Chapter V: Extrapolation Of Scalar Wavefields in an Inhomogeneous Medium

Abstract

A formulation for one-way wave equations is given that is both accurate and stable. The equations are generated from the full wave equation by a continued fraction square root recursion. The solutions are WKBJ-accurate solution in the extrapolation direction. The resulting operators are unconditionally stable. For acoustic wavefields the state variable of the system is pressure divided by the square root of impedance.

The extrapolation equations are illustrated with the problem of computing Love wave modes in a laterally varying medium.

5.1 Introduction

One-way extrapolation operators provide an economical and accurate method for modelling certain types of wave motion. The operators are designed to march the solution from plane to another in a particular spatial extrapolation direction. The basic restriction on the use of extrapolation methods is that only the transmitted wave in the direction of extrapolation is modelled. In many applications (i.e. migration) this restriction is a blessing rather than a deficiency of the method.

For inversion purposes, one-way operators provide a means of backtracking a wave to its point of origin (or reflection). In this case, the operators need to be at least as accurate as the WKBJ solution for the direct wave between any two points in the medium.

The use of one-way operators will entail a few approximations which will lead to errors in the computed solutions. Two of these, anisotropic dispersion, and grid dispersion, are fairly well understood (Claerbout, 1976) and will only be briefly mentioned in this chapter. Anisotropic dispersion occurs with waves that are travelling at a significant angle with respect to the extrapolation direction. The extrapolation operators introduce an artificial angular dependence to the velocity that causes wavefront with large dips to be extrapolated to the wrong place. Anisotropic dispersion can be reduced by using higher order extrapolation operators. Grid dispersion occurs as a result of using finite differences to approximate the partial differential operators in the extrapolation equations. In this case, higher frequency waves travel at a different velocity than lower frequency waves. The effects of grid dispersion can be minimized (at least for modelling purposes) by choosing a finer grid interval.

In this chapter, a set of one-way extrapolation operators are derived for a laterally varying medium. The computational errors with which we will be primary concerned are the amplitude effects of the operators. Associated with this problem however, is the question of stability. As will be shown in the next two sections, the inclusion of all amplitude effects can make the extrapolation operators unstable. Unstable extrapolators are of course, useless.

The extrapolation equations are illustrated with the computation a Love wave modes in a laterally varying medium. This example is a useful test because the modes contain both propagating and evanescent components.

5.2 Recasting The Scalar Wave Equation

The derivation presented in this section is based on the acoustic wave equation

$$\left[\nabla\cdot \frac{1}{\rho}\nabla + \frac{\omega^2}{K}\right] P = 0 \tag{5.1}$$

where ρ is the density, K is the bulk modulus, and P is the pressure wavefield. Analogous results for the scalar SH-displacement equation can be obtained by inspection.

The first step in the derivation is to isolate the ω^2-term from the bulk modulus in equation (5.1). The necessity of this step will not be apparent until we generate the one-way operators. At that point will need to require that the ω^2-term commute with the z −derivatives. To achieve the separation we simply rewrite equation (5.1) as

$$\frac{1}{\sqrt{K}}\left[\sqrt{K}\ \nabla\cdot \frac{1}{\rho}\nabla \sqrt{K} + \omega^2\right]\left[\frac{P}{\sqrt{K}}\right] = 0 \tag{5.2}$$

and cancel the leading $1/\sqrt{K}$ term. The physical variable of the system is now $P/\sqrt{K}$.

To put equation (5.2) in the form of an extrapolator, the x −derivatives and ω^2-term are moved to the right side of the equation. Thus,

$$(-D_z^H D_z)\left[\frac{P}{\sqrt{K}}\right] = \left[-\omega^2 + D_x^H D_x\right]\left[\frac{P}{\sqrt{K}}\right] \tag{5.3}$$

where the spatial derivative operators are defined as

$$D_z = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x}\sqrt{K}, \qquad D_x^H = \sqrt{K}\left[\frac{\partial}{\partial x}\right]^H \frac{1}{\sqrt{\rho}}$$

and

$$D_z = \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial z} \sqrt{K}, \qquad D_z^H = \sqrt{K} \left[\frac{\partial}{\partial z} \right]^H \frac{1}{\sqrt{\rho}}$$

To understand the role of $\left[\frac{\partial}{\partial x}\right]^H$ and $\left[\frac{\partial}{\partial z}\right]^H$ in the definitions of D_x^H and D_z^H, it is necessary to introduce the concept of a directed derivative. Taking the x-derivative for example, we will define the $\frac{\partial}{\partial x}$ by

$$\frac{\partial}{\partial x} f(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x} \tag{5.4}$$

It is important to note in this definition that for the discrete case, the derivative is "spatially causal". That is, it uses only past (smaller x) values. The discrete operator in this case is a bidiagonal matrix whose diagonal and sub-diagonal enteries are $1/\Delta x$ and $-1/\Delta x$ respectively. Clearly, in the discrete case, the $\left[\frac{\partial}{\partial x}\right]^H$ operator is simply the transpose of this matrix. The corresponding continuous definition is then

$$\left[\frac{\partial}{\partial x}\right]^H f(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x+\Delta x)}{\Delta x} \tag{5.5}$$

which one can consider to be "spatially causal" in the $(-x)$-direction.

The eigenvaules of $\partial/\partial x$ and it Hermitian transpose have positive real parts. This means that eigenvalues of the second order operator $\left[\frac{\partial}{\partial x}\right]^H \frac{\partial}{\partial x}$ are strictly positive and real. Also note, that in the limit as $\Delta x \to 0$ in definition (5.5),

$$\left[\frac{\partial}{\partial x}\right]^H = -\frac{\partial}{\partial x}$$

5.3 One-Way Extrapolation Operators

To form one-way equations, we need to take the square root of both sides of equation (5.3). It is not clear how to directly take the square root of the operator $(-D_z^H D_z)$. The approach taken here is to decompose the operator into a square part and a correction term. That is, we write $(-D_z^H D_z)$ as

$$(-D_z^H D_z) = \tilde{D}_z^2 - C(x,z) \tag{5.6}$$

where the square part of the operator is

$$\tilde{D}_z = \sqrt{v}\,\frac{\partial}{\partial z}\sqrt{v} \tag{5.7}$$

and the correction term is

$$C(x,z) = \frac{K}{4\rho}\left[\frac{1}{2}\left(\frac{K_z}{K}\right)^2 + \frac{3}{2}\left(\frac{\rho_z}{\rho}\right)^2 + \frac{1}{2}\left(\frac{\rho_z}{\rho} + \frac{K_z}{K}\right)^2 - \frac{K_{zz}}{K} - \frac{\rho_{zz}}{\rho}\right] \tag{5.8}$$

Note that the correction term $C(x,z)$ is a multiplicative operator because it contains no spatial derivatives that operate on the wavefield itself. It is also second order in length scale. We will neglect the effect of $C(x,z)$ in the remainder of this chapter.

With the z-derivative operator defined by equation (5.6), the wave equation is now

$$\tilde{D}_z^2\left[\frac{P}{\sqrt{K}}\right] = \left[-\omega^2 + D_x^H D_x\right]\left[\frac{P}{\sqrt{K}}\right] \tag{5.9}$$

Formally taking the square root of (5.3) to obtain a one-way wave equation we have

$$\tilde{D}_z\left[\frac{P}{\sqrt{K}}\right] = \sqrt{v}\,\partial_z\sqrt{v}\left[\frac{P}{\sqrt{K}}\right] = \pm S_n\left[\frac{P}{\sqrt{K}}\right] \tag{5.10}$$

where S_n is the n^{th} approximate to the exact square root S_∞,

$$S_\infty = \sqrt{-\omega^2 + D_x^H D_x} \tag{5.11}$$

The approximations can be recursively generated by continued fraction expansion[1]

$$S_n = -i\omega + \frac{D_x^H D_x}{-i\omega + S_{n-1}}, \qquad S_0 = -i\omega \tag{5.12}$$

In taking the square in equation (5.9), have have assumed that the operators $\tilde{D}_z$ and S_∞ commute. Thus, we have neglected the commutator term (F)

$$F = (\tilde{D}_z + S_\infty)^{-1}\,(S_\infty \tilde{D}_z - \tilde{D}_z S_\infty) \tag{5.13}$$

If the medium parameters (K and ρ) are strong functions of z, the commutator term will be large. Neglecting it means that there will amplitude errors in the extrapolation solution.

[1]Francis Muir, personal communication.

To show that equation (5.12) converges to the square root in equation (5.11), let $S_{n-1} \rightarrow S_n \rightarrow S_\infty$ as $n \rightarrow \infty$. Hence equation (5.12) becomes

$$S_\infty = -i\omega + \frac{D_x^H D_x}{-i\omega + S_\infty}$$

Clearing the denominator by operating from either the left or the right with $(-i\omega + S_\infty)$, we have

$$S_\infty^2 = -\omega^2 + D_x^H D_x$$

which confirms the convergence of the recurrence relation.

The choice of sign in equation (5.8) corresponds to up and downgoing waves. In this paper we choose the minus sign which is the appropriate sign for downward wave extrapolation, but a similar result holds for the plus sign.

To put equation (5.10) in the form of an extrapolator, the velocity terms are moved to the right side thus putting the equation in the symmetric form

$$\frac{\partial}{\partial z}\tilde{P} = -\frac{1}{\sqrt{v}} S_n \frac{1}{\sqrt{v}} \tilde{P} \tag{5.14}$$

where $\tilde{P}$ is the state variable and is defined by

$$\tilde{P} = \frac{P}{\left(\rho K\right)^{1/4}} = \frac{P}{\sqrt{r}} \tag{5.15}$$

where r is the acoustic impedance.

5.4 Stability and Accuracy of the Extrapolation Operators

Two points must be considered when judging the suitability of the extrapolators presented in the previous section: accuracy and stability. Stability is the first point to check because a perfectly accurate solution is useless if it is masked by an extraneous solution that is exponentially growing the the direction of extrapolation.

For a solution produced by the extrapolation equation to be stable, we will require that

$$\frac{\partial}{\partial z}\int dx\ \tilde{P}^H \tilde{P} = \frac{\partial}{\partial z}\int dx\ \frac{P^H P}{r} \leq 0 \tag{5.16}$$

Physically, the quantity $P^H P/r$ is the energy flux per unit area. Performing the z-differentiation we have

$$\int dx \left[\left(\frac{\partial}{\partial z} \tilde{P} \right)^H \tilde{P} + \tilde{P}^H \left(\frac{\partial}{\partial z} \tilde{P} \right) \right] \leq 0 \tag{5.17}$$

Substituting from equation (5.11), the condition becomes

$$- \int dx \ \tilde{P}^H \frac{1}{\sqrt{v}} \left(S_n^H + S_n \right) \frac{1}{\sqrt{v}} \tilde{P} \leq 0 \tag{5.18}$$

Equation (5.18) is a quadratic in the variable $v^{-1/2}\tilde{P}$. Consequently, for stable extrapolation we must have that $(S_n^H + S_n)$ be positive definite, or in other words the real parts of the eigenvalues of S_n must be positive.

Before considering whether S_n has positive real eigenvalues, the question of causality must be addressed. The inverse Fourier transform of $(-i\omega)^{-1}$ is $sgn(t)$ which is clearly not a causal integration operator. To preserve causality, we must consider $(-i\omega)$ to have a positive real part, that is, $(-i\omega) \to (-i\omega + \varepsilon)$. This produces causal integration because the inverse Fourier transform of $(-i\omega+\varepsilon)^{-1}$ is $H(t)$, in the limit as $\varepsilon \to 0$.

It is clear from the form of equation (5.10) that all orders of approximations to the exact square root S_∞ have the same eigenvectors. The eigenvectors are those of the operator $D_x^H D_x$. Thus, the recurrence relation only adjust the eigenvalues. To check whether the real parts of the eigenvalues of S_n, $\{\mathrm{Re}[\lambda(S_n)]\}$, are positive, we will proceed by induction, by assuming that $\mathrm{Re}[\lambda(S_{n-1})] \geq 0$. Since $(-i\omega+\varepsilon)$ has a positive real part, and the eigenvalues of $D_x^H D_x$ are purely positive and real, the sign of $\mathrm{Re}[\lambda(S_n)]$ is also strictly positive.

The two terms were neglected in deriving the extrapolator given by equation (5.14). They are the correction term of equation (5.8), and the commutator term of equation (5.13). The result is the that the oneway extrapolators are as accurate as geometric optics in the z-direction, but retain the accuracy of physical optics in the x-direction. To the accuracy in the direction of extrapolation, consider the case of a normally incident plane wave traveling in a medium that contains no lateral variations. Thus all orders of equation (5.10) generate $S_n = -i\omega$. Substituting this into equation (5.11) results in

$$\frac{\partial}{\partial z} \tilde{P} = -i \frac{\omega}{v(z)} \tilde{P} \tag{5.19}$$

or

$$\tilde{P} = e^{i\omega \int_0^z \frac{dz}{v(z)}} \tilde{P}_0 \tag{5.20}$$

In terms of the pressure field we have

$$P = \sqrt{\frac{r(z)}{r(0)}}\, e^{i\omega \int_0^z \frac{dz}{v(z)}} P_0 \tag{5.21}$$

which is the WKBJ solution for P. (See equation 3.18 for a comparison). The examples shown in the next section will demonstrate the accuracy of the solutions in the direction perpendicular to the extrapolation direction.

5.5 An Example of Scalar Extrapolation: Love Wave Modes

As an example of extrapolating scalar wavefields, the problem of computing Love wave modes in a laterally varying medium is considered. The example is useful for testing the modeling method because the modes contain both a propagating and an evanescent component. Also, because the analytic solution for horizontal layers is simple (the mode shape does not change), it is easy to check whether the procedure is producing the correct answer for the layered case.

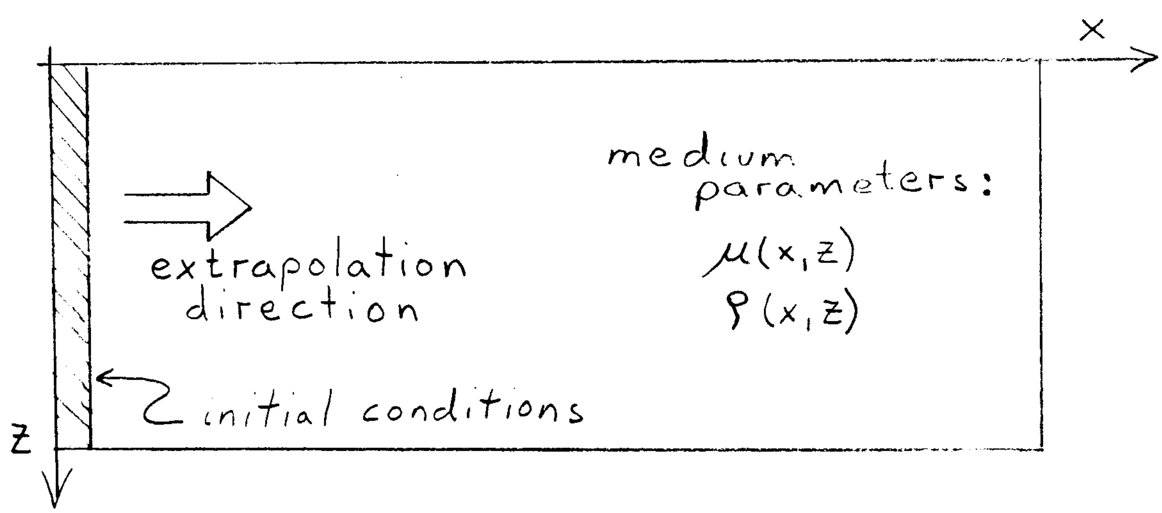

FIG. 5.1. The geometry for extrapolation modeling is shown. The initial conditions are specified on the left, and the solution in the rest of the medium is determined by extrapolation in the x-direction. The top surface is a free surface.

Consider the geometry shown in Figure 5.1. The problem here is the computation of the modes for all x and z given an initial mode shape at $x = 0$. The surface displacements can then be obtained simply by picking off the solution at $z = 0$. The simplest way to apply the results of the previous sections to the case of Love wave mode extrapolation is to interchange the roles of x and z in the extrapolations equations.

The scalar SH-displacement equation is

$$(\rho\omega^2+\nabla\cdot\mu\nabla)u=0 \tag{5.22}$$

where μ is the shear modulus, and u is the displacement in the y-direction. Identifying ρ with μ^{-1} and K with ρ^{-1}, the definitions for D_z and D_z^H become

$$D_z = \sqrt{\mu}\,\frac{\partial}{\partial z}\frac{1}{\sqrt{\rho}} \quad \text{and} \quad D_z^H = \frac{1}{\sqrt{\rho}}\left[\frac{\partial}{\partial z}\right]^H \sqrt{\mu} \tag{5.23}$$

The extrapolation equation is

$$\frac{\partial}{\partial x}\tilde{u} = -\frac{1}{\sqrt{\beta}}S_n\,\frac{1}{\sqrt{\beta}}\tilde{u} \tag{5.24}$$

where β is the shear velocity and the state variable $\tilde{u}$ is

$$\tilde{u} = (\rho\mu)^{1/4}\,u = \sqrt{r}\;u \tag{5.25}$$

The recurrence for S_n is again given by equation (5.10) with x and z interchanged.

The square root approximations given by equation (5.10) are well known to model accurately waves traveling within some cone of the extrapolation direction. The dispersion curves for the first and second approximations (15- and 45-degree) equations are shown in Figure 5.2. The approximations have no evanescent zone in the k_z direction ($|k_z| > \omega/v$), but they do model behavior in the k_x evanescent zone ($|k_x| > \omega/v$) (Landers and Claerbout, 1972). This is shown in the right panel of Figure 5.2, which is simply a replotting of the dispersion relation to show its behavior for both real and imaginary values of k_x.

In a laterally homogeneous medium, Love wave modes have a solution to the SH-displacement equation of the form (Achenbach, 1975, p. 218-220)

$$u(x,z,t) = U(z)\exp i\omega(t - \frac{x}{c}) \tag{5.26}$$

where $u(x,z,t)$ is the displacement in the y direction, $U(z)$ is the mode shape, and c is the phase velocity of the mode. For a layered earth structure the boundary conditions to be satisfied are:

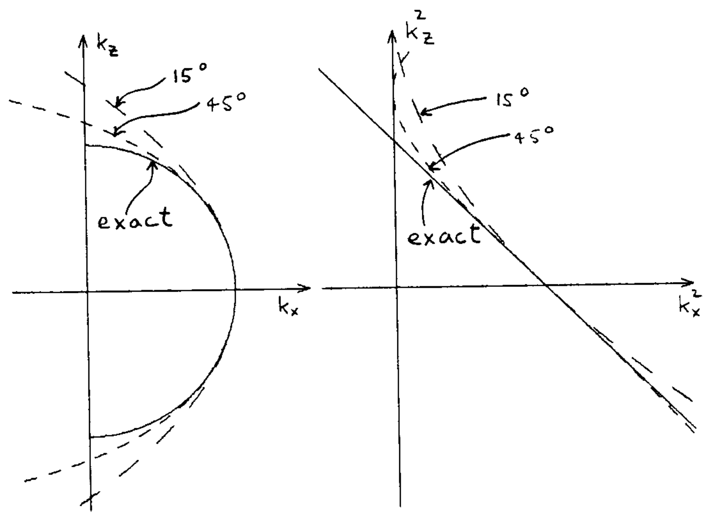

FIG. 5.2. The dispersion relations for the first two square root approximations (15- and 45-degree equations) are shown. The left panel shows the relations plotted in the conventional manner while the right panel shows the relations plotted in a manner which allows for imaginary values of k_z ($k_z^2 < 0$) and k_x ($k_x^2 < 0$). The exact dispersion relation is the straight line in the right plot. The extrapolation equations model waves well into the evanescent zone on the k_x axis.

1) $\partial_z U = 0$ at $z=0$ (zero free surface stress),

2) $[U] = 0$ at layer boundaries (continuity of displacement),

3) $[\mu \partial_z U] = 0$ at layer boundaries (continuity of stress).

The square brackets above indicate differences across layer interfaces.

For the simple case of a layer of thickness h with properties ρ_1 and μ_1 over a half space with properties ρ_2 and μ_2, the mode shape is given by

$$U(z) = \begin{cases} A\cos \nu_1 z & 0 \le z < h \\ A\cos \nu_1 h \; exp[-i\nu_2(z-h)] & z > h \end{cases} \tag{5.27}$$

where

$$\nu_1 = \sqrt{\frac{\omega^2}{\beta_1^2} - k_x^2} \text{ and } \nu_2 = \sqrt{\frac{\omega^2}{\beta_2^2} - k_x^2} \tag{5.28}$$

The boundary conditions also impose the further restriction that

$$cot(\nu_1 h) = \frac{\mu_1}{\mu_2} \frac{\nu_1}{i\nu_2}. \tag{5.29}$$

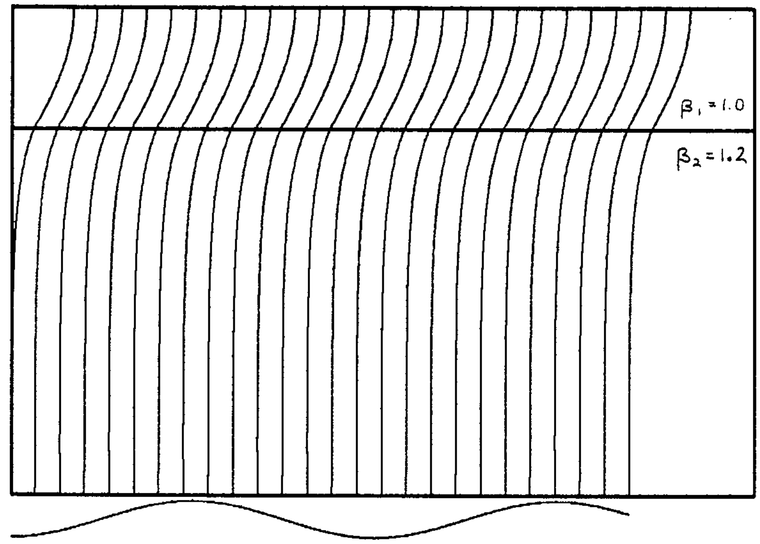

FIG. 5.3. The example shown is the extrapolation of a lower order Love wave mode for a layer over a half-space. The extrapolation is from left to right, with the initial mode shape shown on the left. The position of the layer is superimposed on the plot. The fact that the mode does not change shape significantly in the x-direction after demodulation indicates that the extrapolation solution is correct, and that the mode is being propagated at the correct phase velocity. The sinusoidal function plotted at the bottom is the demodulation function used. The frequency of the mode is 0.18 Hz and its speed is 1.04 km/sec. The plot has a 1 to 2 vertical exaggeration.

Equation (5.29) is the period equation for the simplest type of Love waves, and it has solutions in the range

$$\frac{\omega}{\beta_1} > |k_x| > \frac{\omega}{\beta_2}.$$

With k_x in this range, the mode is propagating in the layer and evanescent in the half space. The period equation determines the relationship between ω and k_x, and since it is nonlinear, the modes are dispersive (velocity depends on frequency). The propagating speed of the mode is given by

$$c(\omega) = \frac{\omega}{k_x(\omega)}. \tag{5.30}$$

To test that the extrapolation works for a layered case, an initial mode shape was specified according to equation (5.27). The solution was then extrapolated in the x-direction with a monochromatic 45 degree equation. For simplicity, the density was assumed constant. At each step in the x-direction the solution was multiplied before plotting by the demodulation factor $\exp[i\omega x / c(\omega)]$, with $c(\omega)$ determined analytically from equation (5.30). This tests two aspects of the solution. First, multiplication by the

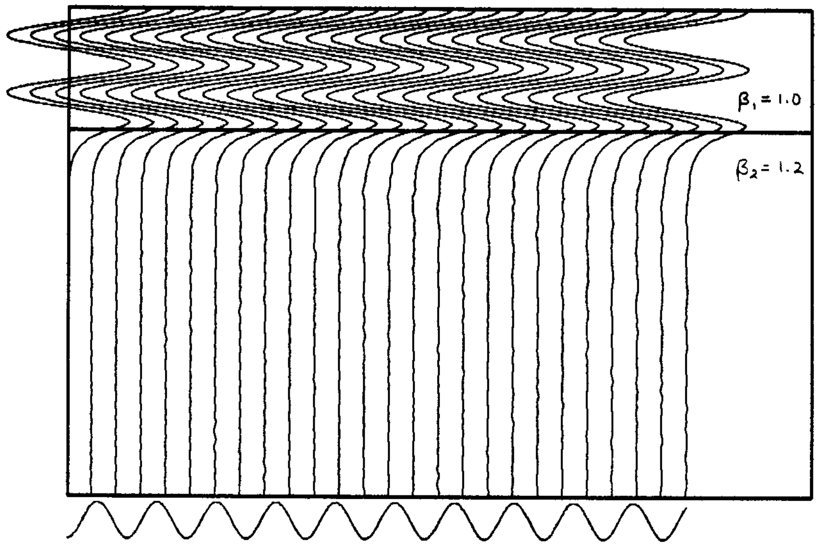

FIG. 5.4. This example is similar to the one shown in Figure 5.3, except that the mode frequency is now 1.17 Hz., and its speed is 1.12 km/sec.

demodulation factor effectively cancels the x-dependence of equation (5.26), leaving only the mode shape. Consequently, the solution if done correctly should be the same at all x-steps. Second, if the solution, after multiplication by the demodulation factor remains invariant with respect to x, then it indicates that the mode is being propagated at the correct phase velocity. The results of propagating a low frequency and a high frequency mode through a layer over a half-space structure are shown in Figures 5.3 and 5.6. The initial mode shape is plotted on the left in each figure and the solutions at various points in x appear to the right of it. The sinusoidal function displayed at the bottom of each figure is a plot of the demodulation factor.

The interesting case is when the model parameters vary laterally. For Love waves we have run two such cases. The first case is a dipping layer which dips down from the initial condition. The solution was again demodulated with a constant phase velocity determined from the initial conditions; but since the medium now varies laterally, this will not be sufficient to make the mode shape invariant with respect to x. The second example is a dipping layer that dips up from the initial conditions. In this example, the energy contained within the layer is diminished and is radiated into the half-space in the form of SH body waves.

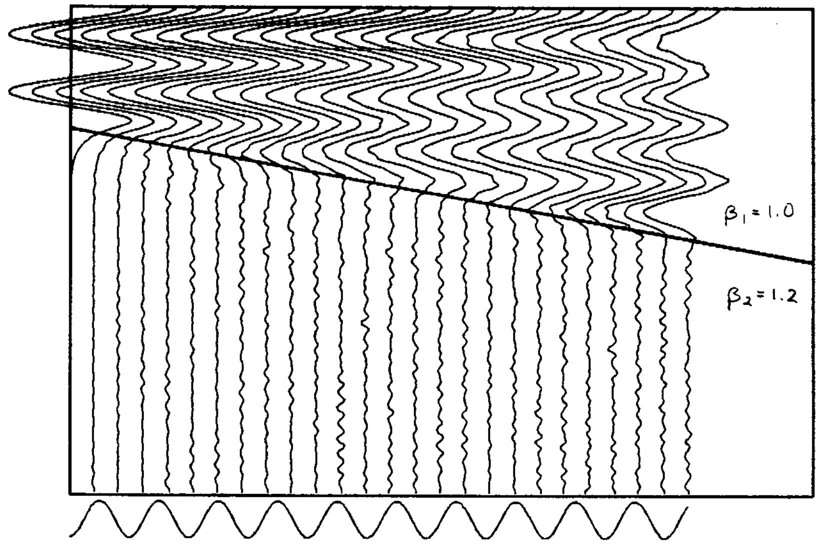

FIG. 5.5. A Love wave mode in a 20 degree down dipping layer. The plot is similar to Figure 5.3. The demodulation factor used is that of Figure 5.3. The characteristic of the mode that seems to be preserved in the x-direction is that its spatial frequency in the layer is preserved. Also, very little energy is radiated into the half-space.

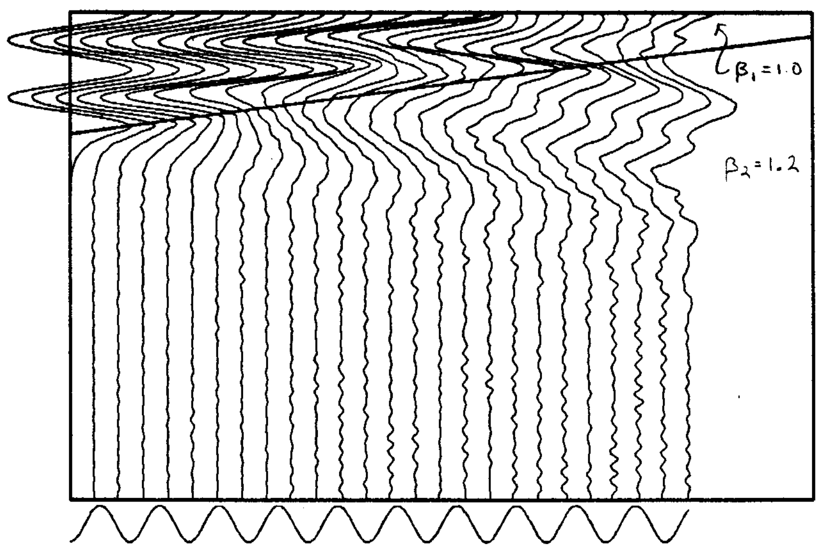

FIG. 5.6. In this example, the layer pinches out toward the surface. This time the mode radiates energy out of the layer into the half-space in the form of SH body waves.

Conclusions

A set a extrapolation operators for the acoustic wave equation and the SH-displacement equation have been presented. The operators can be made unconditionally stable by ignoring certain terms in the recurrence relation. These term are low frequency corrections of a higher order than the usual WKBJ corrections. This means that by omitting them the extrapolators are accurate to geometric optics in the extrapolation direction and physical optics in the perpendicular direction.