Chapter IV: A Born Inversion Method For Elastic Wavefields

Abstract

The inversion of two-dimensional elastic displacement fields can be handled in a manner similar to the acoustic problem. The Born approximation of the Lippmann-Schwinger equation yields a simple relationship in the Fourier-transform domain between the observed horizontal and vertical displacement fields, and the scattering potential. Basically, the observations are a linear combination of the scattering potential evaluated along four different shells. The four shells may be interpreted $P \rightarrow P$, $P \rightarrow S$, $S \rightarrow P$, and $S \rightarrow S$ scattering.

If the source is either purely compressional or purely shear, then one experiment will suffice to invert the forward equation. If the source is a (known) mixture of P and S components, then two experiments with different combinations of P and S components are necessary for the inversion.

4.1 Introduction

In Chapter III, the Born approximation was used to relate the "reflectivity" function to the density and bulk-modulus variations. Here the same is applied approach to the two-dimensional elastic problem. In this case there is a substantial advantage in determining the form of the reflectivity because there are four reflectivity functions, but only three medium parameters.

The field experiment necessary for the inversion method presented here is a standard multi-offset reflection survey with two components of displacement (radial and vertical) recorded at each geophone location. It is (apparently) necessary to cast the elastic inversion method in terms of displacements because exact wave operators for variable media can only be cast in terms of these variables.

The use of the Born approximation will force several restrictions on the procedure. Basically, the background P- and S-wave velocities must be nearly constant. The inversion scheme is limited to sub-critical reflections, and it has no provision for handling multiples.

4.2 The Forward Scattering Equation

The starting point of the derivation is the two-dimensional elastic displacement equation for a linear isotropic medium[1]

$$L\,u = (\partial_z A\,\partial_z + \partial_z B\,\partial_x + \partial_x B^T \partial_z + \partial_x C\,\partial_x + \rho\omega^2 I)\,u = 0 \qquad (4.1)$$

where

$$A = \begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \mu & 0 \\ 0 & \lambda+2\mu \end{pmatrix}$$

and u is the displacement vector $(u,w)^T$. This is the most straightforward form of the operator, however, for the derivation here, it is convenient to rewrite the operator in an equivalent form

$$L = \nabla \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \nabla^T + 2H \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H^T - 2H^T \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} H + \rho\omega^2 I \qquad (4.2)$$

where ∇ and H are the operators

$$\nabla = \begin{pmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{pmatrix} \qquad H = \begin{pmatrix} 0 & \partial_z \\ \partial_x & 0 \end{pmatrix}$$

and $\gamma = \lambda+2\mu$. Note the normalizations for the operators ∇ and H

$$\nabla^T \nabla = \nabla \nabla^T = (\partial_{xx} + \partial_{zz})I = \nabla^2 I \qquad \text{and} \qquad H^T H = H H^T = \partial_x \partial_z I$$

This form of the elastic displacement equation has a number of advantages. First, if the shear modulus is constant, the terms involving the operator H annihilate each other, and the resulting equation is very similar in form to the scalar displacement equation.[2] Second, as will be shown later, the term involving the operator ∇ will give rise to primary scattering (P → P, and S → S), while the terms involving H generate converted scattering (P → S, and S → P). This implies that the converted scattering is primarily governed by the shear modulus.

The operator ∇^T acting on the displacement field produces the divergence and curl of that field, which means that it converts displacements to potentials. The operator ∇ acting on the potential variables produces displacements.

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[1]Here ∂_x and ∂_z are the parial derivatives with respect to x and z.
[2]The scalar equation referred to is the SH displacement equation $(\rho\omega^2 + \nabla\cdot\mu\,\nabla)u = 0$.

In chapter II, the Born approximation of the Lippmann-Schwinger equation was stated to be

$$G = G_0 + G_0 V G_0 \tag{4.3}$$

where $V = L - L_0$. This equation is valid for the elastic case, if we realize that the Green's operators, and the scattering potential are dyadics.

The problem which we will perturb about is the one for which the wave operator is

$$L_o = \rho_o \omega^2 + \nabla \begin{bmatrix} \gamma_o & 0 \\ 0 & \mu_o \end{bmatrix} \nabla^T \tag{4.4}$$

Hence, the scattering potential is

$$V = (\rho - \rho_o)\omega^2 I + \nabla \begin{bmatrix} \gamma - \gamma_o & 0 \\ 0 & \mu - \mu_o \end{bmatrix} \nabla^T + 2H \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix} H^T - 2H^T \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix} H \tag{4.5}$$

For convenience we will define the dimensionless parameters

$$a_1 = \frac{\rho}{\rho_o} - 1, \qquad a_2 = \frac{\gamma}{\gamma_o} - 1, \text{ and } \qquad a_3 = \frac{\mu}{\mu_o} - 1.$$

As with the scalar inversion, we will concentrate on finding the dimensionless functions above, and not worry about reconstructing the actual medium parameters. With the above definitions, the scattering potential becomes

$$V = \rho_o \left[a_1 \omega^2 I + \nabla \begin{bmatrix} \alpha^2 a_2 & 0 \\ 0 & \beta^2 a_3 \end{bmatrix} \nabla^T + 2\beta^2 H \begin{bmatrix} 0 & a_3 \\ a_3 & 0 \end{bmatrix} H^T - 2\beta^2 H^T \begin{bmatrix} 0 & a_3 \\ a_3 & 0 \end{bmatrix} H \right] \tag{4.6}$$

where α and β, defined as

$$\alpha = \sqrt{\frac{\gamma_o}{\rho_o}} \quad \text{and} \quad \beta = \sqrt{\frac{\mu_o}{\rho_o}}$$

are the background P- and S-wave velocities.

In this chapter, we will not consider the presence of a free surface.[3] Instead, we will stop the medium above the datum from scattering by assuming that the $a_i(x,z)$ zero for $z < 0$.

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[3]This is a more significant assumption in the elastic case than the scalar case because we neglect mode conversion on the free surface. Also, in the elastic problem it is not possible to simulate a free surface by a finite combination of free space Green's operators.

For a point source, the observed reflected wave field is related to the scattering potential by

$$D(x_g, x_s, \omega) = G_0 V G_0 F S(\omega) \tag{4.7}$$

where F is a two-component vector representing the relative source strengths in u and ω, and $S(\omega)$ is the transform of the source time function.

4.3 The Scattering Equation in the Frequency Domain

Equation (4.7) has a more useful form in the Fourier-transform domain. Transforming over x_g and x_s we have

$$D(k_g, k_s, \omega) = \langle k_g \mid x_g \rangle \langle x_g, 0 \mid G_0 \mid x', z' \rangle \langle x', z' \mid V \mid x'', z'' \rangle$$

$$\langle x'', z'' \mid G_0 \mid x_s, 0 \rangle \langle x_s \mid k_s \rangle F S(\omega) \tag{4.8}$$

Substituting directly from Appendix A we have

$$D(k_g, k_s, \omega) = \frac{-1}{2\pi} \frac{1}{\rho_0^2 \omega^4} \int dx' \int dz' \int dx'' \int dz'' \; e^{ik_g x'} \left[A_g e^{-i\nu_g |z'|} + B_g e^{-i\eta_g |z'|} \right]$$

$$V(x', z' \mid x'', z'') \; e^{-ik_s x''} \left[A_s e^{-i\nu_s |z''|} + B_s e^{-i\eta_s |z''|} \right] F S(\omega) \tag{4.9}$$

where we have made the following definitions (from Appendix A)

$$\nu = \nu(k_x, \omega) = \frac{\omega}{\alpha} \sqrt{1 - \frac{\alpha^2 k_x^2}{\omega^2}}, \qquad \eta = \eta(k_x, \omega) = \frac{\omega}{\beta} \sqrt{1 - \frac{\beta^2 k_x^2}{\omega^2}},$$

$$A = A(k_x, \nu) = \frac{1}{2\nu} \begin{bmatrix} k_x & -\nu \\ \nu & k_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_x & \nu \\ -\nu & k_x \end{bmatrix}, \tag{4.10}$$

and

$$B = B(k_x, \eta) = \frac{1}{2\eta} \begin{bmatrix} k_x & -\eta \\ \eta & k_x \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_x & \eta \\ -\eta & k_x \end{bmatrix} \tag{4.11}$$

The subscripts g and s in equation (4.9) identify the horizontal wavenumber (k_g or k_s) to be used in the above definitions. Hence,

$$\nu_g = \nu(k_g, \omega) \qquad \nu_s = \nu(k_s, \omega) \qquad \eta_g = \eta(k_g, \omega) \qquad \eta_s = \eta(k_s, \omega)$$

and

$$A_g = A(k_g, \nu_g) \qquad A_s = A(k_s, \nu_s) \qquad B_g = B(k_g, \eta_g) \qquad B_s = B(k_s, \eta_s)$$

The operator A selects the compressional components from the displacement fields. It accomplishes this basically by converting into potential variables, selecting the P component, and then reconverting to displacements. The A operator applied to a purely shear field produces a zero result. In a similar fashion, the B operator selects the shear component of the displacement field.

Since $V(x',z' \mid x'',z'')$ is zero for either $z' < 0$ or $z'' < 0$, the absolute signs in equation (4.9) may be dropped. This allows us to identify each of the terms in equation (4.9) as a four-dimensional Fourier transform over x', z', x'', and z''. Hence,

$$D(k_g,k_s,\omega) = -\frac{2\pi}{\rho_o^2\omega^4}\Big[A_g\, V(k_g,-\nu_g \mid k_s,\nu_s)A_s + A_g\, V(k_g,-\nu_g \mid k_s,\eta_s)B_s$$
$$B_g\, V(k_g,-\eta_g \mid k_s,\nu_s)A_s + B_g\, V(k_g,-\eta_g \mid k_s,\eta_s)B_s\Big]\, F\, S(\omega) \qquad (4.12)$$

Thus, the observed data is a linear combination of the scattering potential evaluated along four different hyper-surfaces or "shells". By noting the positions of the A and B operators, one can identify what type of scattering each shell contributes. For example, the first term involves the operators A_g and A_s, which means that it is P → P type scattering. The next three terms in the sum are respectively S → P scattering, P → S scattering, and finally S → S scattering.

4.4 Inversion of the Scattering Equation

The next logical step is to substitute the Fourier transform of the scattering potential given in Appendix B into equation (4.10). However, since the scattering potential is a sum of three terms, and it appears four times in equation (4.10) with different arguments, we will simplify things first. We will do this by making some assumptions about the nature of the source.

If the source were purely compressional then only two terms in equation (4.10) would be non-zero. Hence,

$$D_P(k_g,k_s,\omega) = -\frac{2\pi}{\rho_o\,\omega^4}\Big[A_g\, V(k_g,-\nu_g \mid k_s,\nu_s)A_s + B_g\, V(k_g,-\eta_g \mid k_s,\nu_s)A_s\Big]\, S(\omega) \qquad (4.13)$$

where D_P is a two-component vector containing the horizontal and vertical components of displacement due to a compressional source.

We can further simplify the problem by exploiting the highly structured from of the operators A and B. It is clear form equations (4.10) and (4.11) that both A and B have a zero eigenvalue, and that it occurs in opposite positions (the 22-position for A, and the

11-position for B). Premultipling either A or B by the eigenvector that corresponds to its zero eigenvalue will annihilate the operator. The operators (which are the appropriate eigenvectors of A and B)

$$e_P = [k_g, \eta_g]^T \tag{4.14}$$

and

$$e_S = [-\nu_g, k_g]^T \tag{4.15}$$

have the properties

$$e_P \cdot B_g = 0 \quad \text{and} \quad e_S \cdot A_g = 0$$

The operators e_P and e_S have, as one might expect, the form of a divergence and a curl operator, respectively. Applying these operators to equation (4.13), we have

$$D_{PP}(k_g, k_s, \omega) = e_P \cdot D_P = -\frac{2\pi}{\rho_o^2 \omega^4} e_P \cdot \left[A_g \, V(k_g, -\nu_g \,|\, k_s, \nu_s) A_s \right] S(\omega) \tag{4.16}$$

and

$$D_{SP}(k_g, k_s, \omega) = e_S \cdot D_P = -\frac{2\pi}{\rho_o^2 \omega^4} e_S \cdot \left[B_g \, V(k_g, -\eta_g \,|\, k_s, \nu_s) A_s \right] S(\omega) \tag{4.17}$$

We have now reduced the problem to the same level as was discussed in Chapter III on acoustic inversion. To proceed from this point one would transform equations (4.16) and (4.17) into midpoint-offset coordinates, and make a change of independent variable $k_z = -\nu_g - \nu_s$ for equation (4.16), and $k_z = -\nu_g - \eta_s$ for equation (4.17). Then after determining the coefficients of the scattering potential given in Appendix B in the new coordinate systems, one could least squares fit for the unknowns $\{a_i\}$.

If the source were purely shear, then the other two terms in equation (4.10) would be the ones that are non-zero. The reduction to two scalar problems is similar in this case.

If the source is a mixture of P and S waves, then two experiments will be required to separate the various contributions. For example, if the source has compressional and shear strengths of p_1 and s_1 for the first experiment, and p_2 and s_2 for the second, then the observed wave fields would be

$$D_1 = \left[p_1 \left(A_g \, VA_s + B_g \, VA_s \right) + s_1 \left(A_g \, VB_s + B_g \, VB_s \right) \right] S(\omega) \tag{4.18}$$

$$D_2 = \left[p_2 \left(A_g \, VA_s + B_g \, VA_s \right) + s_2 \left(A_g \, VB_s + B_g \, VB_s \right) \right] S(\omega) \tag{4.19}$$

For brevity we have omitted the constants found in equation (4.10), and the arguments of V and D. By applying the divergence and curl operators we can reduce these equations to

$$e_P \cdot D_1 = \left[p_1 \, e_P \cdot A_g \, VA_s + s_1 \, e_P \cdot A_g \, VB_s \right] S(\omega) \tag{4.20}$$

$$e_P \cdot D_2 = \left[p_2 \, e_P \cdot A_g \, VA_s + s_2 \, e_P \cdot A_g \, VB_s \right] S(\omega) \tag{4.21}$$

and a similar set for the e_S operator. Solving for $e_P \cdot A_g \, VA_s \; S(\omega)$ and $e_P \cdot A_g \, VB_s \; S(\omega)$ we have

$$e_P \cdot A_g \, VA_s \; S(\omega) = \frac{s_2 e_P \cdot D_1 - s_1 e_P \cdot D_2}{s_2 p_1 - s_1 p_2} \tag{4.22}$$

and

$$e_P \cdot A_g \, VB_s \; S(\omega) = \frac{p_2 e_P \cdot D_1 - p_1 e_P \cdot D_2}{p_2 s_1 - p_1 s_2} \tag{4.23}$$

As long as $p_1 s_2 \neq p_2 s_1$ the problem can be reduced to the scalar case.

Conclusions

The constant background Born inversion scheme presented in Chapter III can be extended to the two-dimensional elastic problem. The observed data field is a linear combination of the scattering potential evaluated along four shells. The four shells correspond to P → P, P → S, S → P and S → S scattering respectively.

To invert the forward, for the case of a general source, it is apparently necessary to conduct two experiments with different source radiation patterns.

APPENDIX A: The Green's Operator For A 2-D Elastic Medium

The equation defining the Green's operator for the 2-D elastic case is

$$\rho_o\left[\omega^2 I + \nabla \begin{pmatrix}\alpha^2 & 0\\ 0 & \beta^2\end{pmatrix} \nabla^T\right] G_o = -\delta(x-x')\,\delta(z-z') \tag{4.A1}$$

where ∇ is defined in equation (4.2). Fourier transforming over x and z in equation (4.A1) we have

$$\rho_o\left[\omega^2 I + \tilde{\nabla} \begin{pmatrix}\alpha^2 & 0\\ 0 & \beta^2\end{pmatrix} \tilde{\nabla}^T\right] G_o = \frac{-1}{2\pi} e^{ik_x x' + ik_z z'} \tag{4.A2}$$

where

$$\tilde{\nabla} = i \begin{pmatrix} k_x & -k_z \\ k_z & k_x \end{pmatrix}$$

This equation may now be solved for G_o

$$G_o = \frac{1}{2\pi\rho_o}\tilde{\nabla}\begin{bmatrix} \dfrac{1}{\alpha^2(k_z-\nu)(k_z+\nu)} & 0 \\ 0 & \dfrac{1}{\beta^2(k_z-\eta)(k_z+\eta)} \end{bmatrix}\tilde{\nabla}^T \frac{e^{ik_x x' + ik_z z'}}{k_x^2+k_z^2} \tag{4.A3}$$

where

$$\nu = \frac{\omega}{\alpha}\sqrt{1 - \frac{\alpha^2 k_x^2}{\omega^2}} \quad \text{and} \quad \eta = \frac{\omega}{\beta}\sqrt{1 - \frac{\beta^2 k_x^2}{\omega^2}}$$

The domain in which we will use the Green's operator is the (z, k_x, ω)-domain. Inverse transforming over k_z we have

$$G_o = \frac{1}{(2\pi)^{3/2}\rho_o}\int dk_z\, \tilde{\nabla}\begin{bmatrix} \dfrac{1}{\alpha^2(k_z-\nu)(k_z+\nu)} & 0 \\ 0 & \dfrac{1}{\beta^2(k_z-\eta)(k_z+\eta)} \end{bmatrix}\tilde{\nabla}^T \frac{e^{ik_x x' + ik_z(z'-z)}}{k_x^2+k_z^2} \tag{4.A4}$$

This integral can be easily evaluated by contour integration in the complex k_z-plane. For the exploding Green's operator we choose the pair of poles that makes $k_z(z'-z) < 0$. To satisfy the radiation condition, the contour is closed in the upper half-plane for $(z'-z) > 0$, and in the lower half-plane for $(z'-z) < 0$. Using the residue theorem we have

$$G_o = \frac{i e^{ik_x x'}}{\sqrt{2\pi}\rho_o\,\omega^2}\left[\tilde{\nabla}_\alpha \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\tilde{\nabla}_\alpha^T \frac{e^{-i\nu|z'-z|}}{-2\nu} + \tilde{\nabla}_\beta \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\tilde{\nabla}_\beta^T \frac{e^{-i\eta|z'-z|}}{-2\eta}\right] \tag{4.A5}$$

where

$$\tilde{\nabla}_\alpha = i\begin{pmatrix} k_x & -\nu \\ \nu & k_x \end{pmatrix} \quad \text{and} \quad \tilde{\nabla}_\beta = i\begin{pmatrix} k_x & -\eta \\ \eta & k_x \end{pmatrix}$$

The first term in the Green's operator depends only on the compressional velocity (α), while the second depends only on the shear velocity (β). This leads to a natural definition for the two terms

$$<k_x,0 \,|\, G_0 \,|\, x',z'> = \frac{i e^{ik_x x'}}{\sqrt{2\pi}\rho_0\, \omega^2}\left[A e^{-i\nu|z'|} + B e^{-i\eta|z'|} \right] \tag{4.A6}$$

where

$$A = \frac{-1}{2\nu}\tilde{\nabla}_\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{\nabla}_\alpha^T \quad \text{and} \quad B = \frac{-1}{2\eta}\tilde{\nabla}_\beta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\nabla}_\beta^T$$

The other Green's operator that we need is the one transformed over the input set of variables

$$<x',z' \,|\, G_0 \,|\, k_x,0> = \frac{i e^{-ik_x x'}}{\sqrt{2\pi}\rho_0\, \omega^2}\left[A e^{-i\nu|z'|} + B e^{-i\eta|z'|} \right] \tag{4.A7}$$

APPENDIX B: Fourier Transform Of The Scattering Potential

The scattering potential may be written as an operator in the form

$$V(x',x'') = \Bigg[a_1\omega^2 I + \nabla' \begin{pmatrix} \alpha^2 a_2 & 0 \\ 0 & \beta^2 a_3 \end{pmatrix} \nabla'^{\,T}$$

$$+ 2\beta^2 H' \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H'^T - 2\beta^2 H'^T \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H' \Bigg] \delta(x'-x'') \tag{4.B1}$$

We now Fourier transform over x' and x'', and integrate (trivially) over x'':

$$V(k',k'') = \frac{1}{(2\pi)^2}\int dx'\, e^{ik'\cdot x'} \Bigg[a_1\omega^2 I + \nabla' \begin{pmatrix} \alpha^2 a_2 & 0 \\ 0 & \beta^2 a_3 \end{pmatrix} \nabla'^{\,T}$$

$$+ 2\beta^2 H' \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H'^T - 2\beta^2 H'^T \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H' \Bigg] e^{-ik''\cdot x'} \tag{4.B2}$$

We now integrate the second through fourth terms by parts to reverse the order of the leading operators and the $exp(ik'\cdot x')$. This allows us to write down the Fourier transform by inspection:

$$V(k',k'') = \frac{1}{(2\pi)^2}\left[a_1\omega^2 I - \nabla' \begin{pmatrix} \alpha^2 a_2 & 0 \\ 0 & \beta^2 a_3 \end{pmatrix} \nabla''^T \right.$$

$$\left. - 2\beta^2 H' \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H''^T + 2\beta^2 H'^T \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix} H'' \right] \tag{4.B3}$$

where $a_i = a_i(k'-k'')$.

The various terms in a_i can be collected together to produce a final form for the scattering potential

$$V(k_x',k_z' \mid k_x'',k_z'') = \frac{\omega^2}{(2\pi)^2}\left[a_1 I + a_2 \frac{\alpha^2}{\omega^2}\begin{pmatrix} k_x' k_x'' & k_x' k_z'' \\ k_z' k_x'' & k_z' k_z'' \end{pmatrix}\right.$$

$$\left. + a_3 \frac{\beta^2}{\omega^2}\begin{pmatrix} k_z' k_z'' & k_z' k_x'' - 2k_x' k_z'' \\ k_x' k_z'' - 2k_z' k_x'' & k_x' k_x'' \end{pmatrix}\right] \tag{4.B4}$$