Computation of Geometrical Spreading by Dynamic Ray Tracing

Vlastislav Červený

1. Introduction

In the evaluation of ray amplitudes and ray synthetic seismograms, it is necessary to know the geometrical spreading. Geometrical spreading can be determined in several ways. For simpler types of media, such as vertically inhomogeneous or radially symmetric media, it can be computed by analytical methods. This is, however, not possible for general laterally inhomogeneous media. In case of laterally inhomogeneous media, another method is often used. The geometrical spreading is computed by direct measurement of distances between individual rays. This method is, however, rather rough and causes difficulties in various situations. It would be better to have some method which would allow us to determine the geometrical spreading by some computations *along just one ray*. Such a possibility is offered by dynamic ray tracing.

Similarly, as in the three preceding papers (Červený 1981a, 1981b and 1981c, hereafter denoted by I, II, and III), we shall consider an arbitrary ray Ω and use the ray centered coordinates (s,n) connected with this ray. The coordinate s measures the arc length along the ray from an arbitrary reference point and n represents a length coordinate in the direction perpendicular to Ω at s . We remember that this system is orthogonal, with scale factors h,1, where $h = 1 + v_{,n}n/v$ (v denotes the velocity and $v_{,n}$ its derivative with respect to n at Ω). In this system, the travel-time field $\tau(s,n)$ in the neighborhood of Ω is given by the relation

$$\tau(s,n) \approx \tau(s,0) + \frac{1}{2}M(s)\,n^2 \ , \qquad (1)$$

where $M(s) = \left[\partial^2\tau(s,n)/\partial n^2\right]_{n=0}$ is a solution of the *dynamic ray tracing equation*

$$\frac{dM(s)}{ds} + vM^2 + \frac{v_{,nn}}{v^2} = 0 \ . \qquad (2)$$

When we introduce new functions $p(s)$ and $q(s)$ by the relations $M(s) = p(s)/q(s)$

and $p(s) = q_{,s}/v(s)$, the system (2) can be rewritten in the following form

$$\frac{dq(s)}{ds} = vp(s) \ , \qquad \frac{dp(s)}{ds} = -\frac{v_{,nn}}{v^2} q(s) \ . \tag{3}$$

It was shown in [III] that the dynamic ray tracing system (3) can also be used to compute rays $n = n(s)$ in the neighborhood of Ω, when we put $q = n$ and $p = p_n$ (p_n is the n-component of the slowness vector $\vec{p}$).

In this paper, we shall show another application of the dynamic ray tracing system (3) -- the evaluation of geometrical spreading. As geometrical spreading is introduced by various authors in different ways, we shall first devote some attention to geometrical spreading itself, without any relation to dynamic ray tracing. As is common in the present literature devoted to ray theory, we shall use, instead of geometrical spreading, the Jacobian of the transformation from Cartesian to ray coordinates (function J), which is strictly and uniquely defined. The various forms of geometrical spreading introduced in seismological literature can be determined from function J. Later we shall show how to use dynamic ray tracing to compute function J. We shall also shortly describe possible means of computing some other related quantities (curvature of the wavefront, radius of curvature of the wavefront, the Laplacian of the travel-time field, etc.).

2. Ray Fields

We shall consider a one-parameter 2-D system of rays. Each ray is specified by a parameter γ. The *ray parameter* γ may have different meanings in different situations. Let us present two important examples.

a) First example

In the case of a *line source* , the ray parameter γ is usually introduced as the angle i_0 , which specifies the initial direction of the ray at the source (see Fig. 1a). Instead of the angle i_0 we can, of course, use any other parameter which specifies uniquely the initial direction of the ray.

b) Second example

In the case of a *selected wavefront* (e.g. wavefront of a plane wave), the initial directions of rays are known - they are perpendicular to the wavefront. Each ray is fully specified by the point of the wavefront to which it is connected. In this way, the ray parameter γ may be considered, for example, as an arc length along the wavefront (see Fig. 1b).

There are, of course, many other possible ways to introduce the ray parameter in various situations; we shall not discuss them here. We shall introduce the *ray coordinates* (ξ,γ). The coordinate γ has the same meaning as shown above; it selects one ray from the whole system of rays. The coordinate ξ specifies the position of a point on the selected ray. Usually, ξ is the arclength along the ray, measured from an arbitrary reference point, or the travel-time along this ray.

Thus, the position of any point $S(x,z)$ can also be expressed in the ray coordinates (ξ,γ). To do this, we must first find the ray which passes through the point S (and thus determine γ) and then find ξ, the coordinate along the ray, see Fig. 1.

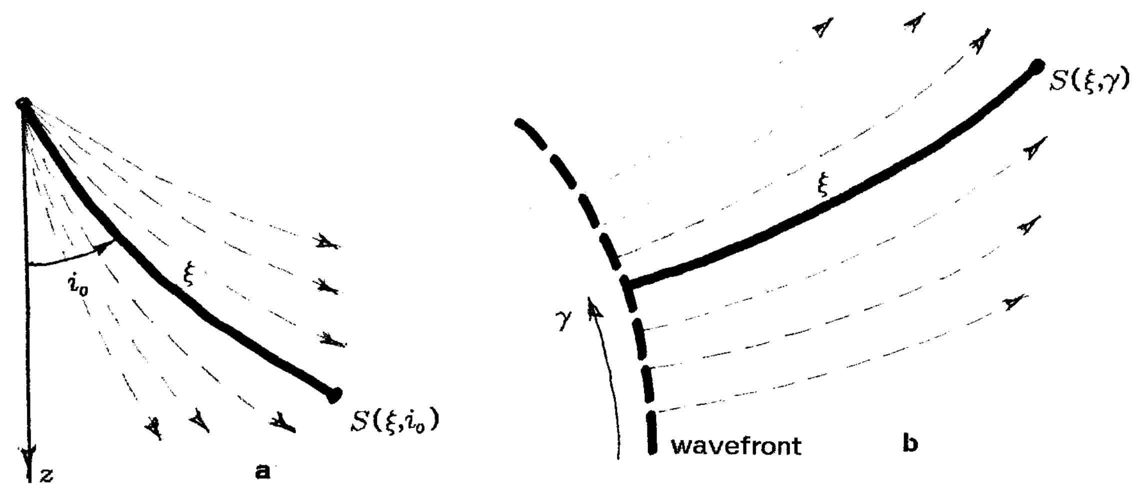

FIG. 1.

Let us emphasize the large difference between the ray coordinates (ξ,γ) of the point S and the ray centered coordinates (s,n) connected with some ray Ω of the same point S (see Fig. 2).

Let us now for a while understand that $\xi = \tau$ (travel-time along the ray). Consider parametric equations

$$x = x(\tau,\gamma) , \qquad z = z(\tau,\gamma) . \tag{4}$$

When γ is fixed and the travel-time τ varies, the system (4) represents the parametric equations of the ray specified by the ray parameter γ. For fixed travel-time τ, as γ varies, these equations are equations of the wavefront.

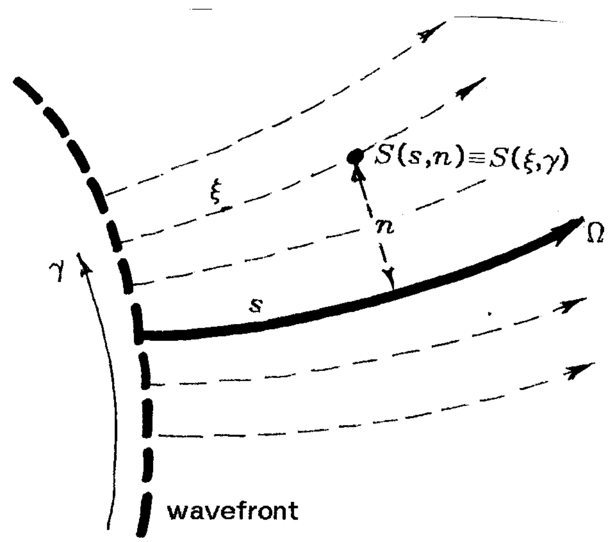

FIG. 2.

We call the ray field *regular* when *one and only one* ray passes through each point. Thus in the regular ray field, the rays cannot intersect and cannot form shadow zones. The same terminology applies to ray coordinates -- they are regular when the ray field is regular, and vice versa.

In the following, we shall again understand that ξ is the arclength along the ray. However, we shall continue to denote it by ξ, in order not to confuse it with the ray-centered coordinate s. A very important role in the theory and applications of ray fields is played by the *Jacobian* of the transformation from Cartesian to ray coordinates. We shall denote it by J and call it, for simplicity, *function* J. As well known, the Jacobian of the transformation is given by the equation

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{vmatrix} . \qquad (5)$$

It will be shown later that function J measures the expansion and contraction of the ray tube. When two neighboring rays intersect, the function J vanishes, $J = 0$. Such points are called *caustic points*. In shadow zones, where rays do not exist, J is not defined. Thus we can say alternatively: when the Jacobian J is defined and does not vanish at any point of the region D, the ray field is regular in the region D. On the other hand, the ray field is called irregular at any point where J is not defined or vanishes at that point.

Now we shall introduce a very important term, the *elementary ray tube.* By the elementary ray tube we mean the family of rays the parameters of which are within the limits $(\gamma, \gamma + d\gamma)$ (see Fig. 3). The term "ray tube" is used mostly in 3-D media, but we shall keep this notation also here, even though we consider only the 2-D situation. At caustics, the width of the elementary ray tube shrinks to zero.

3. Function J in the ray-centered coordinates (s, n)

Function J can be very simply expressed in the ray-centered coordinates (s, n), connected with the ray Ω, especially when we compute it directly at Ω.

Let us first perform the transformation from (x, z) coordinates to the ray-centered (s, n) coordinates, and then from (s, n) coordinates to the ray coordinates (ξ, γ). The Jacobian of the transformation from (x, t) to (ξ, γ) coordinates is then expressed as a product of two corresponding Jacobians,

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial n} & \frac{\partial z}{\partial n} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial s}{\partial \xi} & \frac{\partial n}{\partial \xi} \\ \frac{\partial s}{\partial \gamma} & \frac{\partial n}{\partial \gamma} \end{vmatrix} . \tag{6}$$

As the ray centered coordinate system is orthogonal with scaling factors h, 1, we have

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial n} & \frac{\partial z}{\partial n} \end{vmatrix} = h . \tag{7}$$

Directly at the ray Ω we have $h = 1$, $\partial s / \partial \xi = 1$, $\partial n / \partial \xi = 0$. From this immediately follows

$$J = \frac{\partial n}{\partial \gamma} . \tag{8}$$

This is the final expression for the function J in the ray-centered coordinates (s, n) connected with the ray Ω, directly at Ω. When we keep other relevant quantities fixed, we can replace $\partial n / \partial \gamma$ by $dn / d\gamma$ and rewrite (8) in the form

$$dn = J \, d\gamma , \tag{9a}$$

or, in finite differences

$$\Delta n = J \, \Delta\gamma . \tag{9b}$$

As we are directly at Ω, we have approximately $dn = \Delta n = n$. Then, we can rewrite (9a) and (9b) in the following form:

$$n \approx J \, \Delta\gamma \ . \qquad (9c)$$

It was shown in [III] that the rays Ω_c in the neighborhood of the ray Ω are described by the formula $n = n(s)$, and can be determined from the dynamic ray tracing system (3). Thus, $dn \approx n$ measures the width of the elementary ray tube. Equations (9b) and (9c) clearly demonstrate that function J is a good measure of the expansion and contraction of the ray tube. Equation (9c) also represents the most common approach to the determination of J : $J = n/\Delta\gamma$, where n is the normal distance between two rays specified by ray parameters γ and $\gamma + \Delta\gamma$ (see Fig. 3).

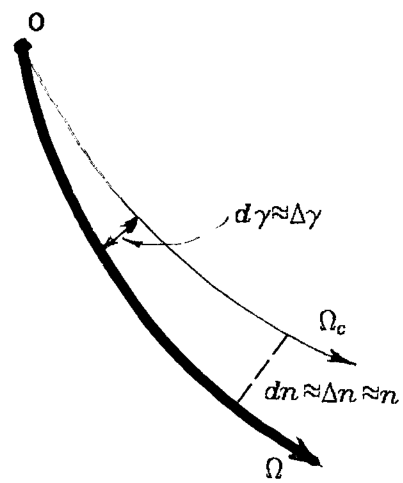

FIG. 3.

4. Evaluation of function J by dynamic ray tracing

We shall return to our dynamic ray tracing system (3). It was shown in [III] that this system is also the ray tracing system for the rays Ω_c which are close to Ω,

$$\frac{dn}{ds} = v p_n \ , \qquad \frac{dp_n}{ds} = -\frac{v_{,nn}}{v^2} n \qquad (10)$$

(see Eq. (III-7)). Differentiating this system with respect to γ, we obtain

$$\frac{dJ}{ds} = vP \ , \qquad \frac{dP}{ds} = -\frac{v_{,nn}}{v^2} J \ , \qquad (11)$$

where

$$J = \frac{\partial n}{\partial \gamma} \ , \qquad P = \frac{\partial p_n}{\partial \gamma} \ . \qquad (12)$$

Here J again denotes the Jacobian (function J), and P is the auxiliary function -- the derivative of the normal component of the slowness vector with respect to the ray parameter γ.

Thus, we have again arrived at the dynamic ray tracing system (see (3)). The dynamic ray tracing along the ray can be used to determine function J and auxiliary function P.

Let us specify the initial conditions for the dynamic ray tracing system (11) when we wish to use it for the determination of function J. We shall again consider two examples: line source and plane source (or plane wavefront). The initial conditions for a line source are given by equations

$$\text{for } s = s_0 : \qquad J(s_0) = 0 \ , \qquad P(s_0) = \frac{1}{v(s_0)} \ , \tag{13}$$

as can be immediately obtained from (III-8a). For a plane source:

$$\text{for } s = s_0 : \qquad J(s_0) = 1 \ , \qquad P(s_0) = 0 \ . \tag{14}$$

These initial conditions follow from (III-8b). As we can again see, these two sets of initial conditions specify two linearly independent solutions of (11). We can call them "the line source spreading" and "the plane source spreading". Any other solution of (11) is obtained as a linear combination of these two solutions. This applies even to complex solutions of (11), which are important in the theory of Gaussian beams.

In [III], analytic solutions of the dynamic ray tracing system were found for three typical simple situations: a) $v_{,nn} = 0$, b) $v_{,nn} = const > 0$, c) $v_{,nn} = const < 0$. The same solutions can be used for function J and the auxiliary function P, when we use appropriate initial conditions (13) or (14).

5. Dynamic ray tracing and the curvature of the wavefront.

In the dynamic ray tracing, we have used the function $M(s)$, which represents the second derivative of the travel-time field with respect to the coordinate n, determined directly at Ω. Alternatively, we can use the curvature of the wavefront $K(s)$ instead of $M(s)$. We shall not give here the detailed derivation of all presented formulae; this part of the text is included only for completeness.

It is possible to show that the relations between the curvature of the wavefront $K(s)$ and $M(s)$ (respectively $p(s)$ and $q(s)$) are as follows:

$$K(s) = vM(s) = \frac{vp(s)}{q(s)} = \frac{d \ln q}{ds} \ . \tag{15}$$

Inserting (15) into (2), we obtain a non-linear ordinary differential equation of first order of the Riccati type for $K(s)$

$$\frac{dK}{ds} - \frac{v_{,s}}{v} K + K^2 + \frac{v_{,nn}}{v} = 0 \ . \tag{16}$$

This formula is slightly more complicated than (2) due to the second term. The solutions of (2) are not quite straightforward in trivial situations. For example, for $v_{,nn} = 0$ we get (see also (II-17)),

$$K(s) = \frac{v(s)\, K(s_0)}{v(s_0) + K(s_0) \int_{s_0}^{s} v(s)\, ds} \ . \tag{17}$$

Curvatures of wavefronts are sometimes used to compute the function J. The relation between J and $K(s)$ is as follows (see (15)),

$$J(s) = J(s_0)\, exp(\int_{s_0}^{s} K(s) ds) \ . \tag{18}$$

(We remember that $q(s)$ has the same meaning as $J(s)$).

It might be also useful to use the radius of the curvature of the wavefront $R(s)$ instead of the curvature of the wavefront $K(s)$,

$$R(s) = \frac{1}{K(s)} = \frac{1}{vM(s)} = \frac{q(s)}{vp(s)} \ . \tag{19}$$

The differential equation for $R(s)$ immediately follows from (16),

$$\frac{dR(s)}{ds} + \frac{v_{,s}}{v} R - \frac{v_{,nn}}{v} R^2 = 1 \ . \tag{20}$$

For $v_{,nn} = 0$, we obtain the solution (see (17)),

$$R(s) = \frac{v(s_0)R(s_0) + \int_{s_0}^{s} v(s) ds}{v(s)} \ . \tag{21a}$$

In an homogeneous medium, this yields a well-known formula

$$R(s) = R(s_0) + (s - s_0) \ , \tag{21b}$$

which has been used in many applications and which has a clear physical interpretation. The relation between the function J and $R(s)$ is as follows,

$$J(s) = J(s_0)\, exp(\int_{s_0}^{s} R^{-1}(s)ds) \ . \tag{22}$$

6. Determination of the Laplacian of the travel-time field by dynamic ray tracing

By Laplacian of the travel-time field we mean the function

$$\nabla^2\tau = \frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial z^2} \ . \tag{23}$$

This function is closely related to the function J and plays an important role in ray methods. It is not complicated to show that this function can be simply computed by the dynamic ray tracing.

The Laplacian $\nabla^2\tau$ can be rewritten in curvilinear orthogonal coordinates ξ_1 , ξ_2 with scale factors h_1 , h_2 as follows,

$$\nabla^2\tau = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial\xi_1}\left[\frac{h_2}{h_1}\frac{\partial\tau}{\partial\xi_1}\right] + \frac{\partial}{\partial\xi_2}\left[\frac{h_1}{h_2}\frac{\partial\tau}{\partial\xi_2}\right]\right\} \ . \tag{24}$$

In the case of ray-centered coordinates (s,n), we have $h_1 = h$, $h_2 = 1$, $\xi_1 = s$, $\xi_2 = n$. This yields

$$\nabla^2\tau = \frac{1}{h}\frac{\partial}{\partial s}\left[\frac{1}{h}\frac{\partial\tau}{\partial s}\right] + \frac{\partial}{\partial n}\left[h\,\frac{\partial\tau}{\partial n}\right] \ . \tag{25}$$

At the central ray Ω we obtain from (25),

$$\nabla^2\tau = \frac{\partial^2\tau}{\partial s^2} + \frac{\partial^2\tau}{\partial n^2} \ . \tag{26}$$

This gives

$$\nabla^2\tau = -\frac{1}{v^2(s)} v_{,s} + M \ . \tag{27}$$

This formula expresses $\nabla^2\tau$ in terms of function M. Note that this equation can be also obtained directly from (II-23). Taking into account that $M(s) = \frac{q_{,s}}{vq}$, we obtain a relation between $q(s)$ (respectively $J(s)$) and $\nabla^2\tau$,

$$\nabla^2\tau = \frac{1}{v}\frac{d}{ds}\ln\left[\frac{J}{v}\right] , \tag{28}$$

respectively

$$J(s) = \frac{v(s)}{v(s_0)} J(s_0) \exp \{ \int_{s_0}^{s} v \nabla^2 \tau ds \} . \tag{29}$$

7. Function J for a point source in a 2-D medium

We have considered here only 2-D media with a line (or plane) source. We are, however, very often interested in wavefields generated by a point source, even though in a 2-D medium (in which the velocity does not depend on one coordinate). The corresponding generalization does not immediately follow from some equations presented here, as they apply only to 2-D symmetries, which do not take into account the spreading in the plane perpendicular to the profile. We shall, however, present here the results of a 3-D investigation of this situation (see e.g. Červený and Hron, 1980). The result is not complicated and can be inductively understood even without a detailed derivation. To obtain the function J corresponding to a point source, the function J for a line source should be multiplied by some "correction" function $J_{\perp}(s)$ (we call it "J perpendicular"), which describes the spreading in the perpendicular direction to the profile. The function $J_{\perp}(s)$ can be again obtained from a dynamic ray tracing system with the coordinate n perpendicular to the (x,z) plane. In a 2-D model, however, $v_{,nn} = 0$ in this case. The solution can be therefore written analytically (see (III-16)), with $n(s) = J_{\perp}(s)$,

$$J_{\perp}(s) = J_{\perp}(s_0) + P_{\perp}(s_0) \int_{s_0}^{s} v(s) ds . \tag{30}$$

For a point source, we have $J(s_0) = 0$ and $P(s_0) = \frac{1}{v(s_0)}$, similarly as in (13). Thus, we get finally

$$J_{\perp}(s) = \frac{1}{v(s_0)} \int_{s_0}^{s} v(s) ds . \tag{31}$$

8. Concluding remarks

To use the differential equations for the computation of geometrical spreading is an old idea (see e.g. Belonosova, Tadzhimukhametova and Alekseyev (1967) , Červený and Pšenčik (1974)). The differential equations were, however, usually rather complicated. Simple dynamic ray tracing systems to compute function J (similar to those presented here) were first derived by Popov and Pšenčik (1978a, 1978b), (see also Červený, Molotkov and Pšenčik (1977)). Curvatures of the wavefront have been used in the interpretation of seismic data by many authors (see e.g. Hubral (1980), Shah (1973), and other references given there).

The equations for the evaluation of ray amplitudes where function J is known are not presented here; we refer, e.g., to Červený, Molotkov and Pšenčik (1977).

REFERENCES

Belonosova, A.V., Tadzhimukhametova, S.S., and Alekseyev, A.S., 1967, Computation of Travel Time Curves and Geometrical Spreading in Inhomogeneous Media: In "Certain Methods and Algorithms in the Interpretation of Geophysical Data", Moscow, Nauka, 124-136. (In Russian.)

Červený, V., 1981a, Dynamic Ray Tracing in 2-D Media: This volume.

Červený, V., 1981b, Determination of Second Derivatives of Travel-time Field by Dynamic Ray Tracing: This volume.

Červený, V., 1981c, Ray Tracing in a Vicinity of a Central Ray: This volume.

Červený, V., and Pšenčik, I., 1974, Computation of Geometrical Spreading of Seismic Body Waves in Laterally Inhomogeneous Media with Curved Interfaces: *Geophys. J. R. Astr. Soc.* Vol. 38, 9-19.

Červený V., and Hron F., 1980, The Ray Series Method and Dynamic Ray Tracing Systems for 3-D Inhomogeneous Media: *Bull. Seismol. Soc. Am.*, Vol. 70, 47-77.

Červený, V., Molotkov, I.A., and Pšenčik, I., 1977, Ray Method in Seismology: Praha, Univ. Karlova.

Hubral, P., 1980, Wavefront Curvatures in 3-D Laterally Inhomogeneous Media with Curved Interfaces: *Geophysics*, Vol. 45, 905-913.

Popov, M.M., and Pšenčik, I., 1978a, Ray Amplitudes in Inhomogeneous Media with Curved Interfaces: In *Geofyzikalni Sbornik*, Vol. 24, 111-129. Praha, Academia.

Popov, M.M., and Pšenčik, I., 1978b, Computation of Ray Amplitudes in Inhomogeneous Media with Curved Interfaces: *Studia Geoph. et Geod.*, Vol. 22, 248-258.

Shah, P.M., 1973, Use of Wavefront Curvature to Relate Seismic Data with Subsurface Parameters: *Geophysics*, Vol. 38, 812-825.

ROGUE RECORD SHEET

Name ______________________
Highest level ______
Highest experience _____________
Highest hit points _________
Highest strength __________
Lowest armor class _________

POSSESSIONS

WEAPONS

_________ _________ _________ _________ _________ _________ _________
_________ _________ _________ _________ _________ _________ _________

ARMORS

_________ _________ _________ _________ _________ _________ _________

POTIONS

_________ _________ _________ _________ _________ _________ _________
_________ _________ _________ _________ _________ _________ _________

MONSTERS KILLED

ANT ___
BAT ___
CENTAUR ___
DRAGON ___
EYE ___
FUNGUS ___
GNOME ___
HOBGOBLIN ___
INVISIBLE STALKER ___
JACKAL ___
KOBOLD ___
LEPRECHAUN ___
MIMIC ___
NYMPH ___
ORK ___
PURPLE WORM ___
QUASIT ___
RUST MONSTER ___
SNAKE ___
TROLL ___
UMBER HULK ___
VAMPIRE ___
WRAITH ___
XORN ___
YETI ___
ZOMBIE ___