Dynamic Ray Tracing in 2-D Media

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1. Introduction

In standard ray tracing, the slowness vector $\vec{p} = \nabla\tau$ is computed at each point of the ray. The slowness vector determines fully the first derivatives of the travel-time field $\tau(x,z)$.

In many applications, it would be very useful to know also the second derivatives of the travel-time field. They have a large importance in themselves, e.g. in the solution of inverse problems, in the solution of two-point ray tracing, in the evaluation of the geometrical spreading and ray amplitudes, etc. They may be also used to determine the basic geometrical characteristics of the wave fronts (e.g. principal curvatures). In fact, they give us some information about the travel-time field not only directly on the ray, but also in its neighborhood.

We shall derive various forms of ordinary differential equations which can be used to compute the second derivatives of the travel time field along any known ray. The ray itself can be evaluated by arbitrary methods. Due to the role of these ordinary differential equations in the computation of dynamic characteristics of seismic waves, we shall call them dynamic ray tracing equations or dynamic ray tracing systems. For simplicity, we shall consider only a 2-D medium.

Let us emphasize that the dynamic ray tracing can be used to determine the geometrical spreading by the computation *along just one ray;* there is no need of the computation of the ray diagram and evaluation of geometrical spreading by the direct measurement of the elementary cross-sectional area of the ray tube from the ray diagram.

Dynamic ray tracing is closely connected to the computation of Gaussian beams (see Červený (1981)). In case of Gaussian beams, the second derivatives become complex-valued. This will be described elsewhere.

2. Ray Centered Coordinates

Let us select an arbitrary ray Ω and introduce an orthogonal coordinate system

$$(s,n), \tag{1}$$

connected with this ray, see Fig. 1.

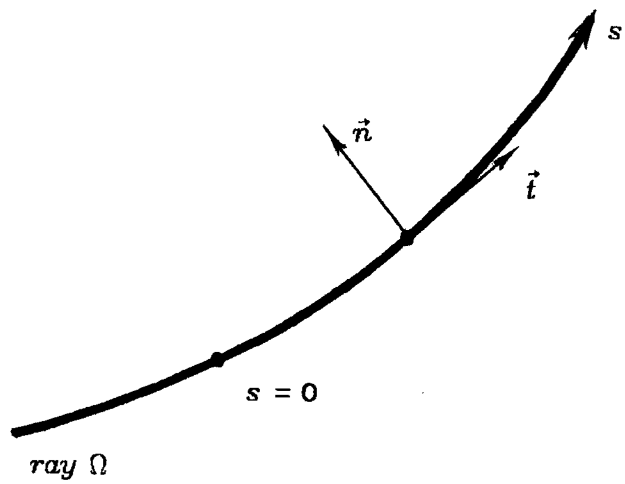

FIG. 1.

The coordinate s measures the arc length along the ray from an arbitrary reference point, n represents a length coordinate in the direction perpendicular to Ω at s. The basis of the new coordinate system is formed by two unit vectors $\vec{t}$ and $\vec{n}$, where $\vec{t}$ is the unit tangent and $\vec{n}$ the unit normal to the ray Ω. Let us now consider a point $S(s,n)$ in the neighborhood of the ray Ω. The position vector $\vec{r}(s,n)$ of the point S can be determined by the relation

$$\vec{r}(s,n) = \vec{r}(s,0) + n\,\vec{n}(s), \tag{2}$$

see Fig. 2.

Now we shall prove that the system (s,n) is orthogonal. To do this, we first find the expression for the infinitesimal length element dl in the coordinate system (s,n), $dl^2 = d\vec{r}\cdot d\vec{r}$. For $d\vec{r}$, we simply obtain from (2),

$$d\vec{r} = \frac{d\vec{r}(s,0)}{ds}ds + n\,\frac{d\vec{n}(s)}{ds}ds + \vec{n}(s)dn.$$

Using the definition that $d\vec{r}(s,0)/ds = \vec{t}$, the equation for $d\vec{r}$ can be rewritten as follows,

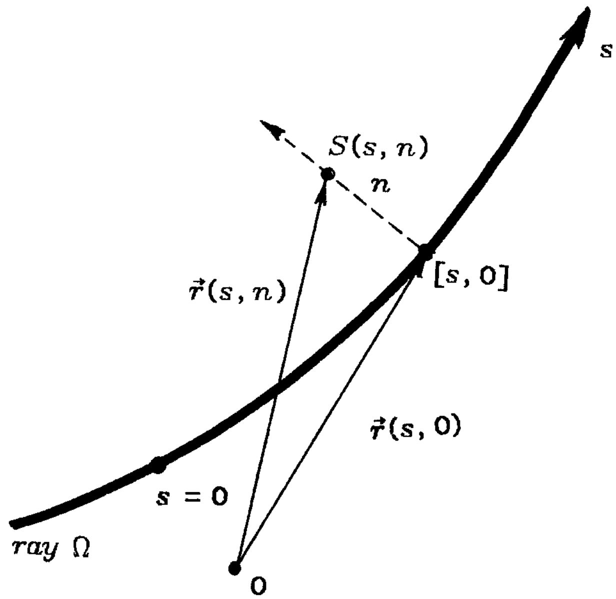

FIG. 2.

$$d\vec{r} = \vec{t}(s)ds + \vec{n}(s)dn + \frac{d\vec{n}(s)}{ds} n \; ds \, . \tag{3}$$

To find the final expression for $d\vec{r}$, it is necessary to determine $d\vec{n}(s)/ds$ along the ray. It would be possible to use some equation from differential geometry, but we shall not do it, we shall derive all equations here. This will make the whole derivation more clear.

As $\vec{t}$ and $\vec{n}$ are two unit mutually perpendicular vectors, it holds $\vec{n}\cdot\vec{n} = 1$, $\vec{n}\cdot\vec{t} = 0$. Differentiating these equations with respect to s gives

$$\vec{n}\cdot\frac{d\vec{n}}{ds} = 0 \, , \qquad \vec{t}\cdot\frac{d\vec{n}}{ds} = -\vec{n}\cdot\frac{d\vec{t}}{ds} \, .$$

It follows from the first equation that $d\vec{n}/ds$ is a vector in a direction of $\vec{t}$ (it has no component of $\vec{n}$). Thus we can write

$$\frac{d\vec{n}}{ds} = -k(s)\vec{t}, \tag{4}$$

where $k(s)$ is some function of s, not yet determined. The sign "-" was chosen for convenience. It follows from the second equation and from (4) that

$$k(s) = \vec{n}\cdot\frac{d\vec{t}}{ds}. \tag{4'}$$

Thus, we have rewritten $d\vec{n}/ds$ in terms of $d\vec{t}/ds$. This is now much better, as the derivative $d\vec{t}/ds$ can be simply determined from the ray tracing system. We remember that the ray tracing system has the form

$$\frac{d\vec{x}}{ds} = V\vec{p} \ , \qquad \frac{d\vec{p}}{ds} = -\frac{1}{V^2}\nabla V ,$$

where $V = V(x,z)$ is the propagation velocity and $\vec{p}$ is the slowness vector $\vec{p} = \vec{t}/V$. Inserting $\vec{p} = \vec{t}/V$ into the second ray tracing equation gives

$$\frac{1}{V}\frac{d\vec{t}}{ds} + \vec{t}\,\frac{d}{ds}\frac{1}{V} = -\frac{1}{V^2}\nabla V .$$

Multiplying this equation by $V\vec{n}$ directly at the ray Ω gives

$$\vec{n}\cdot\frac{d\vec{t}}{ds} = k(s) = -\left[\frac{\vec{n}\cdot\nabla V}{V}\right]_{\Omega} \tag{5}$$

In our ray-centered coordinate system (s,n) we have $V = V(s,n)$. In the following we shall use the following notation

$$v = V(s,0) , \qquad v_{,n} = \left[\frac{\partial V(s,n)}{\partial n}\right]_{n=0} . \tag{6}$$

Thus, v and $v_{,n}$ are measured directly on Ω, they are functions of s only, not of n. Using these symbols simplifies the following equation. In this notation, equation (5) can be rewritten as follows

$$k(s) = -\frac{v_{,n}}{v} . \tag{5'}$$

Inserting (4) and (5′) into (3) gives

$$d\vec{r} = h\,\vec{t}(s)\,ds + \vec{n}(s)\,dn , \tag{7}$$

where h is given by the equation

$$h = 1 + \frac{v_{,n}}{v}n . \tag{8}$$

Then we obtain easily for $dl^2 = d\vec{r}\cdot d\vec{r}$,

$$dl^2 = d\vec{r}\cdot d\vec{r} = h^2 ds^2 + dn^2 .$$

Thus, we can see that the expression for dl^2 contains only terms with dn^2 and ds^2, not with $ds\,dn$. This means that the ray centered coordinate system (s,n) is orthogonal, with scale factors $(h,1)$. Taking this into account, we can simply rewrite any vectorial differential equation in ray centered coordinates. For example,

$$\nabla\tau = \frac{1}{h}\frac{\partial\tau}{\partial s}\vec{t} + \frac{\partial\tau}{\partial n}\vec{n} . \qquad (9)$$

It just remains to say that the quantity $k(s)$ introduced by (4) represents the curvature of the ray. This follows immediately from the Frenet's equation from differential geometry. In the following, however, we shall not work with the curvature of the ray, the expression (8) is more suitable for us.

Note that in case of a curved ray Ω the system of normals constructed to Ω can intersect mutually at larger distances from Ω. In other words, the ray centered coordinate system (s,n) is not regular at large distances from Ω. In the following, we shall consider only a region along Ω at which the ray centered coordinate system (s,n) is regular and call it "the regular region".

3. Eikonal equation in the Ray Centered Coordinates

From (9), the eikonal equation $\nabla\tau\cdot\nabla\tau = 1/V^2(s,n)$ in the ray centered coordinates (s,n) is

$$\frac{1}{h^2}\left[\frac{\partial\tau}{\partial s}\right]^2 + \left[\frac{\partial\tau}{\partial n}\right]^2 = \frac{1}{V^2(s,n)} , \qquad (10)$$

where $V(s,n)$ is the propagation velocity. We shall restrict our attention to the immediate vicinity of the central ray Ω, which is characterized by small values of n. Note that $\partial\tau/\partial n = 0$ at Ω, as the wavefront is perpendicular to the ray. Thus, the Taylor expansion of the phase function $\tau(s,n)$ in the neighborhood of Ω consisting of terms up to the second order in n is

$$\tau(s,n) \approx \tau(s,0) + \frac{1}{2}\left[\frac{\partial^2\tau(s,n)}{\partial n^2}\right]_{n=0} n^2 . \qquad (11)$$

We define

$$M(s) = \left[\frac{\partial^2\tau(s,n)}{\partial n^2}\right]_{n=0} .$$

The function $M(s)$ represents the second derivative of the travel time field in the direction perpendicular to Ω. This is a quantity we are interested to know. Using the definition, we write

$$\tau(s,n) \approx \tau(s,0) + \frac{1}{2}M(s)\, n^2 . \qquad (12)$$

From this we get easily

$$\frac{\partial \tau(s,n)}{\partial s} \approx \frac{d\tau(s,0)}{ds} + \frac{1}{2}\frac{dM(s)}{ds} n^2 , \tag{13}$$

$$\frac{\partial \tau(s,n)}{\partial n} \approx M(s)\, n . \tag{14}$$

It is clear that

$$\frac{\partial \tau(s,0)}{\partial s} = \frac{1}{v(s)} . \tag{15}$$

Equations (13) and (14) can then be written in the form

$$\frac{\partial \tau(s,n)}{\partial s} \approx \frac{1}{v} + \frac{1}{2} M_{,s} n^2 , \tag{16a}$$

$$\frac{\partial \tau(s,n)}{\partial n} \approx Mn . \tag{16b}$$

In (16a), we have used a commonly accepted notation for derivatives, $M_{,s} = dM(s)/ds$. This notation will be used throughout this paper to shorten certain equations.

Now we shall find a similar expansion for $1/V^2(s,n)$, see (10), in the neighborhood of Ω (for small n). We have

$$V(s,n) \approx v + v_{,n}\, n + \frac{1}{2} v_{,nn}\, n^2 , \tag{17}$$

where $v_{,n}$ is given by (6) and $v_{,nn}$ by the equation

$$v_{,nn} = \left[\frac{\partial^2 V(s,n)}{\partial n^2}\right]_{n=0} . \tag{18}$$

Thus, $v_{,n}$ and $v_{,nn}$ are again functions of s only, not of n . Taking into account (8), we can write (17) in the following form

$$V(s,n) \approx vh + \frac{1}{2} v_{,nn} n^2 \approx vh(1 + \frac{v_{,nn}}{2vh} n^2) \approx vh \left(1 + \frac{v_{,nn}}{2v} n^2\right) . \tag{19}$$

From this immediately follows

$$\frac{1}{V^2(s,n)} \approx \frac{1}{v^2 h^2}\left(1 - \frac{v_{,nn}}{v} n^2\right) . \tag{20}$$

Now we return to the eikonal equation (10). We rewrite it,

$$\left(\frac{\partial \tau}{\partial s}\right)^2 + h^2 \left(\frac{\partial \tau}{\partial n}\right)^2 = \frac{h^2}{V^2(s,n)} .$$

Inserting (8), (16) and (20) into this equation, we obtain, with the accuracy up to the second order terms in n ,

$$\frac{1}{v^2}\left[1 + vM_{,s}n^2\right] + M^2n^2 = \frac{1}{v^2}\left[1 - \frac{v_{,nn}}{v}n^2\right] . \tag{21}$$

4. Dynamic Ray Tracing Equations

As we can see, the terms $\frac{1}{v^2}$ at both sides of (21) cancel each other. Then we obtain from (21)

$$\frac{dM(s)}{ds} + vM^2 + \frac{v_{,nn}}{v^2} = 0 \tag{22}$$

This is the final equation for the second derivative of the travel time field $M(s)$. From a mathematical point of view, (22) is an ordinary non-linear differential equation of the first order of the *Riccati type* that, in general, cannot be solved by analytical techniques. It can be, of course, solved numerically without problems.

The Riccati equation (22) can be rewritten in many other forms. We shall present here two other forms, which are especially useful in the investigation of Gaussian beams. Their advantage by comparison with (22) is that they are linear. This substantially simplifies the evaluation of the complex solutions of (22).

We introduce a new function $q(s)$ by the relation

$$M(s) = \frac{1}{vq}q_{,s} . \tag{23}$$

Then we obtain

$$\frac{dM(s)}{ds} = -\frac{1}{v^2}v_{,s}\frac{1}{q}q_{,s} - \frac{1}{vq^2}q_{,s}^2 + \frac{1}{vq}q_{,ss} .$$

Inserting this into the Riccati equation (22) yields

$$-\frac{1}{v^2q}v_{,s}\,q_{,s} - \frac{1}{vq^2}q_{,s}^2 + \frac{1}{vq}q_{,ss} + \frac{1}{vq^2}q_{,s}^2 + \frac{v_{,nn}}{v^2} = 0,$$

As we can see, the two terms with $q_{,s}^2$ cancel each other, and we obtain a linear equation

$$v\frac{d^2q}{ds^2} - v_{,s}\frac{dq}{ds} + v_{,nn}q = 0 . \tag{24}$$

This is an ordinary linear differential equation of second order. This is another form of the dynamic ray tracing equation. When we solve it and determine q and $q_{,s}$, the second derivative of the travel time field is obtained from (23). Equation (24) can also be rewritten in a more compact form

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dq}{ds}\right) + \frac{v_{,nn}}{v^2}q = 0 . \tag{25}$$

We shall now present what seems to be the most useful form of the dynamic ray tracing equations. We introduce a new function $p(s)$ by the relation

$$p(s) = \frac{1}{v(s)}q_{,s} , \quad i.e. \; M(s) = \frac{p(s)}{q(s)} . \tag{26}$$

Then, we obtain from (25) and (26) a *dynamic ray tracing system* in the following form

$$\boxed{\begin{aligned} \frac{dq}{ds} &= vp \\ \frac{dp}{ds} &= -\frac{v_{,nn}}{v^2}q \end{aligned}} \tag{27}$$

Thus, we have arrived at two simple linear ordinary differential equations of first order. We may write them in a more convenient matrix form as

$$\frac{d\mathbf{X}}{ds} = \mathbf{CX} , \tag{28}$$

where **X** as a column vector and **C** is a square 2 x 2 matrix,

$$\mathbf{X} = \begin{bmatrix} q \\ p \end{bmatrix} , \qquad \mathbf{C} = \begin{bmatrix} 0 & v \\ -\frac{v_{,nn}}{v^2} & 0 \end{bmatrix} \tag{29}$$

To complete the dynamic ray tracing equations (or systems), it would be necessary to derive the initial conditions at a source and at interfaces. It would be also useful to learn more about physical meaning of the quantities M,p,q, about their relation to other important quantities (curvature of the wavefront, geometrical spreading, etc) and discuss their possible applications. This will be done elsewhere.

5. Concluding Remarks

The above shown dynamic ray tracing equations and systems for an arbitrarily lateral inhomogeneous media were first derived and investigated in a slightly different form by Babich et al., see e.g. Babich and Buldyrev (1972) and other references given there. They used the ray centered coordinate system and the dynamic ray tracing to study the solution of the wave equation concentrated close to rays (Gaussian beams) and in connection with the theory of resonators. The ray centered coordinate system was introduced to seismology by Popov and Pšenčik (1978a, 1978b). Popov and Pšenčik used the ray tracing systems to determine the geometrical spreading and ray amplitudes of seismic body waves along a ray. (See also Červený, Molotkov and Pšenčik (1977), Červený and Pšenčik (1979), and Červený (1981).)

In a 3-D case, all the dynamic ray tracing equations are very similar, only the quantities M,p, and q have a matrix form (2 x 2 matrices). See more details in Červený and Hron (1980), Hubral (1980), Hubral and Krey (1980), Popov and Pšenčik (1978a, 1978b). The derivation of dynamic ray tracing equations presented here follows in principle the paper Červený and Hron (1980), but several steps are considerably simplified. From this point of view, the derivation of these equations presented here is new.

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