see also Fig. II-3). Let us again emphasize that the unit vector $\vec{m}$ is oriented for all waves (incident, reflected, transmitted) to the left from $\vec{l}$ (i.e., to the left from the tangent to the ray), so that the movement from $\vec{l}$ to $\vec{m}$ is always counter-clockwise (see Fig. 2).

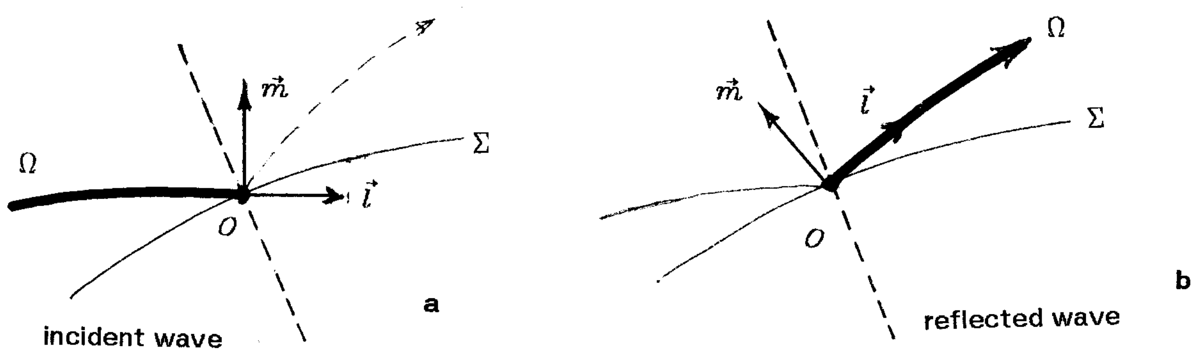


FIG. 2.

The case of a transmitted wave is not shown in Fig. 2; the situation is quite analogous to the incident wave in this case.

Now we shall introduce a new auxiliary Cartesian coordinate system at $O$, with one axis tangent to $\Sigma$ at $O$, and the second perpendicular to $\Sigma$ at $O$. To be able to use the formulae derived earlier, we shall introduce it in the same way as we introduced the rectangular system $(x,z)$ in [II] (see Fig. II-3). We shall, however, use a new notation for the system, since we wish to keep the notation $(x,z)$ for the absolute fixed Cartesian coordinate system ($z$...depth, $x$...length coordinate along the profile). We shall denote the new auxiliary Cartesian coordinate system by $(g,h)$, with unit basis vectors $\vec{g}$, $\vec{h}$ (see Fig. 3).

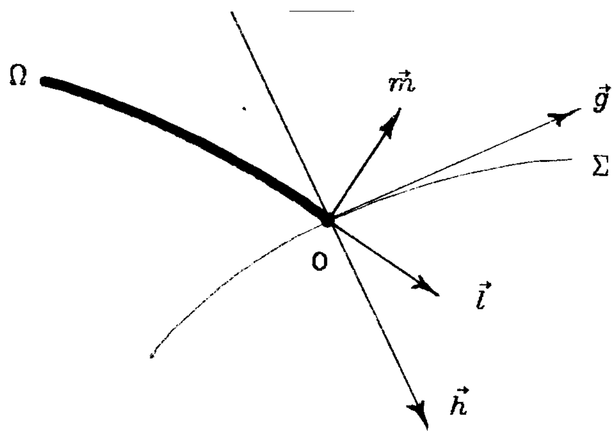


FIG. 3.

The unit vector $\vec{h}$ is the normal vector to the interface $\Sigma$ at $O$; it makes an acute angle with the unit vector $\vec{l}$ corresponding to the incident wave (see Fig. 3). The unit vector $\vec{g}$ is oriented along the tangent to the interface $\Sigma$ at $O$, in such a way that it makes an acute angle with the unit tangent $\vec{l}$ corresponding to the incident wave.

Now we shall use the formulae (II-17) and (II-18) we derived for an arbitrary rotated Cartesian coordinate system. It is obvious that we can put $x-x_0 = g$ , $z-z_0 = h$ , $i = \vartheta$ for the incident wave, $i = \tilde{\vartheta}$ for the transmitted wave, but $i = \pi - \tilde{\vartheta}$ for the reflected waves. Thus, we shall write

$$\sin i = \sin \tilde{\vartheta} \qquad \cos i = \pm \cos \tilde{\vartheta} \ . \tag{4}$$

Here the upper sign corresponds to the transmitted wave, the lower to the reflected wave.

Then we can rewrite Eq.(II-17) for the travel-time field in the neighborhood of $O$ as follows

$$\tau(g,h) \;\approx\; \tau(0,0) + \frac{1}{v} g \sin\vartheta \pm \frac{1}{v} h \cos\vartheta + \frac{1}{2}\mathbf{g}^T \mathbf{W}\,\mathbf{g} \;, \tag{5}$$

where

$$\mathbf{g} = \begin{pmatrix} g \\ h \end{pmatrix} \;,\quad \mathbf{W} = \frac{1}{v^2}\begin{pmatrix} \pm\cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \pm\cos\vartheta \end{pmatrix}\begin{pmatrix} v^2 M & -v_{,m} \\ -v_{,m} & -v_{,l} \end{pmatrix}\begin{pmatrix} \pm\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \pm\cos\vartheta \end{pmatrix} \;, \tag{6}$$

and $\mathbf{g}^T$ is the transpose of $\mathbf{g}$. This equation is applicable for all the waves (incident, reflected, transmitted). For incident and transmitted waves, we again use the upper sign, for the reflected wave the lower sign. For reflected and transmitted waves, all the relevant quantities in (5) and (6) should be denoted by a tilde.

Now we shall find the expression for $\tau$ along the interface $\Sigma$, in the vicinity of $O$. We shall specify the interface $\Sigma$ by the equation $h = h(g)$. In the neighborhood of $O$, we can use the Taylor expansion and write, with accuracy up to second order in $h$ and $g$,

$$h \;\approx\; \frac{1}{2}\left[\frac{\partial^2 h}{\partial g^2}\right]_{h=g=0} \cdot g^2 \;=\; \frac{1}{2} G\, g^2 \;, \tag{7}$$

where we put $G = [\partial^2 h/\partial g^2]_{h=g=0}$. It is simple to recognize that $G$ is the curvature of the interface $\Sigma$ at $O$. We also see that $G$ is positive for those parts of the interface $\Sigma$ that are seen as convex by an observer located in the medium containing the incident wave (see Fig. 3). Inserting (7) into (5) yields, with accuracy up to second order in $g$ and $h$,

$$\tau_\Sigma \;\approx\; \tau(0,0) + \frac{1}{v} g \sin\vartheta + \frac{1}{2} g^2 \left[\pm\frac{1}{v} G\cos\vartheta + W_{11}\right] \;, \tag{8}$$

as all other terms are of higher order. For $W_{11}$ we have (see (6)),

$$W_{11} \;=\; M\cos^2\vartheta \mp 2\frac{v_{,m}}{v^2}\sin\vartheta\cos\vartheta - \frac{v_{,l}}{v^2}\sin^2\vartheta \;. \tag{9}$$

Inserting (9) into (8) yields the final formula for the travel-time field $\tau_\Sigma$ along the interface $\Sigma$,

$$\tau_\Sigma \;\approx\; \tau(0,0) + \frac{1}{v} g \sin\vartheta + \frac{1}{2} g^2 F \;, \tag{10}$$

where

$$F \;=\; M\cos^2\vartheta \pm G\,\frac{\cos\vartheta}{v} \mp 2\frac{v_{,m}}{v^2}\sin\vartheta\cos\vartheta - \frac{v_{,l}}{v^2}\sin^2\vartheta \;. \tag{11}$$

Eq. (10) gives the travel-time field $\tau$ along $\Sigma$ in the vicinity of $O$, both for incident and reflected/transmitted waves. For reflected/transmitted waves, all of the quantities should be written with tildes. For incident and transmitted waves, we again take the upper sign, for reflected waves the lower sign.

The phase matching method requires that $\tau_\Sigma$ be the same for incident and for reflected/transmitted waves. Thus, we obtain from (10),

$$\frac{1}{v}\sin\vartheta = \frac{1}{\tilde{v}}\sin\tilde{\vartheta} \ , \tag{12}$$

$$F = \tilde{F} \ . \tag{13}$$

Equation (12) is the well-known Snell's Law. We have only derived it in a different way than it is usually done. The second equation is more important for us. Inserting (11) into (13) yields

$$M\cos^2\vartheta + G\,\frac{\cos\vartheta}{v} - 2\frac{v_{,m}}{v^2}\sin\vartheta\cos\vartheta - \frac{v_{,l}}{v^2}\sin^2\vartheta =$$

$$\tilde{M}\cos^2\tilde{\vartheta} \pm G\,\frac{\cos\tilde{\vartheta}}{\tilde{v}} \mp 2\frac{\tilde{v}_{,m}}{\tilde{v}^2}\sin\tilde{\vartheta}\cos\tilde{\vartheta} - \frac{\tilde{v}_{,l}}{\tilde{v}^2}\sin^2\tilde{\vartheta} \ . \tag{14}$$

From this we easily obtain the final result -- an equation for $\tilde{M}$ ,

$$\tilde{M} = M\,\frac{\cos^2\vartheta}{\cos^2\tilde{\vartheta}} + \left\{ G\left[\frac{\cos\vartheta}{v} \mp \frac{\cos\tilde{\vartheta}}{\tilde{v}}\right] - 2\,\frac{\sin\vartheta}{v}\left[\frac{v_{,m}}{v}\cos\vartheta \mp \frac{\tilde{v}_{,m}}{\tilde{v}}\cos\tilde{\vartheta}\right] - \frac{\sin^2\vartheta}{v^2}(v_{,l} - \tilde{v}_{,l})\right\}\frac{1}{\cos^2\vartheta} \tag{15}$$

This equation will be discussed in greater detail in the following sections.

It would also be possible to replace the velocity derivatives $v_{,l}$ , $v_{,m}$ , $\tilde{v}_{,l}$ and $\tilde{v}_{,m}$ by the derivatives $v_{,x}$ , $v_{,z}$ , $\tilde{v}_{,x}$ and $\tilde{v}_{,z}$ , determined in the fixed Cartesian coordinate system $(x,z)$ . For this purpose, it would be possible to use Eqs. (II-19), where $i$ is defined as shown in Figs. II-2 and II-3. We shall not, however, do it here.

### 3. Discussion of Equation (15) for $\tilde{M}$

Eq. (15) shows that the function $M$ changes discontinuously across the interface. As soon as the ray $\Omega$ strikes an interface, Eq. (15) must be used to determine a new $\tilde{M}$ for reflected and/or transmitted waves.

To make Eq. (15) more understandable, we shall rewrite it in a slightly different form. We denote

$$C = G\left(\frac{\cos\vartheta}{v} \mp \frac{\cos\tilde{\vartheta}}{\tilde{v}}\right)\frac{1}{\cos^2\tilde{\vartheta}} , \qquad (16)$$

$$I = -\frac{\sin\vartheta}{v\cos^2\tilde{\vartheta}}\left[2\left(\frac{v_{,m}}{v}\cos\vartheta \mp \frac{\tilde{v}_{,m}}{\tilde{v}}\cos\tilde{\vartheta}\right) + \frac{\sin\vartheta}{v}(v_{,l}-\tilde{v}_{,l})\right] , \qquad (17)$$

and call $C$ the *interface curvature term* and $I$ the *inhomogeneity term*. Then we have (see (15)),

$$\tilde{M} = M\frac{\cos^2\vartheta}{\cos^2\tilde{\vartheta}} + C + I . \qquad (18)$$

When the interface $\Sigma$ is a plane in the neighborhood of $O$, the term $C$ vanishes. Similarly, when the medium in which the incident ray and the reflected/transmitted ray under consideration are situated is homogeneous in the neighborhood of $O$, the term $I$ vanishes. Thus, in the case of a plane interface between two homogeneous media, we have the simple relation

$$\tilde{M} = M\frac{\cos^2\vartheta}{\cos^2\tilde{\vartheta}} . \qquad (19)$$

Eq. (18) can be simplified even in certain other situations. For example, for a reflected unconverted wave we have $C = 2G/(v\cos\vartheta)$, and the expression for $I$ is also simpler. In the case of the interface of second order, we have a simple expression for the transmitted unconverted wave,

$$\tilde{M} = M + I , \qquad (20)$$

as $\vartheta = \tilde{\vartheta}$ , $C = 0$ . Even the expression for $I$ can be, of course, simplified in this case. We leave all these special cases as an exercise for the reader.

From (18), we immediately obtain corresponding formulae for the curvature of the wavefront $\tilde{K}$, when we use the relation $K = vM$ , $\tilde{K} = \tilde{v}\tilde{M}$ (see IV - 15). We obtain

$$\tilde{K} = K\frac{\tilde{v}\cos^2\vartheta}{v\cos^2\tilde{\vartheta}} + \tilde{v}C + \tilde{v}I . \qquad (21)$$

The discussion of this equation is again left to the reader.

**4. Quantities $q$ and $p$ across curved interfaces**

It is not difficult to recognize how the quantity $q$ changes across the interface. It was shown in [III] that the dynamic ray tracing system (3) corresponds also to the ray tracing system for rays $\Omega_c$ which are close to $\Omega$. The quantity $q$ has in this case the meaning of normal distance $n$ of the ray $\Omega_c$ from $\Omega$. It is simple to see how $n$ changes during the process of reflection/transmission (see Fig. 4). By direct inspection of both pictures, we obtain $\tilde{n}/n = \pm\cos\tilde{\vartheta}/\cos\vartheta$, as the length $OO'$ is the same for the two triangles $OO'O_1$ and $OO'O_2$. In the case of reflection, the ray $\Omega_c$ shifts to the other side of $\Omega$, so that there is a minus sign.

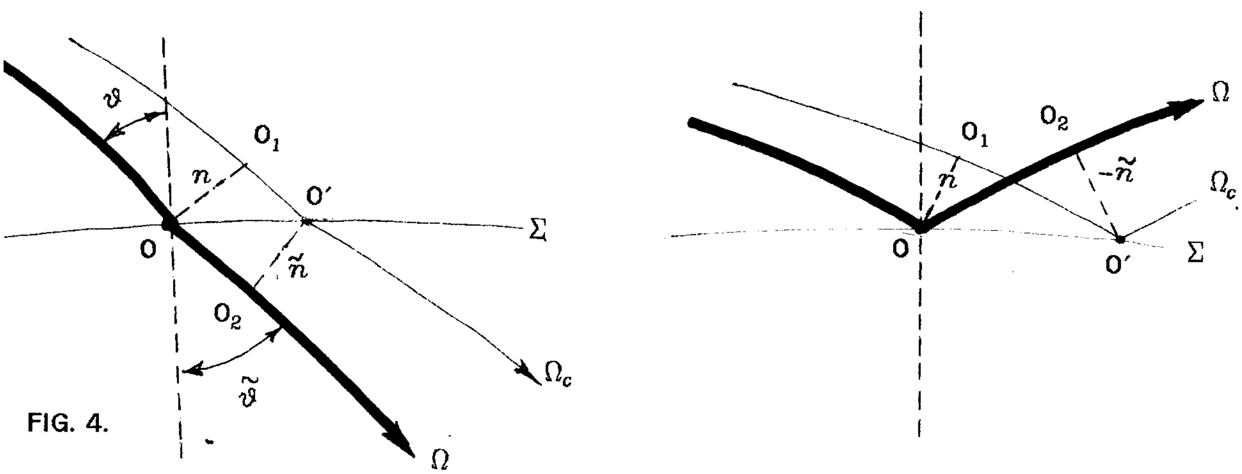


FIG. 4.

Returning to $q$, we have

$$\tilde{q} = \pm q \frac{\cos\tilde{\vartheta}}{\cos\vartheta} . \tag{22}$$

Now we determine $\tilde{p}$. Inserting $M = p/q$ and $\tilde{M} = \tilde{p}/\tilde{q}$ into (18) yields

$$\frac{\tilde{p}}{\tilde{q}} = \frac{p}{q}\frac{\cos^2\vartheta}{\cos^2\tilde{\vartheta}} + C + I . \tag{23}$$

Using (22) we obtain

$$\tilde{p} = \pm \frac{\cos\tilde{\vartheta}}{\cos\vartheta}\left\{ p \frac{\cos^2\vartheta}{\cos^2\tilde{\vartheta}} + (C + I)q \right\} . \tag{24}$$

When we denote, as in (I-29)

$$\mathbf{X} = \begin{pmatrix} q \\ p \end{pmatrix} \quad \tilde{\mathbf{X}} = \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} , \tag{25}$$

we can rewrite (22) and (24) in matrix form

$$\tilde{\mathbf{X}} = \pm \frac{\cos \tilde{\vartheta}}{\cos \vartheta} \mathbf{A}\,\mathbf{X} \tag{26}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ C+I & \cos^2 \vartheta / \cos^2 \tilde{\vartheta} \end{bmatrix} . \tag{27}$$

**5. Dynamic ray tracing across an interface**

Equation (26) expresses the jump in the quantities $p$ and $q$ directly at the point $O$ at the interface $\Sigma$. In this section, we shall determine $p$ and $q$ at $S$, when we assume that we know them at $S_0$ (see Fig. 5). It is possible to solve this problem step-by-step: first from $S_0$ to $O$, then across the interface, and again from $O$ to $S$. We shall, however, try to find some solution in explicit form.

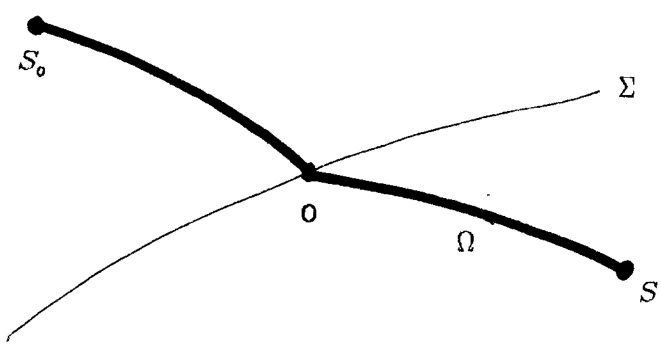


FIG. 5.

Let us now for a while return to our dynamic ray tracing system in the matrix form (I-28), $d\mathbf{X}/ds = \mathbf{C}\,\mathbf{X}$, where $\mathbf{C}$ is given by (I-29). Let us specify two linearly independent solutions by initial conditions $q_1(s_0) = 1$, $p_1(s_0) = 0$ and $q_2(s_0) = 0$, $p_2(s_0) = 1$. We denote the matrix of two linearly independent solutions of the dynamic ray tracing system (I-28) with the initial conditions indicated above by $\mathbf{B}(s, s_0)$. The matrix $\mathbf{B}$ is a function not only of $s$, but also of $s_0$, as the initial conditions are specified at that point. It is obvious that

$$\mathbf{B}(s_0,s_0) = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{28}$$

Each column of **B** is one linearly independent solution of the equation $d\mathbf{X}/ds = \mathbf{C}\,\mathbf{X}$ .

For arbitrary initial conditions specified by the following equation,

$$\mathbf{X}(s_0) = \begin{pmatrix} q(s_0) \\ p(s_0) \end{pmatrix} , \tag{29}$$

we obtain the solution of (I-28) at $s$ as

$$\mathbf{X}(s) = \mathbf{B}(s,s_0)\,\mathbf{X}(s_0) .$$

Now we can solve easily our problem. To make the following formulae clearer, we shall write in the arguments of individual matrices directly the points, not the ray-centered coordinate $s$ (e.g. $\mathbf{B}(O,S_0)$ ).

Then, we can start at $S_0$ with initial conditions $\mathbf{X}(S_0)$ . At $O$ we have

$$\mathbf{X}(O) = \mathbf{B}(O,S_0)\,\mathbf{X}(S_0) . \tag{30}$$

At the other side of the interface $\Sigma$ , but still at $O$ , we have

$$\tilde{\mathbf{X}}(O) = \pm \frac{\cos \tilde{\vartheta}(O)}{\cos \vartheta(O)} \mathbf{A}(O)\,\mathbf{B}(O,S_0)\,\mathbf{X}(S_0) , \tag{31}$$

and at $S$ ,

$$\mathbf{X}(S) = \pm \frac{\cos \tilde{\vartheta}(O)}{\cos \vartheta(O)} \mathbf{B}(S,O)\,\mathbf{A}(O)\,\mathbf{B}(O,S_0)\,\mathbf{X}(S_0) . \tag{32}$$

This is the final expression for the quantities $q(s)$ and $p(s)$ at the point $S$ , when we know the same quantities at the point $S_0$ .

## 6. Multi-layered medium

The above formulae (32) can be simply generalized for seismic waves in multi-layered media. Denote the initial point by $O_0$ , the points of reflection/transmission by $O_1$ , $O_2$ , $\cdots$ , $O_N$ and the final point by $O_{N+1}$ . Thus, we have $N$ points of reflection/transmission (see Fig. 6).

Then we obtain by successive application of (32) from one interface to another,

$$\mathbf{X}(O_{N+1}) = \prod_{j=1}^{N} \left\{ (-1)^{\varepsilon_j} \frac{\cos \tilde{\vartheta}(O_j)}{\cos \vartheta(O_j)} \right\} \mathbf{B}(O_{N+1},O_N) \prod_{j=1}^{N} \left\{ \mathbf{A}(O_j)\,\mathbf{B}(O_j,O_{j-1}) \right\} \mathbf{X}(O_0) . \tag{33}$$

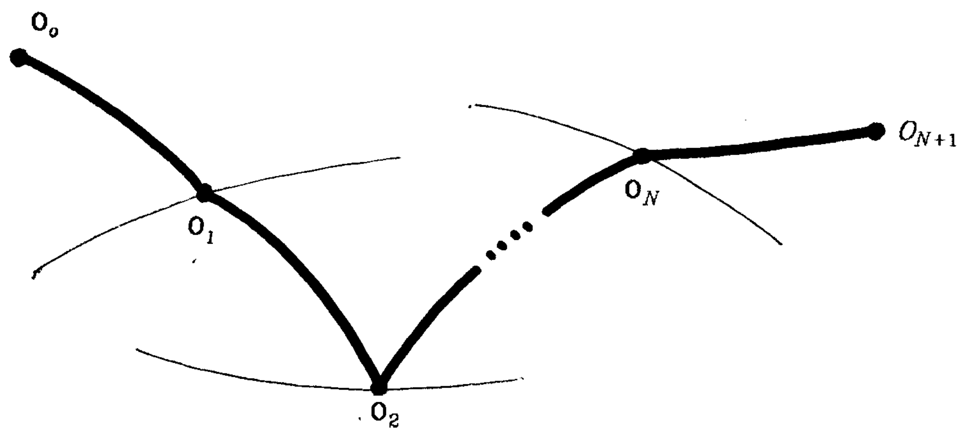


FIG. 6.

Here $\varepsilon_j = 1$ for the reflection at $O_j$ ; $\varepsilon_j = 0$ for the transmission at $O_j$ .

This is the final equation for dynamic ray tracing in a 2-D medium with an arbitrary number of interfaces. It is applicable to any multiply-reflected, P, S, or converted wave (including refracted waves, of course). It may be used in any application discussed in previous papers [I-IV] (i.e. in the evaluation of the travel-time field in the neighborhood of $\Omega$, in ray tracing in the neighborhood of $\Omega$, and in the evaluation of the function $J$ and geometrical spreading). It will be shown later that the same formula is directly applicable in the evaluation of 2-D Gaussian beams.

To solve any of the above-mentioned problems, we must properly specify the initial conditions at $O_0$ (see $\mathbf{X}(O_0)$ in (33)). To obtain both two linearly independent solutions with the initial conditions given by the matrix $\mathbf{I}$ (see (28)), we get from (33),

$$\mathbf{B}(O_{N+1},O_0) = \prod_{j=1}^{N} \left\{ (-1)^{\varepsilon_j} \frac{\cos\tilde{\vartheta}(O_j)}{\cos\vartheta(O_j)} \right\} \mathbf{B}(O_{N+1},O_N) \prod_{j=1}^{N} \left\{ \mathbf{A}(O_j)\,\mathbf{B}(O_j,O_{j-1}) \right\} . \qquad (34)$$

When we compute the function $J$, we put $q(O_0) = 0$ , $p(O_0) = 1/v(O_0)$ for a line source and $q(O_0) = 1$ , $p(O_0) = 0$ for a plane source in $\mathbf{X}(O_0)$ in (33). Then the quantity $q(O_{N+1})$ in $\mathbf{X}(O_{N+1})$ corresponds to the function $J$. One additional note to the evaluation of the function $J$ : When we evaluate the ray amplitudes, there appears an additional factor $\prod_{j=1}^{N} \left\{ (-1)^{\varepsilon_j} \cos\vartheta(O_j)/\cos\tilde{\vartheta}(O_j) \right\}$. This factor just cancels the first factor in (33). The formulae then become considerably simpler.

Note that the formulae for the function $M(O_{N+1})$ and the curvature of the wavefront $K(O_{N+1})$ can be obtained from (33) immediately.

**7. Layers with constant velocity gradients**

In this case, we have (see (III-14) and (III-15))

$$\mathbf{B}(s,s_0) = \begin{pmatrix} 1 & \int_{s_0}^{s} v(s)ds \\ 0 & 1 \end{pmatrix} . \qquad (35)$$

Then Eq. (33) can be used to evaluate $\mathbf{X}(s)$ analytically, without any solution of differential equations. Eq. (33) with (35) can be, of course, also used in the case of homogeneous layers. Then, even $I = 0$ in Eq. (27) for **A** .

**8. Concluding remarks**

The problem discussed in this paper has an old history. In seismology, similar expressions written in terms of radii of curvature were probably first derived by Gel'chinskiy (1961), and are discussed in detail by Červený and Ravindra (1971). Similar expressions were derived and discussed by Shah (1973), Popov and Pšenčik (1978a,1978b), Hubral (1979), Gol'din (1979), Červený and Hron (1980) and many others. The derivation presented here follows mainly from the paper by Červený and Hron (1980), where similar formulae were derived for a 3-D case.

For a broader bibliography, see Shah (1973), Červený and Hron (1980), and Hubral (1980).

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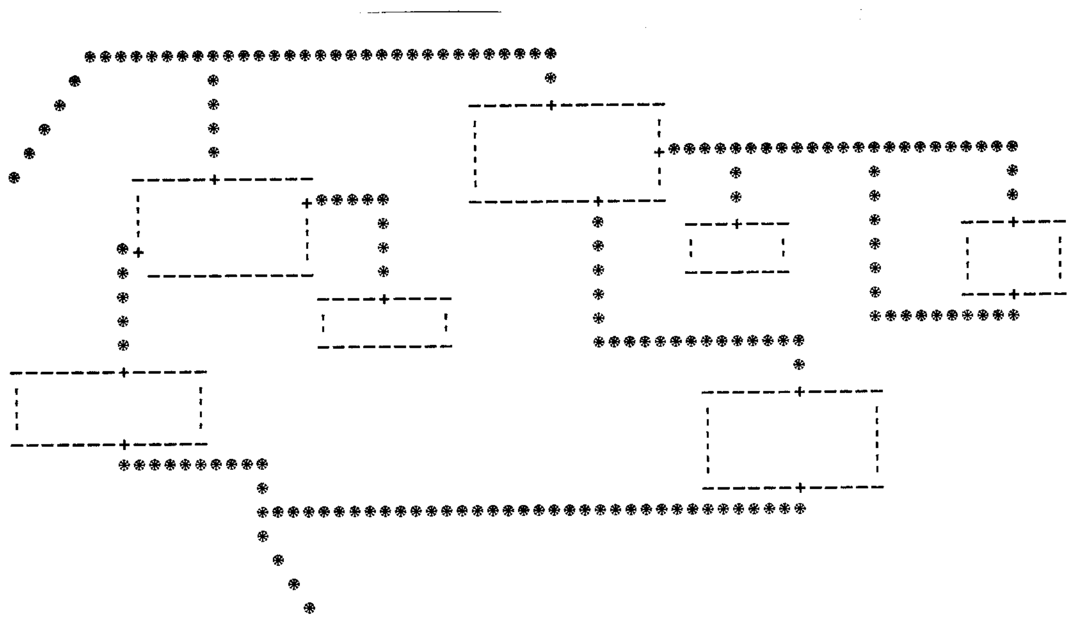

Top Ten Adventurers:

| Rank | Score | Name |
|---|---|---|
| 1 | 6756 | Clay the Bravissimo: killed on level 24 by a mimic. |
| 2 | 5960 | Fraderick the Grate: killed on level 18 by an umber hulk. |
| 3 | 5769 | Thor the Pious: killed on level 22 by a violet fungi. |
| 4 | 5270 | Zippy the pinhead: killed on level 21 by a dragon. |
| 5 | 5194 | Grederick the Freight: killed on level 19 by an umber hulk. |
| 6 | 5133 | ferdy: killed on level 18 by an umber hulk. |
| 7 | 4847 | Gredrick the Freight: killed on level 19 by a xorn. |
| 8 | 4841 | Frederick the Grizzle: killed on level 18 by an umber hulk. |
| 9 | 4679 | ferdy: killed on level 19 by a xorn. |
| 10 | 4559 | ferdy: quit on level 15. |