Determination of Second Derivatives of Travel-time Field by Dynamic Ray Tracing

Vlastislav Červený

1. Introduction

In the previous paper (Červený, 1981), various forms of the dynamic ray tracing equations for a 2-D laterally inhomogeneous media were derived. Here we shall use the solutions of dynamic ray tracing equations to determine the travel-time field in a close neighborhood of the ray and to determine the second derivatives of the travel-time field. We shall again select an arbitrary ray Ω and use the ray centered coordinate system (s,n) corresponding to this ray. We remember that s measures the arc length along the ray from an arbitrary reference point and n represents a length coordinate in the direction perpendicular to Ω at s. Such a system is orthogonal, with scale factors h,1, where $h = 1 + v_{,n} n / v$ (v denotes the velocity and $v_{,n}$ its derivative with respect to n at Ω). In this system, we can determine the travel-time field $\tau(s,n)$ in the vicinity of Ω using the equation

$$\tau(s,n) \approx \tau(s,0) + \frac{1}{2} M(s)\, n^2 \ , \tag{1}$$

where $M(s) = \left[\partial^2 \tau(s,n) / \partial n^2\right]_{n=0}$. It was shown in Červený (1981) that the function $M(s)$ is a solution of the "dynamic ray tracing equation",

$$\frac{dM(s)}{ds} + vM^2 + \frac{v_{,nn}}{v^2} = 0 \ , \tag{2}$$

where $v_{,nn}$ is the second derivative of velocity with respect to n, determined at Ω. Equation (2) represents a nonlinear ordinary differential equation of first order of the Riccati type, which can be used to determine $M(s)$ along any known ray. Several alternative forms of (2) are derived in Červený (1981).

In the following, we shall use (1) to derive equations for the determination of the travel-time field in a neighborhood of a specified point $s = s_0$ at Ω. We are interested mostly in the determination of second derivatives of the travel-time field, which cannot be obtained from a standard ray tracing.

The equations derived here will be used elsewhere to study the dynamic ray tracing and the Gaussian beam tracing *across curved interfaces.* They have, however, large importance in themselves.

2. Determination of the Travel-time Field in the Neighborhood of a Specified Point of the Ray Ω

Equation (1) is rather general and may be used in many applications. We shall assume that the ray Ω is known. Usually it is specified by points with a constant increment in the travel time ($\Delta\tau$) or a constant increment in the arc length (Δs). The way in which it was determined is not important for us. The procedure how to determine the travel-time field $\tau(s,n)$ at the point $S(s,n)$ situated close to Ω is as follows:

a) We perform the dynamic ray tracing along Ω and determine $M(s)$ along the whole ray, (i.e., at all points of the ray), using (2).

b) We determine the ray centered coordinates (s,n) of the point $S(s,n)$. In case of a curved ray Ω, this must be done numerically.

c) By the interpolation between individual points of Ω, we determine $\tau(s,0)$ and $M(s)$ corresponding to the point S.

d) We use (1) to determine $\tau(s,n)$.

The above-described procedure can be more simplified. We shall now assume that the travel-time field and other related quantities ($M(s)$, $v(s)$, $v_{,n}(s)$, etc.) are known at some specified point $s = s_0$ of the ray Ω, which is situated close to the point S. Then we can use a Taylor series and expand the quantities $\tau(s,0)$ and $M(s)$ in (2) in the powers of $(s - s_0)$. Neglecting all higher terms than $(s - s_0)^2$, n^2 and $(s - s_0)n$, we can write, see (1),

$$\tau(s,n) \approx \tau(s_0,0) + \left[\frac{\partial\tau(s,0)}{\partial s}\right]_{s = s_0} (s - s_0) + \frac{1}{2}\left[\frac{\partial^2\tau(s,0)}{\partial s^2}\right]_{s = s_0} (s - s_0)^2 + \frac{1}{2} M(s_0)\, n^2 \ . \tag{3}$$

It is simple to see that

$$\left[\frac{\partial\tau(s,0)}{\partial s}\right]_{s = s_0} = \frac{1}{v(s_0)} \ , \qquad \left[\frac{\partial^2\tau(s,0)}{\partial s^2}\right]_{s = s_0} = -\frac{1}{v^2(s_0)} v_{,s}(s_0) \ . \tag{4}$$

In the following, we shall not use the argument s_0 in the quantities $v(s_0)$, $v_{,s}(s_0)$ and $M(s_0)$, and we shall remember that all of them are determined at $s = s_0$ and $n = 0$. We shall also use the notation $\tau(s_0,0) = \tau_0$. Then we get

$$\tau(s,n) \approx \tau_0 + \frac{1}{v}(s - s_0) - \frac{1}{2v^2}v_{,s}(s - s_0)^2 + \frac{1}{2}Mn^2 \quad . \tag{5}$$

In the ray centered coordinates (s,n) , this is the final equation, which can be used to determine $\tau(s,n)$ at any point $S(s,n)$ close to the point $s = s_0$ of the ray Ω .

To remove the difficulty connected with the determination of the ray centered coordinates (s,n) of the point S , we shall introduce the rectangular Cartesian coordinate system (l,m) with the origin at the point $s = s_0$. The basis of the new rectangular coordinate system is formed by two unit vectors $\vec{l}$ and $\vec{m}$, which coincide with $\vec{t}$ and $\vec{n}$ at the point $s = s_0$, see Fig. 1. The only difference between the two systems is that the system (l,m) is Cartesian, whereas the ray centered coordinate system (s,n) is curviliniar (albeit orthogonal). The difference between (s,n) and (l,m) can be clearly seen in Fig. 1.

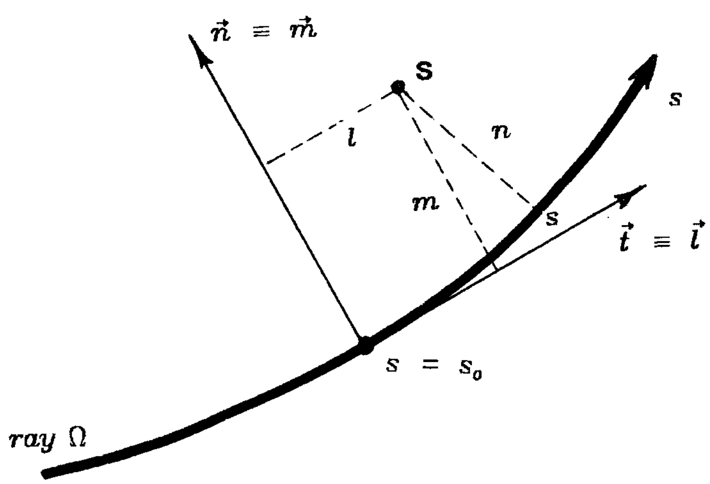

FIG. 1.

The length element dr is given by $\sqrt{h^2ds^2 + dn^2}$ in the ray centered coordinate system (s,n) and by $\sqrt{dl^2 + dm^2}$ in the Cartesian coordinate system (l,m) . Thus, to rewrite Eq. (5) in the Cartesian coordinate system (l,m) , we must use the relation $ds = h^{-1}dl$. Assume that $(s - s_0)$ and l are small, then we can consider $ds \approx s - s_0$ and $dl \approx l$ and write

$$s - s_0 = l/h \approx l/(1 + \frac{v_{,n}}{v}n) \approx l - \frac{v_{,n}}{v}ln \quad . \tag{6}$$

The quantity m is also different from n, see Fig. 1, but the differences would appear only in higher order terms of (5) and we can neglect them. Then we can rewrite (6) as follows

$$s - s_0 \approx l - \frac{v_{,m}}{v}lm \quad . \tag{6'}$$

Inserting (6′) into (5), we obtain

$$\tau(l,m) \approx \tau_0 + \frac{l}{v} - \frac{v_{,m}}{v^2}lm - \frac{1}{2}\frac{v_{,l}}{v^2}l^2 + \frac{1}{2}Mm^2 \quad . \tag{7}$$

This is the final equation for the travel-time field $\tau(l,m)$ in the neighborhood of some specified point $s = s_0$ of the ray Ω in the Cartesian coordinate system (l,m). All the quantities $\tau_0, v, v_{,m}\ v_{,l}$ and M are determined at $s = s_0$. The determination of the coordinates (l,m) of the point S is straightforward.

It is easy to see that Eq. (7) can be rewritten in a very convenient matrix form,

$$\tau(l,m) \approx \tau_0 + v^{-1}l + \frac{1}{2}\mathbf{m}^T\mathbf{B}\mathbf{m} \ , \tag{8}$$

where

$$\mathbf{m} = \begin{bmatrix} m \\ l \end{bmatrix} , \quad \mathbf{m}^T = [m \quad l] \qquad \mathbf{B} = \frac{1}{v^2}\begin{bmatrix} v^2M & -v_{,m} \\ -v_{,m} & -v_{,l} \end{bmatrix} \tag{9}$$

($\mathbf{m}^T$ is transpose of $\mathbf{m}$).

3. Second Derivatives of the Travel-time Field in the Neighborhood of a Specified Point of the Ray Ω

Equations (7) and (8) can be simply used to determine the first and the second derivatives of $\tau(l,m)$ with respect to l and m *at any point* $s = s_0$ *of the ray* Ω *, along which the dynamic ray tracing was performed.* From (7), we easily obtain

$$\frac{\partial^2\tau}{\partial l^2} = -\frac{1}{v^2}v_{,l} \quad , \quad \frac{\partial^2\tau}{\partial m^2} = M \quad , \quad \frac{\partial^2\tau}{\partial l\,\partial m} = -\frac{1}{v^2}v_{,m} \quad . \tag{10}$$

For the first derivatives of τ, we obtain from (7) certain equations, valid even in some neighborhood of the ray Ω,

$$\frac{\partial\tau}{\partial l} \approx \frac{1}{v} - \frac{1}{v^2}v_{,m}m - \frac{1}{v^2}v_{,l}l \ , \tag{11}$$

$$\frac{\partial \tau}{\partial m} \approx -\frac{1}{v^2} v_{,m}\, l + Mm \quad . \tag{12}$$

Here again the quantities $v, v_{,m}\, v_{,l}$ and M are taken at the point $s = s_0$. Especially the second equation (12) is very interesting. It gives us the component of the slowness vector in the direction perpendicular to the ray. We shall use and discuss (12) in more detail later.

4. Travel-time field in the Neighborhood of the Ray Ω in (x,z) Coordinates

In practical applications, we usually do not work in coordinates connected with some curved ray, but in a fixed rectangular coordinate system. In this section, we shall consider the most common coordinate system, i.e., the Cartesian coordinate system (x,z) , where z represents depth (increasing downwards) and x the length coordinate along the profile (increasing from the left to the right). It is easy to transform Eq. (8) into the coordinates (x,z) . It is just necessary to rotate the system (m,l) . Before we do it, we strictly define individual directions and the angle involved in the rotation. The vector $\vec{l}$ is defined in a standard way, it is tangent to the ray Ω at the point $s = s_0$ and is oriented in the direction of increasing s . The vector $\vec{m}$, perpendicular to $\vec{l}$, is oriented to the left from $\vec{l}$ (so that the movement from $\vec{l}$ to $\vec{m}$ is counterclockwise). We introduce the angle i between the vector $\vec{l}$ and the z-axis (which is oriented downwards). The angle i is measured positively when the movement from the z-axis to $\vec{l}$ is counterclockwise, see two examples in Fig. 2.

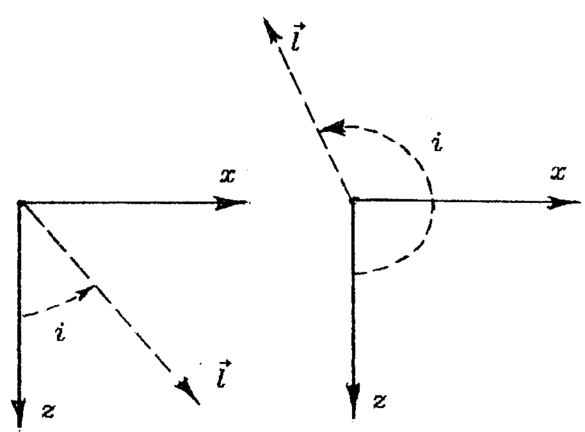

FIG. 2.

Thus, i may reach any value from 0 to 2π. Assume that the point $s = s_0$ corresponds to the point $[x_0, z_0]$ in the Cartesian coordinate system (x,z). Then it can be easily verified in Fig. 3 that the relations between m,l and $(x - x_0)$, $(z - z_0)$ are as follows:

$$m = (x - x_0)\cos i - (z - z_0)\sin i \ ,$$

$$l = (x - x_0)\sin i + (z - z_0)\cos i \ . \qquad (13)$$

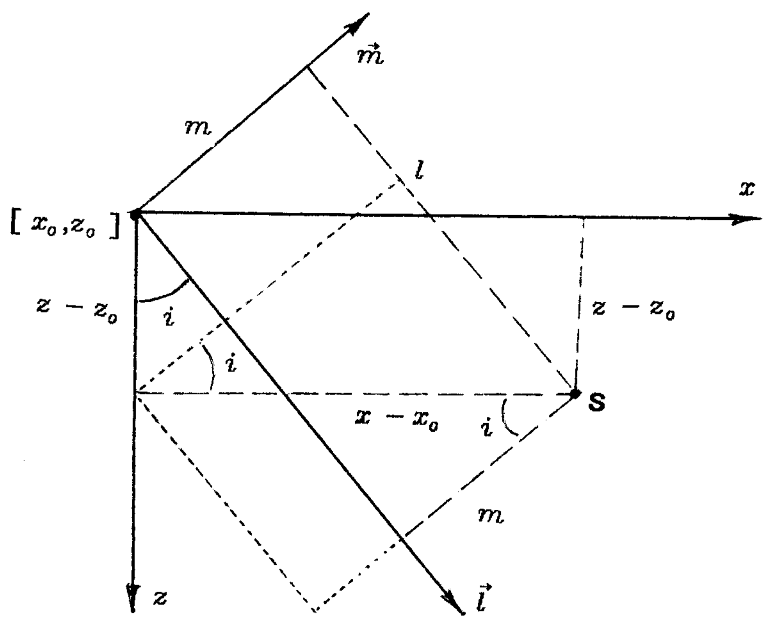

FIG. 3.

We now denote the transformation matrix by **A** ,

$$\mathbf{A} = \begin{pmatrix} \cos i & -\sin i \\ \sin i & \cos i \end{pmatrix} . \qquad (14)$$

When we further denote

$$\mathbf{x} = \begin{pmatrix} x - x_0 \\ z - z_0 \end{pmatrix} , \qquad (15)$$

we can rewrite (13) as follows

$$\mathbf{m} = \mathbf{A}\mathbf{x} \ , \qquad (16)$$

where $\mathbf{m}$ is given by (9). Taking into account that $\mathbf{m}^T = \mathbf{x}^T \mathbf{A}^T$, where T denotes a transpose, we get from (8),

$$\tau(x,z) \approx \tau(x_0,z_0) + \frac{1}{v}(x - x_0)\sin i + \frac{1}{v}(z - z_0)\cos i + \frac{1}{2}\mathbf{x}^T\mathbf{W}\mathbf{x} , \tag{17}$$

where

$$\mathbf{W} = \mathbf{A}^T\mathbf{B}\mathbf{A} = \frac{1}{v^2}\begin{pmatrix} \cos i & \sin i \\ -\sin i & \cos i \end{pmatrix} \begin{pmatrix} v^2 M & -v_{,m} \\ -v_{,m} & -v_{,l} \end{pmatrix} \begin{pmatrix} \cos i & -\sin i \\ \sin i & \cos i \end{pmatrix} . \tag{18}$$

This is the final equation for the determination of the travel-time field in the fixed coordinate system (x,z) at some point $[x,z]$ situated close to the point $[x_0,z_0]$ of the ray Ω. The quantities $i, v, v_{,m}, v_{,l}$ and M in (17) and (18) are determined at the point $[x_0, z_0]$. Let us note that the derivatives $v_{,m}$ and $v_{,l}$ can be simply expressed in terms of derivatives $v_{,x}$ and $v_{,z}$,

$$v_{,m} = v_{,x}\cos i - v_{,z}\sin i ,$$

$$v_{,l} = v_{,x}\sin i + v_{,z}\cos i . \tag{19}$$

It might be useful to write explicit equations for individual elements of the matrix $\mathbf{W}$. They are as follows:

$$W_{11} = M\cos^2 i - v_{,x}\, v^{-2}(\cos^2 i + 1)\sin i + v_{,z}\, v^{-2}\sin^2 i\cos i ,$$

$$W_{22} = M\sin^2 i - v_{,z}\, v^{-2}(\sin^2 i + 1)\cos i + v_{,x}\, v^{-2}\sin i\cos^2 i , \tag{20}$$

$$W_{12} = W_{21} = -M\sin i\cos i - v_{,x}\, v^{-2}\cos^3 i - v_{,z}\, v^{-2}\sin^3 i .$$

For completeness, we shall also write here the expression for $v_{,nn}$ in terms of $v_{,xx}$, $v_{,xz}$ and $v_{,zz}$, which simply follows from the above transformation equations,

$$v_{,nn} = v_{,xx}\cos^2 i - 2v_{,xz}\sin i\cos i + v_{,zz}\sin^2 i . \tag{21}$$

This expression is needed in the dynamic ray tracing equation (2).

5. Second Derivatives of the Travel-time Field with Respect to (x,z) Coordinates

It is now easy to determine the first and the second derivatives of the travel-time field $\tau(x,z)$ with respect to x and z. For the first derivatives, we obtain from (17),

$$\frac{\partial \tau}{\partial x} = \frac{1}{v}\sin i + (x - x_0)\, W_{11} + (z - z_0)\, W_{12} ,$$

$$\frac{\partial \tau}{\partial z} = \frac{1}{v}\cos i + (x - x_0)\, W_{12} + (z - z_0)\, W_{22} . \qquad (22)$$

These equations allow us to determine the first derivatives of $\tau(x,z)$ even in some neighborhod of Ω. For the second derivative , at the point $[x_0,z_0]$ of the ray Ω, we obtain the following equations,

$$\frac{\partial^2 \tau}{\partial x^2} = W_{11} , \qquad \frac{\partial^2 \tau}{\partial z^2} = W_{22} , \qquad \frac{\partial^2 \tau}{\partial x \partial z} = W_{12} . \qquad (23)$$

6. Concluding Remarks

Many applications of second derivatives of the travel-time field in seismology and seismic prospecting can be found in Gol'din (1979) and in Hubral and Krey (1980).

The derivation of equations for the second derivatives of the travel-time field presented in the first part of this paper follows mainly the paper by Červený and Hron (1980). In that paper, an equation equivalent to (7), but for a 3-D case, was derived.

REFERENCES

Červený, V., 1981, Dynamic Ray Tracing in 2D Media, this volume.

Červený, V., and Hron, F., 1980, The Ray Series Method and Dynamic Ray Tracing Systems for 3-D Inhomogeneous Media: *Bull. Seimol. Soc. Am.*, Vol. 70, 47-77.

Gol'din, S.V., 1979, Interpretation of Seismic Data Obtained by the Method of Reflected Waves: Moscow, Nedra. (In Russian.)

Hubral, P., and Krey, Th., 1980, Interval Velocities from Seismic Reflection Time Measurements: *SEG Monograph Series* No. 3, Tulsa.