Ray Tracing in a Vicinity of a Central Ray

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1. Introduction

The dynamic ray tracing along the ray Ω can be very useful not only for the determination of the travel-time field τ in the neighborhood of Ω, but also in the computation of rays which are close to Ω. In the following, we shall show how the dynamic ray tracing along Ω can be used to determine these rays. The results will also help us considerably to understand the physical meaning of individual quantities involved in the dynamic ray tracing.

We shall again use some results from the two previous papers, Červený (1981a) and Červený (1981b). For simplicity, we shall refer to these two papers as paper [I] and paper [II]. We shall also refer to equations from these papers in the same way, e.g. Eq. (I-22).

We shall select an arbitrary ray Ω and use the ray centered coordinate system (s,n) corresponding to this ray. The coordinate s measures the arc length along the ray Ω from an arbitrary reference point, and n is a length coordinate in the direction perpendicular to Ω at s. In this coordinate system, the travel-time field $\tau(s,n)$ in the vicinity of Ω is given by the approximate equation

$$\tau(s,n) \approx \tau(s,O) + \frac{1}{2} M(s) n^2 \ , \tag{1}$$

where $M(s) = \left[\partial^2 \tau(s,n)/\partial n^2\right]_{n=0}$. The quantity $M(s)$ is a solution of the "dynamic ray tracing" equation,

$$\frac{dM(s)}{ds} + vM^2 + \frac{v_{,nn}}{v^2} = 0 \ , \tag{2}$$

where $v_{,nn}$ is the second derivative of velocity with respect to n, determined at Ω. It was also shown in [1] that this nonlinear ordinary differential equation (of Riccati type) can be rewritten to two linear differential equations of first order

$$\frac{dq}{ds} = vp \ , \qquad \frac{dp}{ds} = -\frac{v_{,mn}}{v^2} q \ , \tag{3}$$

when we put $M(s) = p(s)/q(s)$ with $p(s) = q_{,s}/v(s)$. We did not, however, explain what is the physical meaning of $p(s)$ and $q(s)$. Both these quantities have a very important physical meaning and can be used in various applications (computation of ray amplitudes, Gaussian beams, etc.). Therefore, we shall devote much attention to these quantities.

2. Slowness Vector in Ray-Centered Coordinates

In all ray tracing systems, an important role is played by the slowness vector. Again we shall denote the slowness vector by $\vec{p}$. We remember that it is given by equation $\vec{p} = \nabla\tau$. It has a direction perpendicular to the wavefront and the magnitude 1/velocity (i.e., slowness). In the ray-centered coordinate system (s,n), the slowness vector is given by the relation

$$\vec{p} = h^{-1}\tau_{,s}\vec{t} + \tau_{,n}\vec{n} \ ,$$

see (I-9). Here h is the scale factor given by the equation $h = 1 + v_{,n}n/v$, $\vec{t}$ is the unit tangent and $\vec{n}$ unit normal to Ω. From this, we easily obtain the equations for both components of the slowness vector, p_s and p_n, with the accuracy up to linear terms in n,

$$p_s = h^{-1}\tau_{,s} \approx \frac{1}{v} - \frac{v_{,n}}{v^2}n \ , \qquad (4)$$

$$p_n = \tau_{,n} \approx Mn \ , \qquad (5)$$

see (I-16a) and (I-16b). In the following, we shall be mainly interested in the component p_n, which has a large practical importance. In [II], we introduced a rectangular Cartesian coordinate system (l,m) with basis vectors $\vec{l}$, $\vec{m}$ which coincide at the point $s = s_0$ of the ray Ω with unit vectors $\vec{t}$ and $\vec{n}$. We found the expressions for the components of the slowness vector $p_l = \partial\tau/\partial l$ and $p_m = \partial\tau/\partial m$ at any point $S(l, m)$, see Eqs. (II-11) and (II-12). We shall introduce again a similar system and look for the components of the slowness vector directly at the m-axis (i.e. for $l = 0$), see Fig. 1. Comparing (4) with (II-11) and (5) with (II-12) we can see that $p_n = p_m$ and $p_s = p_l$ in this case, with the accuracy up to linear terms in m.

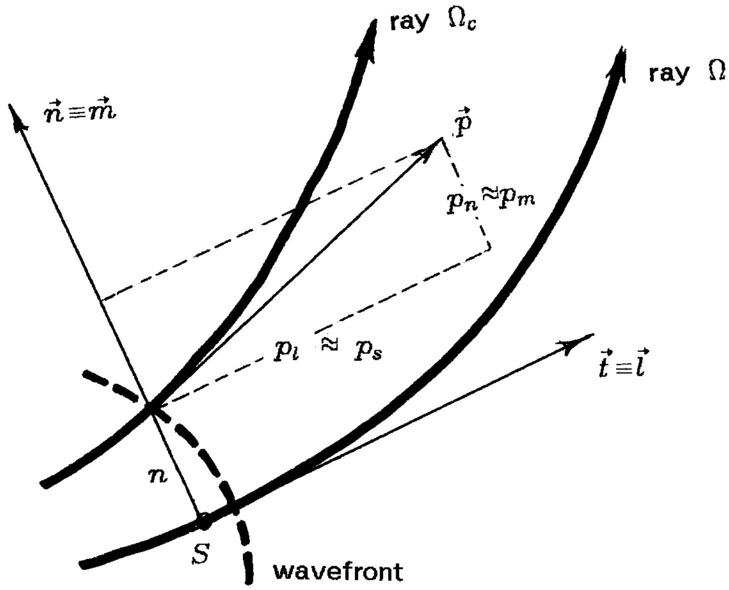

FIG. 1.

3. Rays in a Vicinity of the Central Ray

In this section, we shall derive the ray tracing equations for a ray, which is situated close to the central ray Ω. We shall denote such a ray by Ω_c, see Fig. 1.

We know that the rays are perpendicular to the wavefront, i.e. they have the direction of the slowness vector. In the Cartesian coordinate system (l, m), this condition can be expressed as follows

$$\frac{dm}{dl} = \frac{p_m}{p_l},$$

see Fig. 1. Here dm is the change of m along the ray Ω_c when the change in l is dl. This equation can be replaced, with the accuracy up to linear terms in n by the equation

$$\frac{dn}{ds} = \frac{p_n}{p_s}.$$

The differences between corresponding quantities $(n \approx m, dl \approx ds, p_m \approx p_n$ and $p_l \approx p_s)$ are of higher order in n and need not be considered. Taking into account also (4), we finally obtain

$$\frac{dn}{ds} = vp_n \quad . \tag{6a}$$

This is the first equation in the ray tracing system.

Now, we shall find how the quantity p_n changes along the ray. We can write,

$$\frac{dp_n}{ds} = \frac{\partial p_n}{\partial s} + \frac{\partial p_n}{\partial n}\frac{\partial n}{ds}.$$

From (5) we obtain $\partial p_n / \partial s = M_{,s} n$ and $\partial p_n / \partial n = M$. Using also (6a), we get

$$\frac{dp_n}{ds} = n\left(M_{,s} + vM^2\right) .$$

However, we know from the dynamic ray tracing equation that $dM / ds + vM^2 = -v_{,nn} / v^2$, so that

$$\frac{dp_m}{ds} = -\frac{v_{,nn}}{v^2} n . \tag{6b}$$

Thus, we have derived the ray tracing system for any ray Ω_c , which is close to the central ray Ω ,

$$\frac{dn}{ds} = vp_n , \qquad \frac{dp_n}{ds} = -\frac{v_{,nn}}{v^2} n . \tag{7}$$

The system is written in the ray centered coordinate system (s,n) , corresponding to the central ray Ω .

The ray tracing system (7) we obtained is rather surprising - it corresponds fully to the dynamic ray tracing system (3), when we put $q = n$ and $q = p_n$.

Now we can easily see the physical meaning of quantities $q(s)$ and $p(s)$ that were introduced rather formally to simplify the dynamic ray tracing equation (2). In fact, the dynamic ray tracing system (3) is again a ray tracing system, for rays Ω_c situated close to the central ray Ω . The quantity of q corresponds to the distance of the ray Ω_c from Ω , the quantity p is the component of the slowness vector in the direction perpendicular to the ray Ω .

What is the advantage of our ray tracing system (7) in comparison with the standard ray tracing system? In 2-D media, the standard ray tracing system is formed by *three non-linear* ordinary differential equations of first order. The right-hand side of these equations must be evaluated independently at any point of each ray. Our ray tracing system (7) is formed by *two linear* ordinary differential equations of first order. The quantities $v_{,nn} / v^2$ and v at the right-hand sides of (7) are functions of the coordinates along the central ray Ω only. They can be determined along Ω only once for a whole family of rays Ω_c close to Ω . As soon as v and $v_{,nn} / v^2$ along Ω are determined, we can compute arbitrary number of rays Ω_c practically without any work.

Let us specify the initial conditions for the ray tracing system (7) :

$$\text{for } s = s_o : \quad n(s_o) = n_o , \quad p_n(s_o) = p_{n_o} . \tag{8}$$

The first condition $(n(s_o) = n_o)$ specifies the distance of the ray Ω_c from the central ray Ω at $s = s_0$, and the second condition $(p_n(s_0) = p_{n_0})$ specifies its initial direction at the same point $s = s_0$. The second condition can be, of course, also replaced by some initial angle of the ray at $s = s_0$, e.g. by the angle i_c between Ω and Ω_c at $s = s_0$. The determination of $p_n(s_0)$ is thus straightforward. It is simple to see that for a line source situated at $s = s_0$ we have

$$n_0 = 0 , \qquad p_{n_0} = \sin i_c / v(s_0) . \tag{8a}$$

In the case of a plane wavefront at $s = s_0$, we have

$$n_0 \neq 0 , \qquad p_{n_0} = 0 . \tag{8b}$$

We shall call the solution of (7) which corresponds to the initial conditions (8a) the "line source solution", and the solution which corresponds to (8b) the "plane source solution". Both these solutions are linearly independent, so that any solution of (7) can be constructed as a linear combination of these two solutions.

4. Rays as Solutions of the One-Dimensional Helmholtz Equation

Another useful form of the ray tracing equations (7) is easily obtained from (7) when we insert p_n from the first to the second equations. We get

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dn}{ds}\right) + \frac{v_{,nn}}{v^2} n = 0 . \tag{9}$$

This is, of course, fully equivalent to (I-25) when we put $q = n$. Equation (9) can be even more simplified when we introduce a new variable η along the central ray Ω instead of s by the relation

$$d\eta = v ds , \qquad \text{i.e.,} \qquad \eta = \eta_0 + \int_{s_0}^{s} v(s) ds . \tag{10}$$

Then, multiplying (9) by v^{-1} , we immediately obtain

$$\frac{d^2 n}{d\eta^2} + \frac{v_{,nn}}{v^3} n = 0 . \tag{11}$$

But this is a one-dimensional Helmholtz equation! The result is really surprising. We remember that the one-dimensional Helmholtz equation (i.e., the one-dimensional wave equation for harmonic waves) has the following form

$$\frac{d^2 u}{dy^2} + k^2 u = 0 , \tag{12}$$

where u may represent various physical quantities in different wave propagation problems, k is so-called wave number, $k = 2\pi/\lambda = \omega/c$, where λ is the wavelength, ω is circular frequency and c velocity. Let us consider the case that $v_{,nn} > 0$. Comparing (11) with (12) we can see that the "effective wave number" k_e and "effective wavelength" λ_e in our ray tracing system (11) are given by relations

$$k_e = \sqrt{\frac{v_{,nn}}{v^3}} , \qquad \lambda_e = 2\pi\sqrt{\frac{v^3}{v_{,nn}}} . \tag{13}$$

All these quantities have, of course, sense only when $v_{,nn} > 0$.

5. Simple Examples of Rays in the Vicinity of the Central Ray Ω

We shall show here three elementary examples, corresponding to three different types of rays Ω_c in the vicinity of a central ray Ω .

a) First example: $v_{,nn} = 0$

This is a very important example, as $v_{,nn} = 0$ not only in homogeneous media, but also in media with constant velocity gradients, see (II-21). In this case, the solution of (11) is as follows:

$$n(s) = n(\eta o) + a(\eta - \eta_o) = n(s_o) + a \int_{s_o}^{s} v(s)ds , \tag{14}$$

where a is time constant. For $p_n(s)$, we obtain from (8),

$$p_n(s) = p_n(s_o) . \tag{15}$$

Thus, $p_n(s)$ does not change along the ray in this case. As $p_n(s) = v^{-1}(s)dn(s)/ds$, we immediately obtain $p_n(s_o) = a$. Thus, we can finally write (14) in the following form

$$n(s) = n(s_o) + p_n(s_o) \int_{s_o}^{s} v(s)ds . \tag{16}$$

The rays Ω_c of this type are schematically shown in Fig. 2a.

Note that the function $M(s)$ is given in this case by the equation $(M(s) = p(s)/q(s) = p_n(s)/n(s))$

$$M(s) = \frac{p_n(s_o)}{n(s_o) + p_n(s_o) \int_{s_o}^{s} v(s)ds} = \frac{M(s_o)}{1 + M(s_o) \int_{s_o}^{s} v(s)ds} \tag{17}$$

For $n(s_o) = 0$ we have $M(s_o) = \infty$; the result which was, of course, expected.

b) Second example: $v_{,nn}/v^3 = k_e^2 > 0$, **where** k_e **is constant**

This is a typical situation when the central ray Ω lies on the axis of a symmetrical waveguide, along which the velocity does not change. The solution of (11) is then as follows:

$$n(s) = n(s_o)\cos\left[k_e \int_{s_o}^{s} v(s)ds\right] + p_n(s_o)k_e^{-1}\sin\left[k_e \int_{s_o}^{s} v(s)ds\right],$$

$$p_n(s) = p_n(s_o)\cos\left[k_e \int_{s_o}^{s} v(s)ds\right] - k_e n(s_o)\sin\left[k_e \int_{s_o}^{s} v(s)ds\right], \qquad (18)$$

where $k_e = \sqrt{v_{,nn}/v^3}$. The formula for $M(s)$ is straightforward. As we can see from (18), the rays Ω_c oscillate around Ω, see Fig. 2b. The above introduced "effective wavelength λ_e" has a clear sense in this case.

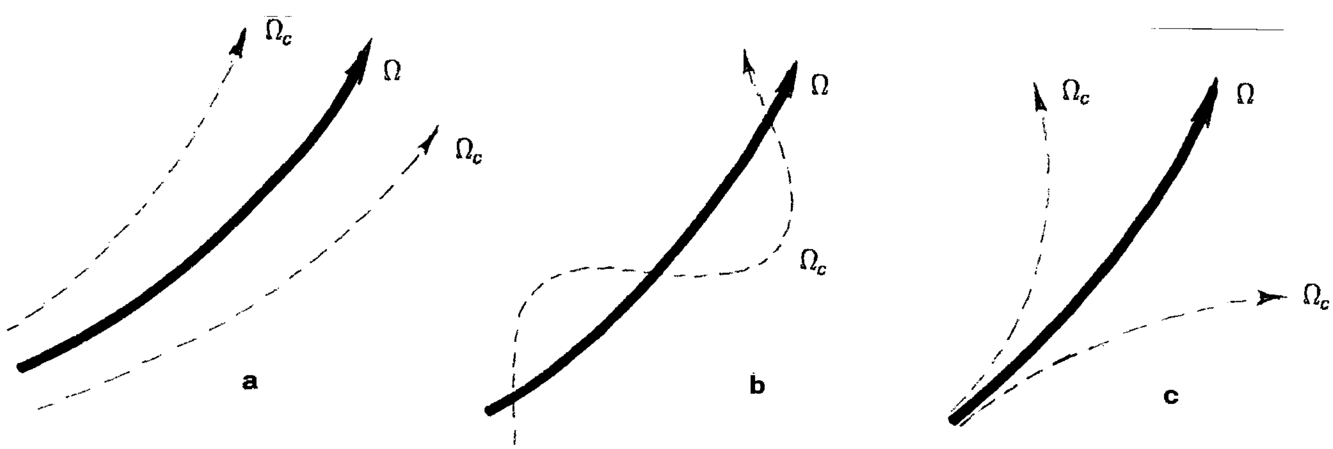

FIG. 2.

c) Third example: $v_{,nn}/v^3 = -k^2$, **where** k **is constant.**

This is a typical situation when the central ray Ω lies on the axis of some symmetrical high-velocity layer. In this case, the solution of (11) is as follows:

$$n(s) = A\exp\left[k\int_{s_o}^{s} v(s)ds\right] + B\exp\left[-k\int_{s_o}^{s} v(s)ds\right],$$

$$p_n(s) = \left\{ A \exp\left[k\int_{s_o}^{s} v(s)ds \right] - B \exp\left[-k\int_{s_o}^{s} v(s)ds \right] \right\} k \ , \tag{19}$$

where

$$A = \frac{1}{2}\left[kn(s_o) + p_n(s_o) \right], \ B = \frac{1}{2}\left[kn(s_o) - p_n(s_o) \right], \ k = \sqrt{-v_{,nn}/v^3} \ .$$

The rays Ω_c exponentially deviate from the central ray Ω (or exponentially approach the central ray Ω). The situation is again shown schematically in Fig. 2c.

In many seismological situations, the velocity usually varies very smoothly and $v_{,nn}$ is rather small. Then the situation is similar to the case shown in Fig. 2a.

It would be possible to write analytical solutions of (11) even for more complicated cases. For example, when $v_{,nn}/v^3$ is a linear function of s , the solution of (11) can be expressed in terms of Airy functions. For general laterally inhomogeneous media, however, Eq. (11) or equivalently Eqs. (7), must always be solved numerically.

6. Derivation of Ray Tracing Equations by Fermat's Principle

In section 3, we derived the ray tracing system (7) for rays Ω_c which are close to the central ray Ω. We derived them using the assumption that the rays are orthogonal trajectories to the wavefronts. We know that in isotropic media such an assumption is correct, but it may be useful to show that these equations also follow simply from the Fermat's principle.

Let us write the Fermat's integral in the ray centered coordinates (s,n) (which are connected with the ray Ω),

$$\tau = \int_{S_o}^{S} \frac{\sqrt{h^2 ds^2 + dn^2}}{V(s,n)} \ , \tag{20}$$

where S_o and S are some points which are close to the ray Ω. We remember that the quantity $\sqrt{h^2 ds^2 + dn^2}$ represents an elementary length element, written in ray centered coordinates (s,n) , see [l]. We can rewrite it in the following form

$$\sqrt{h^2 ds^2 + dn^2} = h\sqrt{1 + h^{-2}(n')^2}ds \ ,$$

where $n' = dn/ds$ is a small quantity. We can therefore write

$$\sqrt{h^2 ds^2 + dn^2} \approx h\left[1 + \frac{1}{2h^2}(n')^2\right] ds \approx h\left[1 + \frac{1}{2}(n')^2\right] ds \ . \tag{21}$$

Similarly, we can write for $1/V(s,n)$, see (I-20),

$$\frac{1}{V(s,n)} \approx \frac{1}{vh}\left(1 - \frac{v_{,nn}}{2v}n^2\right) . \tag{22}$$

Then we can rewrite the Fermat's integral in the following form

$$\tau = \int_{S_0}^{S} F(s,n,n')ds \ , \tag{23}$$

where

$$F(s,n,n') = \frac{1}{v}\left(1 + \frac{1}{2}(n')^2\right)\left(1 - \frac{v_{,nn}}{2v}n^2\right) \approx \frac{1}{v}\left(1 + \frac{1}{2}(n')^2 - \frac{v_{,nn}}{2v}n^2\right) \tag{24}$$

Euler's equations for the extremal curve (ray) of the functional (23) has the form

$$\frac{d}{ds}\left(\frac{\partial F}{\partial n'}\right) - \frac{\partial F}{\partial n} = 0 \ . \tag{25}$$

In our case, from (24) we immediately obtain

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dn}{ds}\right) + \frac{v_{,nn}}{v^2}n = 0 \ . \tag{26}$$

This is fully equivalent to (9). The reduction of (26) into (7) is straightforward.

Concluding Remarks

Similar ray tracing equations for rays Ω_c situated in the neighborhood of some central ray Ω were probably first written in a different form by Babich et al., see e.g. Babich and Buldyrev (1972), Babich and Kirpichnikova (1974). They called these rays "rays in the first approximation" (luchi v pervom priblizheniyi).

In seismological literature, the ray tracing equations in the ray centered coordinate system (s,n) connected with some central ray were first discused in Červený and Pšenčik (1979). The ray tracing equations presented in Červený and Pšenčik (1979) are exact, without any assumption that the points S_0 and S are close to Ω. They are, of course, rather more complicated.

The derivation presented here is new.

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