Chapter III: A Born-WKBJ Inversion Method For Acoustic Reflection Data

Abstract

A method is presented for determining density and bulk-modulus variations in the earth from standard reflection surveys. Explicit formulas are given which utilize the amplitude-versus-offset information present in the observed wave fields. The method automatically accounts for dipping reflectors, but since it is based on a Born approximation of the scattering equation, it is restricted to subcritical reflections.

For the inversion, the medium is considered to be composed of a known low-spatial frequency variation (the background) plus an unknown high-spatial frequency variation in bulk modulus and density (the reflectivity). The division between the background and the reflectivity depends on the frequency content of the source.

For constant background parameters, the computations are done in the Fourier domain, where the first part of the algorithm includes a frequency shift identical to that in an F-K migration. The modulus and density variations are then determined by observing in a least-squares sense amplitude versus offset wavenumber.

For a spatially variable background WKBJ Green's operators that model the direct wave in a medium with a smoothly varying background are used. A downward continuation with these operators removes the effects of the variable velocity from the problem, and consequently the remainder of the inversion essentially proceeds as if the background were constant. If the background is strictly depth dependent, then the WKBJ Green's operators are analytic, and consequently the inversion can be expressed in closed form.

3.1 Introduction

In seismic reflection data, there are basically two sources of information about the subsurface: traveltimes and amplitudes. The traveltimes of the various wavefronts in the wave field provide information about the low-spatial frequency components (the background) of the medium parameters. The amplitudes of the wavefronts, on the other hand, are sensitive to the high-spatial frequency components (the reflectivity). The two types of information sample different aspects of the medium. In this chapter, the amplitude variations are used to determine the fine scale variations in the density and modulus and it will be assumed that the background can be determined by independent means. The field experiment necessary to provide the data for the method is a "standard" (or perhaps slightly super-standard) reflection survey with multiple offset coverage.

Our basic approach is similar to that of Cohen and Bleistein (1977, 1979), Phinney and Frazer (1978), and Raz (1980). We use a Born approximation of the Lippmann-Schwinger equation to develop a forward equation relating the surface data to a scattering potential. The scattering potential is an operator which depends on the medium parameters, and essentially represents the reflectivity of the medium. The details of this approach are outlined in the second section of the chapter.

The use of the Born approximation will entail several assumptions about the nature of the medium and the wave phenomena that is to be modeled. First the Born approximation is limited to primary subcritical reflections only. Also, since it is based on a perturbation of the true medium about the background variations, it is necessary to be able to construct accurate solutions for the background variations. In this chapter, we use the WKBJ solutions for the background which are discussed in the third section.

The remaining sections of the chapter deal with the inverse problem. In the fourth section an inversion scheme is presented for the case when the background variations are assumed constant. In this case, the problem may be cast in the Fourier domain where the observed wave field can be algebraically related to the variations in the medium parameters.

In the fifth section the inverse problem in a laterally varying medium is treated. It is shown that a migration of the data essentially removes the effects of the variable background, and the remainder of the inversion proceeds as in the constant background case. A special case of this where the background variation is strictly depth dependent is given in the final section. This case is of interest because the WKBJ Green's operators are analytical.

We will assume the source used in the experiment is band-limited. This usually causes problems with inversion methods because at some point in the inversion scheme, the source has to be deconvolved. This, of course, can only be successfully done within a limited passband, and attempts to invert data outside this passband will usually cause instabilities. We will bypass this problem by only reconstructing the parameter variations within a limited spatial frequency range.

We will also assume that the sources and receivers used in our experiment have no spatial extension (i.e. they are "points") and are of infinite aperature (that is, for a given source, receivers cover the whole of the earth's surface, and vice versa). This, of course, does not conform to current practice, and we acknowledge that some more analysis is required to establish the correspondence between our experiment and that actually performed.

Finally, we assume that the amplitude information in the data is retained. Since we are not attempting here to unite the rapid earth parameter variations with the slow ones, it is not necessary to know the absolute amplitude of the data. However, if we are to sort density from modulus variations, we must know accurately how amplitude varies with offset and, perhaps less accurately, how it varies with time.

3.2 The Forward Scattering Equation

In this section we derive the Lippmann-Schwinger equation for acoustic problems. The Born approximation of this equation will lead to a simple relationship between the observed data and the scattering potential.

The derivation starts with the linear isotropic acoustic wave equation

$$L\,P = \left[\frac{\omega^2}{K} + \nabla\cdot\,\frac{1}{\rho}\nabla\right]P = 0 \tag{3.1}$$

where P is the pressure field, K is the bulk modulus, and ρ is the density. Associated with the wave operator L, is the Green's operator or resolvent, which we formally define as (Taylor, 1972, p. 129)

$$G = -L^{-1} \tag{3.2}$$

There are actually many Green's operators that satisfy equation (3.2). They are distinguished from each other by the manner in which the inverse of L is evaluated. If we replace $-\omega^2$ in equation (3.1) with $(-i\omega+\varepsilon)^2$, and consider L to be a function of the variable ε, then we can define two independent Green's operators

$$G^+ = \lim_{\varepsilon\downarrow 0} \frac{-1}{L(\varepsilon)} \tag{3.3}$$

and

$$G^- = \lim_{\varepsilon\uparrow 0} \frac{-1}{L(\varepsilon)} \tag{3.4}$$

The *exploding* Green's operator G^+, projects a wavefront a positive distance from the source point, as time increases. The *imploding* Green's operator G^-, moves the wavefront a negative distance as time increases, or equivalently, if we keep distances positive, then G^- projects backward in time.

In this chapter we will employ free-space Green's operators. If the problem has external boundary conditions such as a free surface, then the Green's operators should satisfy them. For acoustic problems, this can usually be accomplished by a linear combination of the free-space Green's operators.

In general, we cannot analytically determine the Green's operator for arbitrary variations in ρ and K. Instead, solutions are usually cast as a perturbation about a simpler problem for which analytic solutions are available, or at least can be easily computed. In this chapter we will perturb about a reference problem for which the wave operator is

$$L_r = \left[\frac{\omega^2}{K_r} + \nabla\cdot \frac{1}{\rho_r}\nabla\right] \tag{3.5}$$

where K_r and ρ_r are the reference bulk modulus and density respectively. The reference density and bulk modulus will be chosen to be the slow variations (the background) in the true density and bulk modulus. By slowly varying we mean that the scale length of the variations is much greater than the wavelength of the waves under consideration.

To relate G and G_r (the Green's operator for L_r), we employ the simple identity

$$A = B + B\,(B^{-1} - A^{-1})\,A$$

and associate G with A and G_r with B. Hence, if we define $V = L - L_r$ then

$$G = G_r + G_r\,V\,G \tag{3.6}$$

Equation (3.6) is the Lippmann-Schwinger equation for G, and V is termed the scattering potential. It is valid for any choice of G_r that satisfies the same external boundary conditions as G.

As written, equation (3.6) is implicit in G, but it can be formally solved.

$$G = (I - G_r V)^{-1}\,G_r \tag{3.7}$$

The Born series is an expansion of the right- hand side of equation (3.7) in powers of the operator VG_r.

$$G = G_r \sum_{i=0}^{\infty} (VG_r)^i \tag{3.8}$$

The Born approximation of the Lippmann-Schwinger equation is the first two terms of the series

$$G = G_r + G_r V G_r \tag{3.9}$$

In this section we are constructing a model for the observed data so it is appropriate to use the exploding Green's operators (G^+ and G_r^+).

In Figure 3.1 the Born series and the Born approximation are represented in terms of Feynman diagrams. It is clear from this figure that if the source and receiver are above the scattering points, then the Born approximation models primary reflections only, while the next two terms include the effects of transmission and first order multiples.

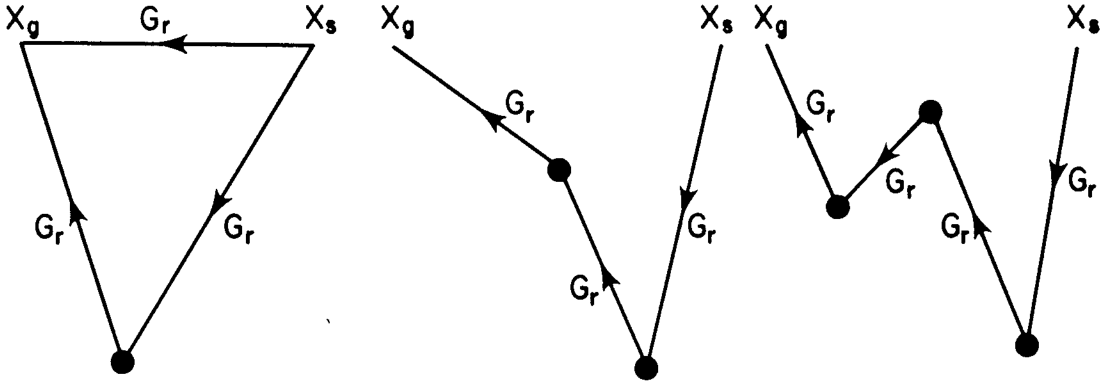

FIG. 3.1. A schematic interpretation of the Born series is shown. The left panel shows the first two terms in the Born series (the Born approximation). It contains a single scattering point, and hence models only the effects of primary reflections. The total response at the receiver x_g due to the source at x_s, is the integration of the scattering point over all points in the subsurface. The addition of another term in the series adds another scattering point, as shown in the center panel. This term accounts for first order transmission effects. The right panel shows the next term, which includes the effects of first order multiples.

The suitability of the Born approximation depends on how well the reference Green's operator models the direct wave between any two points in the medium. If it is a good approximation then the higher order terms have the interpretation given in Figure 3.1. Thus it is clear what physical effects we are neglecting by omitting the higher order terms. If the reference Green's operator is a poor approximation to the direct wave, then the higher order terms contain corrections for the direct wave. In this case the series is very inefficient to sum up, and the suitability of the Born approximation is doubtful.

For acoustic problems, the scattering potential is simply the difference of the wave operators in equations (3.1) and (3.5)

$$V = \omega^2 \left[\frac{1}{K} - \frac{1}{K_r} \right] + \nabla \cdot \left[\frac{1}{\rho} - \frac{1}{\rho_r} \right] \nabla \qquad (3.10)$$

For convenience, we will introduce the dimensionless medium parameters

$$a_1 = \left[\frac{K_r}{K} - 1 \right] \text{ and } a_2 = \left[\frac{\rho_r}{\rho} - 1 \right] \qquad (3.11)$$

where a_1 represents the spatial variations in bulk modulus relative to the reference modulus, and a_2 represents the variations in density. For the remainder of the chapter, we will consider a_1 and a_2 as the medium variations, and not worry about reconstructing the actual modulus and density variations from them. With these definitions the scattering potential becomes

$$V(x,z) = \omega^2 \frac{a_1}{K_r} + \nabla \cdot \frac{a_2}{\rho_r} \nabla \qquad (3.12)$$

The presence of derivatives in equation (3.12) represents a departure from basic scattering theory, in which V is a simple function of the spatial variables rather than a differential operator. As it turns out, however, the structure of V will not greatly complicate the problem.

The observations of the wavefield response are made on the horizontal surface ($z_s = z_g = 0$). In the 2-D problem they are functions of the receiver location x_g, the source location x_s, and frequency. It is convenient to define the data wavefield D as $D = G - G_r$, which is the total recorded wavefield minus the direct wave from the source to the receiver. Using the Born approximation, the relationship between the data field and the scattering potential is[1]

$$D(x_g,x_s,\omega) = \langle x_g,0 \,|\, G_r^+ \,|\, x',z' \rangle \; V(x',z') \; \langle x',z' \,|\, G_r^+ \,|\, x_s,0 \rangle \; S(\omega) \qquad (3.13)$$

Equation (3.13) is a forward equation in the sense that given the parameters variations a_1 and a_2, the observed data wavefield can be computed. For the remainder of this chapter, we will be concerned with the inverse problem: finding a_1 and a_2 from measurements of D on the surface.

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[1] In this chapter, repeated dummy variables will generally signify an implied integration. For example, equation (3.13) is really a total volume integral over the intermediate points x', z'.

3.3 WKBJ Solutions for the Direct Wave

The suitability of the Born approximation depends on how well the reference Green's operator models the direct wave in the medium. Since the effects of reflections, transmissions, and multi-pathing are best handled by the Born series itself (Stolt and Jacobs, 1980), we can ignore these effects when constructing the reference Green's operator. This makes the solution for the direct wave a candidate for the WKBJ approximation.

To find the two-dimensional Green's operators for the reference problem $L_r G_r^{\pm} = -\delta(x)\delta(z)$, they are cast as an asymptotic expansion of the form[2] (Yedlin, 1980)

$$G_r^{\pm}(x,z,\omega) = \pm H_0^{\left(\substack{1\\2}\right)}[\omega\, \Theta(x,z)] \sum_{n=0}^{\infty} \frac{A_n(x,z)}{(i\omega)^n} \tag{3.14}$$

Under the WKBJ approximation, we retain only the first term in the expansion. Hence,

$$G_r^{\pm}(x,z,\omega) = \pm H_0^{\left(\substack{1\\2}\right)}[\omega\, \Theta(x,z)]\, A_0(x,z) \tag{3.15}$$

As $(x,z) \to 0$, we require that $G_r^{\pm}$ approach the constant background form. Thus $\Theta(x,z) \to \sqrt{x^2+z^2}/v_r(0,0)$ and $A_0(x,z) \to \rho_r(0,0)/4i$. Applying the reference wave operator L_r to equation (3.15), the following equations are generated for Θ and A_0 by matching powers of ω.

$$(\nabla\, \Theta)^2 = \frac{\rho_r}{K_r} \tag{3.16}$$

and

$$\left[\nabla \cdot \frac{1}{\rho_r} \nabla\, \Theta - \frac{1}{K_r \Theta}\right] A_0 = \frac{-2}{\rho_r} \nabla\, \Theta \cdot \nabla\, A_0 \tag{3.17}$$

The first equation is the Eikonal equation and its solution for Θ governs the travel time of the wavefronts. The solution for A_0 from the second equation determines the amplitudes of the wavefronts. The higher order terms in the expansion correct for the low frequency behavior of the solution. The WKBJ solutions will be accurate if the wavelength of the waves is considerably shorter than the scale length of the variations in the medium. This is the motivation for choosing the background parameters to be slowly varying.

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[2] $H_0^{(1)}$ is the Hankel function of the first kind , and $H_0^{(2)}$ is the Hankel function of the second kind.

For a constant parameter medium, the Green's operators have a simple analytical form which is given in the next section. For the slightly more general case of a depth variable background, the Green's operators are

$$G_r^{\pm}(x,z,\omega) = \frac{\sqrt{\rho_r(z)\rho_r(0)}}{2\pi} \int dk_x \, e^{ik_x x} \, \frac{e^{\pm i \int_0^z dz' q(z')}}{\mp 2i \sqrt{q(z)q(0)}} \tag{3.18}$$

where

$$q(z) = \frac{\omega}{v_r(z)} \sqrt{1 - \frac{k_x^2 v_r^2(z)}{\omega^2}} \tag{3.19}$$

Equation (3.18) points out that the WKBJ solution is not valid near turning points $[q(z) = 0]$.

For a laterally variable background, the WKBJ solutions must be obtained numerically. The straight forward construction of $G_r^{\pm}$ using equations (3.15), (3.16), and (3.17) is certainly possible. However, finite-difference solutions of one-way wave equations (Claerbout, 1976; Clayton and Engquist, 1980) may provide a better approach provided the tendency of current formulations to overlook amplitude effects is corrected or compensated for. In Chapter VI, one-way extrapolation operators are derived which include amplitude effects to at least the same order as the WKBJ solutions.

3.4 Constant Background Inversion

In this section, an inversion method is presented for the case when the reference parameters K_r and ρ_r are assumed to be constant. The solution in this case is simple because the WKBJ Green's operators have an exact analytical form. The resulting inversion will contain a frequency shift which is identical to F-K migration (Stolt, 1978).

The first step is to Fourier transform[3] the data wavefield [equation (3.13)] over x_g and x_s.

$$D(k_g,k_s,\omega) = \langle k_g | x_g \rangle \langle x_g,0 | G_r^+ | x',z' \rangle$$

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[3] We adopt here the Fourier transform conventions (for 2-D)

$$\langle x | k \rangle = \frac{e^{ik \cdot x}}{2\pi} \text{ ,and } \langle k | x \rangle = \langle x | k \rangle^*$$

Note that this means that Fourier transforms over source coordinates have the opposite sense to those over receiver coordinates.

$$\cdot\ V(x',z')\ \langle x',z' \mid G_r^+ \mid x_s,0\rangle\ \langle x_s \mid k_s\rangle\ S(\omega) \qquad (3.20)$$

In the two dimensional problem (line sources and receivers), x_g, x_s, k_g, and k_s are scalars. The equations that follow will hold for the three- dimensional problem if we consider them to be two component vectors, and adjust the occasional factor of 2π.

For constant background parameters, the Green's operators in equation (3.20) have the analytical expressions

$$\langle k_g,0 \mid G_r^+ \mid x',z'\rangle = \frac{i\rho_r}{\sqrt{2\pi}}\ \frac{e^{-i(k_g x' - q_g |z'|)}}{2q_g} \qquad (3.21)$$

and

$$\langle x',z' \mid G_r^+ \mid k_s,0\rangle = \frac{i\rho_r}{\sqrt{2\pi}}\ \frac{e^{i(k_s x' + q_s |z'|)}}{2q_s} \qquad (3.22)$$

where

$$q_g = \frac{\omega}{v_r}\sqrt{1 - \frac{v_r^2 k_g^2}{\omega^2}} \text{ and } q_s = \frac{\omega}{v_r}\sqrt{1 - \frac{v_r^2 k_s^2}{\omega^2}} \qquad (3.23)$$

In the expressions for q_g and q_s we have intentionally factored an ω outside the square roots to indicate that q_g, q_s, and ω have the same sign.

We will now use the fact that the Green's operators look very much like the kernel of a Fourier transform to obtain a simple equation relating the data field to the scattering potential. Substituting equations (3.21) and (3.22) into equation (3.20) we have

$$D(k_g,k_s,\omega) = \frac{-\rho_r^2}{2\pi}\int dx' \int dz'\ \frac{e^{-i(k_g x' - q_g |z'|)}}{2q_g}\ V(x',z')\ \frac{e^{i(k_s x' + q_s |z'|)}}{2q_s} \cdot S(\omega) (3.24)$$

We now assume that $a_1(x,z)$ and $a_2(x,z)$ are zero for $z<0$. This will allow us to drop the absolute signs in equation (3.24) because $V(x',z')$ will be zero for $z'<0$. Actually, removing the absolute signs will mean that any scatters located above the datum plane $z=0$ will only contribute to D in negative time. This point is discussed further in the next section. Using the definition (3.12) of V and integrating equation (3.24) by parts yields

$$D(k_g,k_s,\omega) = \frac{-\rho_r}{2\pi}\int dx' \int dz'\ \frac{e^{-i[(k_g - k_s)x' - (q_g + q_s)z']}}{4q_g q_s}$$

$$\cdot \left[\frac{\omega^2}{v_r^2} a_1(x',z') + (q_g q_s - k_g k_s)\ a_2(x',z')\right] S(\omega) \qquad (3.25)$$

The two integrals in (3.25) are recognizable as Fourier transforms over x' and z'. Thus

$$D(k_g,k_s,\omega) = \frac{-\rho_r}{4q_g q_s}\left[\frac{\omega^2}{v_r^2} a_1(k_g-k_s,-q_g-q_s) + (q_g q_s - k_g k_s)\, a_2(k_g-k_s,-q_g-q_s)\right] S(\omega) \qquad (3.26)$$

That is, the triple Fourier transform of D is a linear combination of the double Fourier transforms of a_1 and a_2. Counting variables on both sides of (3.26) indicates the inverse problem is overdetermined. That is, there should be more than enough information in D to solve for a_1 and a_2. If V were a more general operator, things would have been different. V would then be a function of two sets of coordinates [$V(x,z) \rightarrow V(x,z\,|\,x',z')$] and equation (3.26) would have the form

$$D(k_g,k_s,\omega) = -\frac{2\pi\rho_r^2}{4q_g q_s} V(k_g,-q_g\,|\,k_s,q_s)\cdot S(\omega) \qquad (3.27)$$

That is, the triple Fourier transform of D would then be proportional to the quadruple Fourier transform of V. Counting variables again, we see the problem is underdetermined and consequently there would be no way to calculate V given D.

The first step to solving for a_1 and a_2 is to change to midpoint-offset coordinates. The midpoint wavenumber (k_m) and the half-offset wavenumber (k_h) are defined by[4]

$$k_m = k_g - k_s \quad \text{and} \quad k_h = k_g + k_s \qquad (3.28)$$

In the space domain, these substitutions correspond to a midpoint (x_m), and a half-offset (x_h) defined as

$$x_m = \frac{x_g + x_s}{2} \quad \text{and} \quad x_h = \frac{x_g - x_s}{2} \qquad (3.29)$$

Also, since a_1 and a_2 depend on $-(q_g+q_s)$, a new independent variable (k_z) is defined

$$k_z = -q_g - q_s = -\frac{\omega}{v_r}\sqrt{1-\frac{v_r^2 k_g^2}{\omega^2}} - \frac{\omega}{v_r}\sqrt{1-\frac{v_r^2 k_s^2}{\omega^2}} \qquad (3.30)$$

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[4] These definitions of midpoint and offset wavenumber differ from other authors (c.f. Yilmaz and Claerbout, 1980), because we have used a conjugate rather than a symmetric relationship between source and receiver. This arises directly from the operator notation used in this chapter. In the physical domain [equation (3.29)], the relations for midpoint and offset are the same with both approaches.

After a little algebra, equations (3.28) and (3.30) may be combined to obtain expressions for ω, q_s, and q_g in terms of the new variables k_m, k_h, k_z.

$$\omega = -\frac{v_r k_z}{2}\sqrt{\left[1 + k_m^2/k_z^2\right]\left[1 + k_h^2/k_z^2\right]} \equiv \omega(k_m,k_h,k_z) \tag{3.31}$$

$$q_g = -\frac{k_z}{2}\left[1 - k_m k_h/k_z^2\right] \tag{3.32}$$

$$q_s = -\frac{k_z}{2}\left[1 + k_m k_h/k_z^2\right] \tag{3.33}$$

Combining equations (3.26), (3.31), (3.32), and (3.33) we obtain

$$D(k_m,k_h,k_z) = -\rho_r \left[\sum_{i=1}^{2} A_i(k_m,k_h,k_z)\, a_i(k_m,k_z)\right]\cdot S(\omega) \tag{3.34}$$

where

$$A_1(k_m,k_h,k_z) = \frac{1}{4}\,\frac{(k_z^2 + k_h^2)\,(k_z^2 + k_m^2)}{k_z^4 - k_m^2 k_h^2} \tag{3.35}$$

and

$$A_2(k_m,k_h,k_z) = \frac{1}{4}\,\frac{(k_z^2 - k_h^2)\,(k_z^2 + k_m^2)}{k_z^4 - k_m^2 k_h^2} \tag{3.36}$$

In equation (3.34), it is understood that ω obeys the functional relationship given in equation (3.31), which is identical to the frequency shift used in F-K migration (Stolt, 1978).

To invert equation (3.34), we start by deconvolving the source $S(\omega)$. Thus we define

$$D'(k_m,k_h,k_z) = \frac{-1}{\rho_r}\,\frac{D(k_m,k_h,\omega)}{S(\omega)}\,. \tag{3.37}$$

Since in general, $S(\omega)$ will be bandlimited, this operation can not be accomplished exactly without introducing instabilities. This is the point where Gel'fand-Levitan inverse methods (Ware and Aki, 1969; Jacobs and Stolt, 1980) have problems. To avoid the instabilities, we simply set D' to zero outside the frequency bandwidth of $S(\omega)$, which means we will only be able to resolve the variations in a_1 and a_2 within the passband

$$\omega_1 \le \omega(k_m,k_h,k_z) \le \omega_2 \tag{3.38}$$

where ω_1 and ω_2 are the lower and upper limits of the passband of $S(\omega)$. In Figure 3.2 the region of resolution is illustrated for $k_h=0$. It is interesting to note that by increasing the ratio k_h/k_z the circles in this figure will shrink in radius. Hence, it is possible to partially fill in the low frequency variations in the parameters by increasing the offset in the experiment.

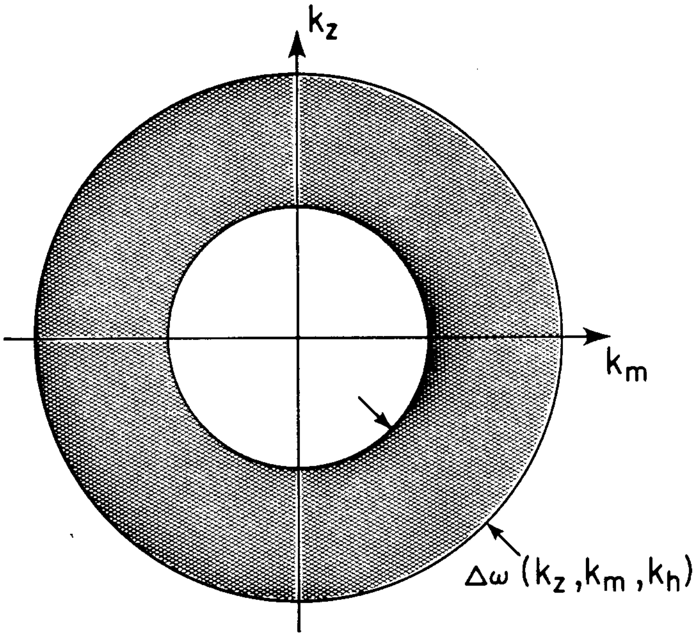

FIG. 3.2. The shaded ring shows the region of resolution of the bulk modulus and density variations. Here k_z, k_m, and k_h are respectively the vertical, the midpoint, and the offset wavenumbers. The width and radius of the ring depend on the passband $\Delta\omega$ of the source time function. As the ratio k_h/k_z is increased, the radius of the ring shrinks. This corresponds in the physical domain to increasing the source-receiver offset relative to the depth to the reflector.

With the (partial) deconvolution of equation (3.34), the inverse problem reduces to

$$D'(k_m,k_h,k_z) = \sum_{i=1}^{2} A_i(k_m,k_h,k_z)\, a_i(k_m,k_z) \tag{3.39}$$

Since the a_i's are independent of k_h, the measurement of D' at any two distinct values of k_h will suffice to determine a_1 and a_2. In a standard reflection survey however, D' is usually determined at several values of k_h, and therefore a more robust evaluation is possible. For example, a least-squares determination is given by the solution to the equation

$$\begin{bmatrix} \sum A_1^2 & \sum A_1 A_2 \\ \sum A_1 A_2 & \sum A_2^2 \end{bmatrix} \begin{bmatrix} a_1(k_m,k_z) \\ a_2(k_m,k_z) \end{bmatrix} = \begin{bmatrix} \sum A_1 D' \\ \sum A_2 D' \end{bmatrix} \tag{3.40}$$

In this equation, the summations are taken over k_h with the restriction that

$$|k_h\, k_m| < |k_z|^2 \tag{3.41}$$

The necessity for the restriction lies in the fact that the Born approximation as used in this chapter, is not adequate in the evanescent zone. This restriction is sufficient to avoid both evanescent zones in equation (3.30), and to avoid turning points in both the up and downgoing paths by keeping both q_g and q_s strictly negative in equations (3.32) and (3.33).

Thus far we have been concerned with the 2-D problem which has line sources and receivers. The full 3-D problem with point sources and receivers is only slightly different. In the usual seismic experiment the data is recorded with (assumed) point sources and receivers, but along a line on the free surface. In the Appendix the results presented in this section are modified for this case.

3.5 Inversion with a Variable Background

For a realistic earth model, we must assume that the background parameters will vary from one location to another. If we ignore this variation as we did in the previous section, then the inversion scheme will incorrectly locate the parameter variations. Fortunately, if the background variations are known, their effects may be removed from the inversion problem by a downward continuation. This step is actually a migration of the data prior to the inversion.

The migration is based on the representation integral over a closed surface S. If we assume that P is a solution to the wave equation $L_r P = -F$, where F is a volume source, and $G_r^{\pm}$ are the Green's operators associated with L_r, then the representation integral is

$$I^-(x) = \int ds\; G_r^-(x\,|\,s)\,T(s)\,P(s) \tag{3.42}$$

where

$$T(s) = \underleftarrow{\frac{\partial}{\partial n}}\,\frac{1}{\rho_r} - \frac{1}{\rho_r}\,\underrightarrow{\frac{\partial}{\partial n}}$$

and n is the normal to the surface. The arrows in the definition of $T(s)$ have the following meaning

$$A\left[\underleftarrow{\frac{\partial}{\partial n}}\,\frac{1}{\rho_r} - \frac{1}{\rho_r}\,\underrightarrow{\frac{\partial}{\partial n}}\right]B = \left[\frac{\partial}{\partial n}A\right]\frac{B}{\rho_r} - \frac{A}{\rho_r}\left[\frac{\partial}{\partial n}B\right]$$

The imploding Green's operator is used in equation (3.42) because, since it projects backwards in time, it is the proper operator to backtrack a wave to its point of origin. If we wanted to extrapolate waves away from their point of origin then G_r^+ would replace G_r^- in equation (3.42). Using the divergence theorem, we may convert I^- to a volume integral

$$\begin{aligned} I^-(x) &= \int_V dx' G_r^-(x \mid x') \left[\underleftarrow{\nabla \cdot \frac{1}{\rho_r} \nabla} - \underrightarrow{\nabla \cdot \frac{1}{\rho_r} \nabla} \right] P(x') \\ &= \int_V dx' G_r^-(x \mid x') \left[\underleftarrow{L_r} - \underrightarrow{L_r} \right] P(x') \end{aligned}$$

where V is the volume bounded by S and $x' \in V$. Applying the fact that $L_r P = -F$ and $L_r G_r^- = -1$, I^- is found to be

$$I^-(x) = \begin{cases} P(x) + \int_V dx'\ G_r^-(x \mid x')\ F(x') & \text{for } x \in V \\ \int_V dx'\ G_r^-(x \mid x')\ F(x') & \text{for } x \not\in V \end{cases} \tag{3.43}$$

If there are no sources inside the volume then $I^-(x)$ is a representation of $P(x)$ inside the volume, and is zero outside. When sources are present, they contribute to both the inner and outer solutions.

Consider applying the representation to a field point outside the volume. The geometry is shown in Figure 3.3. The closed surface integral can be broken up into two line integrals, if we assume that the edges are sufficiently far away that their contribution is zero. Hence we can write by equation (3.43)

$$\begin{aligned} I^-(x) &= I_0(x) - I_z(x) \\ &= \int ds_0\, G_r^-(x \mid s_0) T(s_0) P(s_0) - \int ds_z\, G_r^-(x \mid s_z) T(s_z) P(s_z) \\ &= \int_V dx' G_r^-(x \mid x') F(x') \end{aligned} \tag{3.44}$$

Suppose the source distribution F is concentrated at zero time, then the volume integral in (3.44) is zero for all time greater than zero. Consequently, the two surface integrals in (3.44) are identical at all positive times. We now assume that P has been generated by sources partly within V and partly beneath it. Thus we have

$$\begin{aligned} P(x) &= \int_V dx' G_r^+(x \mid x') F(x') + \int_{V^c} dx' G_r^+(x \mid x') F(x') \\ &= P_V(x) + P_L(x) \end{aligned} \tag{3.45}$$

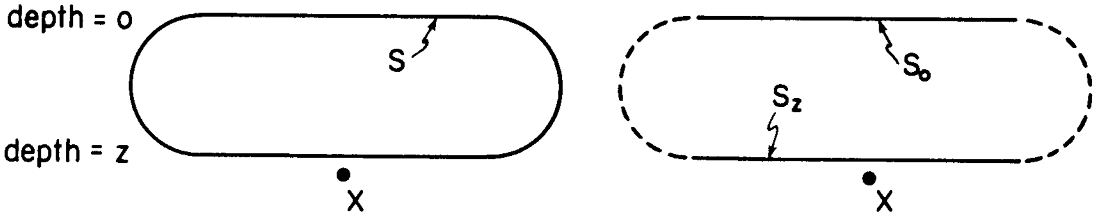

FIG. 3.3. The left panel shows the closed contour (s) to be used in the representation of the response at x. The closed contour is then broken into two line integrals over s_0 and s_z shown in the right panel. The contribution from the edges is assumed to be zero. In the text, it is shown that for positive time, the response at x can related to a line integral of the recorded data along s_0.

If we now take x to lie infinitesimally below the surface s_z, then we can evaluate $I_z(x)$ to a very good approximation as

$$I_z(x) = \int ds_z\, G_r^-(x \mid s_z) T(s_z) P(s_z) = P_L(x) \tag{3.46}$$

Contributions from source above s_z are filtered out by this surface integral because G_r^- can only project backwards in time. With this result, equation (3.44) can be rewritten as

$$I_0(x) = \int ds_0\, G_r^-(x \mid s_0) T(s_0) P(s_0) = P_L(x) + \int_V dx'\, G_r^-(x \mid x') F(x') \tag{3.47}$$

The substance of equation (3.47) is that it is a prescription for downward continuation of P from the surface s_0 to the point x. The construction of the surface integral on s_0 yields the portion P_L of the field P at x which is due to sources below x, plus the time reversal of the portion of P from sources above x. Note that the sources above the plane s_z contribute only in negative time.

Now we generalize the representation to the data wavefield given by the Born approximation [equation (3.13)]. The appropriate I^- in this case is

$$I^-(x_g \mid x_s) = -\int ds\; G_r^-(x_g \mid s) T(s) \int ds'\; D(s \mid s') T(s') G_r^-(s' \mid x_s) \tag{3.48}$$

where s and s' are located on the closed surface S. Applying the divergence theorem twice, the representation can be converted to the volume integral

$$I^{-}(x_g \,|\, x_s) = -\int_V dx \int_V dx' G_r^{-}(x_g \,|\, x)(\underset{\leftarrow}{L_r} - \underset{\rightarrow}{L_r}) D(x \,|\, x')(\underset{\leftarrow}{L_r} - \underset{\rightarrow}{L_r}) G_r^{-}(x' \,|\, x_s) \quad (3.49)$$

where x and x' V. For x_g and x_s below the volume equation (3.49) reduces to

$$I^{-}(x_g \,|\, x_s) = \int_V dx \int_V dx' G_r^{-}(x_g \,|\, x) \underset{\rightarrow}{L_r} D(x \,|\, x') \underset{\leftarrow}{L_r} G_r^{-}(x' \,|\, x_s)$$

$$= \int_V dx \, G_r^{-}(x_g \,|\, x) V(x) G_r^{-}(x \,|\, x_s) \quad (3.50)$$

We can use the previous analysis to construct a downward continuation operator from the representation in (3.48). To do this we apply the steps of equations (3.44) through (3.45) to each of the surface integrals (s and s'). The result are (provided x_g and x_s are infinitesimally below the surface s_z)

$$I_z(x_g \,|\, x_s) \equiv \int ds_z \int ds_z' \, G_r^{-}(x_g \,|\, s_z) T(s_z) D(s_z \,|\, s_z') T(s_z') G_r^{-}(s_z' \,|\, x_s)$$

$$= D_L(x_g \,|\, x_s) \quad (3.51)$$

and

$$I_0(x_g \,|\, x_s) \equiv \int ds_0 \int ds_0' \, G_r^{-}(x_g \,|\, s_0) T(s_0) D(s_0 \,|\, s_0') T(s_0') G_r^{-}(s_0' \,|\, x_s)$$

$$= D_L(x_g \,|\, x_s) + \int dx \, G_r^{-}(x_g \,|\, x) V(x) G_r^{-}(x \,|\, x_s) \quad (3.52)$$

where D_L is the reflection data from points below s_z. The volume integral involving $G^{-}VG^{-}$ is zero in positive time. Thus for $t > 0$, I_z and I_0 are identical, and represent the downward continuation of the data field. The $G^{-}VG^{-}$ term in I_0 simply represents the well known fact in migration that the response from reflectors above the datum plane is pushed into negative time.

The result obtained in equation (3.52) can be easily generalized to move the data wavefield between any two planes. To move D from the depth $z-\varepsilon$ to the depth z we have for $t > 0$.

$$D_L(s_z \,|\, s_z') = \int ds \int ds' G^{-}(s_z \,|\, s) T(s) D(s \,|\, s') T(s') G^{-}(s' \,|\, s_z') \quad (3.53)$$

where s and s' lie on the surface $s_{z-\varepsilon}$.

We are now in a position to invert the data for the scattering potential. First we define the migrated wavefield M at depth z to be

$$\rho_r(x_m, z) v_r(x_m, z) \, M(x_g, x_s, z) = \lim_{t \downarrow 0} \int d\omega \, e^{-i\omega t} D_L(x_g, z \,|\, x_s, z; \omega)$$

$$= \int d\omega \, I_0(x_g, z \,|\, x_s, z; \omega) \tag{3.54}$$

The presence of ρ_r and v_r in this definition will simplify things later on. For now we just note that with this definition, the triple Fourier transform of M is dimensionless. We may now use equation (3.53) to relate the data field in equation (3.54) to the data field a small distance $(z-\varepsilon)$ above. Writing this out for the 2-D case we have

$$\rho_r v_r \, M(x_g, x_s, z) = \lim_{t \downarrow 0} \int d\omega \, e^{-i\omega t} \int dx_g' \int dx_s' \; G_r^-(x_g, z \,|\, x_g', z-\varepsilon; \omega) \, T(x_g', z-\varepsilon)$$

$$\cdot D_L(x_g', z-\varepsilon \,|\, x_s', z-\varepsilon) \, T(x_s', z-\varepsilon) \, G_r^-(x_s', z-\varepsilon \,|\, x_s, z; \omega) \tag{3.55}$$

As time goes to zero, the region of support for the x_g' and x_s' integrals shrinks to a small region centered around the midpoint between x_g and x_s. Under the assumption of a smoothly varying background, G_r^- and D_L will assume their constant parameter forms with the relevant parameters being $K_r(x_m, z)$ and $\rho_r(x_m, z)$. Substituting in the Green's operators from the previous section [equations (3.21) and (3.22)] and performing the derivatives in the T operators we have

$$\rho_r v_r \, M(x_g, x_s, z) = \int d\omega \int dk_g \int dk_s \, D_L(k_g, z-\varepsilon \,|\, k_s, z-\varepsilon; \omega)$$

$$\cdot e^{i(k_g x_g - k_s x_s)} \, e^{-i\varepsilon(q_g + q_s)} \tag{3.56}$$

Substituting in the constant parameter form for D_L [equation (3.34)] we have

$$v_r \, M(x_g, x_s, z) = -\int d\omega \int dk_g \int dk_s \; e^{i(k_g x_g - k_s x_s)} \, e^{-iz(q_g + q_s)}$$

$$\cdot \sum_{i=1}^{2} A_i(k_g, k_s, q_g, q_s) \; a_i(k_g - k_s, -q_g - q_s) \tag{3.57}$$

Even though it is not explicitly mentioned in equation (3.57), the coefficients A_i depend on x_m and z via the background modulus and density $K_r(x_m, z)$ and $\rho_r(x_m, z)$.

Equation (3.57) looks suspiciously like a Fourier transform and indeed, we can put it in that form. Changing integration variables in (3.57) from (ω, k_g, k_s) to (k_z, k_m, k_h) yields

$$M(x_m, x_h, z) = -\int dk_m \int dk_h \; \frac{1}{2} \left| \frac{d\omega}{dk_z} \right| \, e^{i(k_m x_m + k_h x_h + k_z z)}$$

$$\cdot \sum_{i=1}^{2} \frac{A_i(k_g, k_s, q_g, q_s)}{v_r} a_i(k_m, k_z) \tag{3.58}$$

With the forms (3.35) and (3.36) for A_1 and A_2, plus the relation

$$\frac{d\omega}{dk_z} = \frac{v_r}{8}\sqrt{1+\frac{k_m^2}{k_z^2}}\sqrt{1+\frac{k_h^2}{k_z^2}}\frac{1}{A_1(k_g,k_s,q_g,q_s)} \tag{3.59}$$

we obtain

$$M(x_m,x_h,z) = -\frac{1}{(2\pi)^{3/2}}\int dk_m \int dk_h \int dk_z \; e^{i(k_m x_m + k_h x_h + k_z z)}$$

$$\cdot \sum_{i=1}^{2} a_i(k_m,k_z) B_i(k_m,k_h,k_z) \tag{3.60}$$

where

$$B_1(k_m,k_h,k_z) = \frac{(2\pi)^{3/2}}{16}\sqrt{1+\frac{k_m^2}{k_z^2}}\sqrt{1+\frac{k_h^2}{k_z^2}} \tag{3.61}$$

and

$$B_2(k_m,k_h,k_z) = B_1(k_m,k_h,k_z)\,\frac{k_z^2-k_h^2}{k_z^2+k_h^2} \tag{3.62}$$

Note that B_1 and B_2 do not depend on the spatial coordinates. Equation (3.60) is in fact a 3-D Fourier inverse transform over k_m, k_h, and k_z. Taking the Fourier transform of both sides we arrive at the final result

$$M(k_m,k_h,k_z) = \sum_{i=1}^{2} a_i(k_m,k_z) B_i(k_m,k_h,k_z) \tag{3.63}$$

Thus, just as in the constant background case, the 3-D Fourier transform of the migrated field is a linear combination (with known coefficients) of the 2-D Fourier transforms of a_1 and a_2.

3.6 Inversion With A Depth Variable Background

In this section we consider a special case of the previous section in which the background parameters are allowed to vary only in the depth direction. The WKBJ Green's operators in this case are analytic and consequently explicit formulas can be derived for the inverse problem.

For a depth variable medium, the WKBJ Green's operators are given by equation (3.18). With these we can form the Born-WKBJ approximation of the data field.

$$D(k_g,z_g=0\,|\,k_s,z_s=0;\omega) = \frac{-\rho_r(0)}{8\pi\sqrt{q_g(0)q_s(0)}}\int_0^{\infty} dz \; \frac{e^{\,i\int_0^z dz'[q_g(z')+q_s(z')]}}{\sqrt{q_g(z)q_s(z)}}$$

$$\cdot\left[\frac{\omega^2}{v_r^2(z)}a_1(k_g-k_s,z) + [q_g(z)q_s(z) - k_gk_s]a_2(k_g-k_s,z)\right] \quad (3.64)$$

where q_g and q_s are the same as in equation (3.23) except that now the velocity is a function of z.

Equation (3.52) for the downward continued field I_0 can be evaluated explicitly in this case. Fourier transforms over the lateral coordinates yields

$$I_0(k_g,z\,|\,k_s,z) = -\int ds G_r^-(k_g,z\,|\,s)T(s)\int ds' D(s\,|\,s')T(s')G_r^-(s'\,|\,k_s,z) \quad (3.65)$$

In this expression we have set the continuation depths for both the sources and receivers equal to z. To evaluate this expression we need only to substitute in the explicit form (3.18) for each G_r^-, and do the derivatives in each T. The result is

$$I_0(k_g,z\,|\,k_s,z) = \frac{\rho_r(z)}{\rho_r(0)}\sqrt{\frac{q_g(0)q_s(0)}{q_g(z)q_s(z)}}\,e^{i\int_0^z dz'(q_g+q_s)}\cdot D(k_g,0\,|\,k_s,0) \quad (3.66)$$

In the derivation of this equation the derivatives of q_g and q_s were neglected in comparison to the derivatives of the phase terms. Note that as $z \to 0$, the downward continued field I_0 approaches the data field D, as it must. According to equation (3.66), downward continuation is achieved in the vertically varying case by multiplying the data by the phase factor $\exp[-i\int_0^z dz'(q_g+q_g)]$, and adjusting the amplitude of the data. The phase factor is that used in the Gazdag phase-shift migration method (Gazdag, 1978). The amplitude factor has no analog.

By equation (3.54), migration is achieved by integrating the downward continued field I_0 over all frequencies and dividing by $\rho_r v_r$. Formally,

$$M(k_g,k_s,z) = \frac{1}{v_r(z)\rho_r(0)}\int d\omega\sqrt{\frac{q_g(0)q_s(0)}{q_g(z)q_s(z)}}\,e^{i\int_0^z dz'(q_g+q_s)}\cdot D(k_g,0\,|\,k_s,0) \quad (3.67)$$

According to equation (3.63), the Fourier transform over z of this quantity is a linear combination of the double Fourier transforms of the desired quantities a_1 and a_2. In the appendix, the necessary modifications are given to incorporate point sources and receivers into the solutions presented in this section.

Conclusions

An inversion scheme has been presented to determine the rapid variations in bulk modulus and density from the amplitude versus offset information present in a seismic reflection survey. The procedure consists of two steps.

First, a migration of the data is performed with WKBJ Green's operators for an assumed slowly varying background variation in the medium parameters. The migration essentially removes the effects of the background from the inversion by transforming the recorded wave field from the time domain to the depth domain. For a constant background, this step is similar to F-K migration. For a depth variable background a phase shift migration is used. For a laterally variable background the WKBJ Green's operators have to be constructed numerically.

The second step is to determine the parameter variation from the migrated data. It is shown that the triple Fourier transform of the migrated data is a linear combination (with known coefficients) of the double Fourier transform of the bulk modulus and density variations. Thus, a simple least squares solution can be used to invert the data.

APPENDIX: Incorporating Point Sources and Receivers in the Two-Dimensional Solution

The solutions given in the text are for a two-dimensional medium. However, it is trivial to modify the solutions for the full three-dimensional case. For example, the three-dimensional equivalent of the constant background equation (3.26) is

$$D(k_g, k_y \mid k_s, k_y' ; \omega) = \frac{-1}{4\pi^2} \frac{\rho_r^2}{4qq'} \left[\frac{\omega^2}{v_r^2} a_1(k_g - k_s, k_y - k_y', -q - q') \right.$$

$$\left. + (qq' - k_g k_s - k_y k_y') a_2(k_g - k_s, k_y - k_y', -q - q') \right] S(\omega) \qquad (3.A1)$$

where

$$q = \frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2}{\omega^2}(k_g^2 + k_y^2)} \quad \text{and} \quad q' = \frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2}{\omega^2}(k_s + k_y'^2)}$$

In this equation primed variables refer to the source location, while unprimed variables refer to the receiver location.

The seismic experiment is usually conducted along a line (say $y = y' = 0$), and the medium parameters are assumed to be invariant in the y direction. In this case the a_i have the form

$$a_i(k_g - k_s, k_y - k_y', -q - q') \rightarrow a_i(k_g - k_s, -q - q')\, \delta(k_y - k_y') \qquad (3.A2)$$

To restrict the 3-D problem to one that can be handled by the 2-D algorithm outlined in the text, we start by inverse transforming over k_y and $k_y{}'$, and evaluating the data field along $y = y' = 0$.

$$D(k_g, 0 \,|\, k_s, 0; \omega) = \int dk_y \int dk_y' \; D(k_g, k_y \,|\, k_s, k_y'; \omega) \qquad (3.A3)$$

The integral over k_y' can be evaluated trivially because of the assumed form of a_i in equation (3.A2).

$$D(k_g, 0 \,|\, k_s, 0; \omega) = \int dk_y \; D(k_g, k_y \,|\, k_s, k_y; \omega) \qquad (3.A4)$$

To remove the remaining integral over k_y, we express the a_i as a Fourier transform over z. That is

$$a_i(k_g - k_s, -q - q') = \int dz \; e^{-i(q+q')z} \; a_i(k_g - k_s, z) \qquad (3.A5)$$

Substituting equation (3.A5) into equation (3.A4), and interchanging the order of integration we have

$$D(k_g, 0 \,|\, k_s, 0; \omega) = \int dz \sum_{i=1}^{2} \int dk_y A_i(k_g, k_s, k_y, q, q')$$

$$\cdot a_i(k_g - k_s, z)\; e^{-i(q+q')z} \qquad (3.A6)$$

where the A_i are the 3-D analogs of the factors defined by equations (3.35) and (3.36). If we assume the A_i are slowly varying with respect to the exponential, then we can evaluate the k_y integral by stationary phase. To do this $q + q'$ is expanded about the point where its derivative with respect to k_y is zero, which in this case is the point $k_y = 0$. Thus,

$$q + q' = k_z + k_y^2 \, k_z''$$

where k_z is given by equation (3.30), and

$$k_z'' = \frac{-k_z}{q_g q_s}$$

In the last expression q_g and q_s are the two-dimensional vertical wavenumbers defined by equation (3.23).

Using the standard stationary phase formulas, equation (3.A4) may be expressed as

$$D(k_g,0\,|\,k_s,0;\omega) = \sum_{i=1}^{2} \tilde{A}_i\, \tilde{\alpha}_i \tag{3.A7}$$

where the $\tilde{\alpha}_i$ are scaled versions of the α_i used in the text

$$\tilde{\alpha}_i(x,z) = \frac{\alpha_i(x,z)}{\sqrt{z}} \tag{3.A8}$$

and the factors $\tilde{A}_i$ are related to the A_i of equations (3.35) and (3.36) by

$$\tilde{A}_i = \sqrt{\frac{q_g q_s}{ik_z}}\, A_i \tag{3.A9}$$

The result is, of course, subject to the approximations used in the stationary phase evaluation of the k_y integral. However, since most seismic data is far-field, we expect the approximation to be reasonably accurate.

For the vertically varying medium, a similar argument leads to a modification of the multiplicative factor in the downward continuation algorithm. We obtain

$$I_0(k_g,z\,|\,k_s,z) \rightarrow I_0(k_g,z\,|\,k_s,z)\cdot\left\{ i\int_0^z dz'\, \frac{\omega^2}{v_r^2}\left[\frac{1}{q_g^3(z')} + \frac{1}{q_s^3(z')}\right]\right\}^{1/2} \tag{3.A10}$$

The rest of the inversion proceeds as before.

The modification required to adapt the laterally varying algorithm to point sources and receivers, will be left as an exercise for the reader.