An Inverse-Q Filter

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Abstract

Statistical deconvolutions such as predictive deconvolution, Minimum Entropy Deconvolution, etc. perform best when applied to stationary data. The typical seismic trace, however, is highly non-stationary; and one persistent source of non-stationarity is attenuation. Toward the objective of enhancing stationarity, a mathematical model of attenuation is described which is consistent with currently popular models and, most importantly, leads to a computationally efficient method for "backing out" attenuation.

Introduction

A phenomenon commonly observed in seismic traces is that the dominant frequency in a seismogram decreases with time. This attenuation of high frequencies is perhaps best illustrated by a synthetic example. Let the spike train labeled "x" in Figure 1 represent a reflectivity sequence for a layered earth, and suppose we record the earth's response to an impulsive, vertically propagating plane-wave. If the earth were a non-attenuating, "perfect", oscillator, the response (neglecting multiple reflections and transmission losses) would be x. In a "lossy" earth, however, the higher frequencies contained in the initially impulsive wavefront are progressively attenuated, resulting in both a broadening and a decrease in amplitude of the wavefront with traveltime. The trace labeled "q" in Figure 1 illustrates this progressive wavefront distortion at traveltime intervals of 200 msec. The corresponding earth-response is labeled "y".

Now suppose we want to "deconvolve" y to recover x. Any statistical deconvolution method based on the assumption that y is stationary (as do most routinely used methods) will yield a poor result. Traces "dq" and "dy", for example, are the results of applying "spiking", predictive deconvolution to q and y, respectively. As expected, a time-invariant

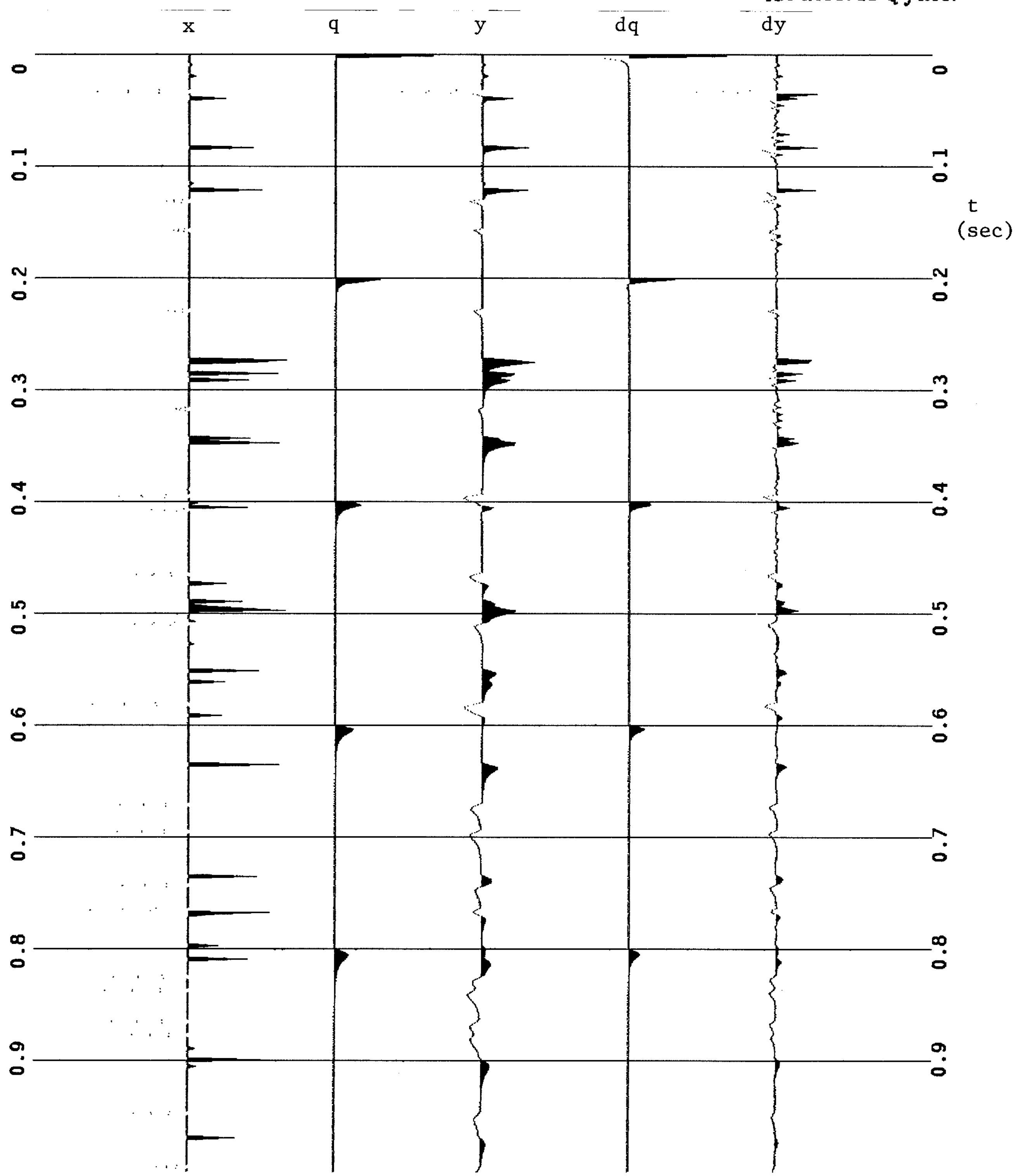

FIG. 1. A synthetic example of attenuation. x represents a reflectivity series, q illustrates the progressive distortion of an initially impulsive plane-wave in an attenuating earth (Q_0 = 100), and y is the corresponding earth response. dq and dy are the spiking deconvolutions of q and y, respectively.

operator cannot deconvolve a time-varying waveform; the operator has done too much at early traveltimes and too little at late traveltimes.

Seismic data processors, of course, have developed ways of circumventing non-stationarity. One simple technique is to divide a trace into several smaller windows, assume stationarity within each window, deconvolve each window separately, and then blend the windows together. However, because statistical deconvolution processes generally perform best when given long, stationary windows from which to estimate an unknown seismic wavelet, one must choose window lengths carefully. Indeed, papers by Wang (1969) and Foster et al (1968) treat the subject of "optimum" window lengths in great detail. Adaptive deconvolution is another recourse for non-stationary data. But even with adaptive methods one must choose an adaptation rate with the same considerations as those given to choosing window lengths. [See, for example, Griffiths et al (1977).] The point is this: *any* statistical deconvolution process should perform better with stationarity than without stationarity.

How, then, can we make seismic traces more stationary? We routinely compensate for spherical divergence, one source of non-stationarity, by applying time-variable gain to our traces. We might hope to further improve the stationarity of seismic traces by compensating for attenuation or, more specifically, by "inverting" the attenuation process. The purpose of this paper is to describe an efficient method for inverting attenuation in seismic traces.

Modeling

Before we discuss inversion, we must first specify a mathematical model of attenuation. Suppose a horizontal reflector at a depth z in the earth "explodes" at time $t=0$. A commonly used model of attenuation [e.g., Trorey (1962)] states that the waveform recorded at the surface would have a Fourier transform proportional to

$$B(\omega) \equiv \exp\left[-\frac{|\omega| z}{2Q_0 v} - i\frac{\omega z}{v}\right]$$

where v is propagation velocity and Q_0 is an intrinsic property of the earth which parameterizes attenuation. Both v and Q_0 are assumed constant with respect to z and ω. (Although the following discussion can be generalized to depth-variable Q_0, Q_0 will, for simplicity, be assumed constant.) The corresponding seismic trace would be a zero-phase waveform centered at $t = z/v$. Such a waveform is physically unacceptable for it implies propagation of energy at a velocity greater than v. A more realistic model is

$$B(\omega) \equiv \exp\left[-\frac{|\omega| z}{2Q_0 v} - i\frac{\omega z}{v} - i\varphi(\omega)\right] \tag{1}$$

where $\varphi(\omega)$ is chosen to make the waveform *beginning* at $t = z/v$ minimum-phase. A thorough, physical justification for equation (1) is given by Aki and Richards (1980, p. 167-185). [Choosing $\varphi(\omega)$ to be the minimum-phase corresponds to including velocity dispersion in the model, and v in equation (1) must now strictly be thought of as the high-frequency limit of velocity.] Further motivation for the choice of phase in equation (1) lies in the resulting efficiency of the modeling and, particularly, the inverse-filtering process.

To see how equation (1) can be used to model attenuation, first replace z/v in that equation by two-way, vertical traveltime τ:

$$B(\tau,\omega) \equiv \exp\left[-\frac{|\omega| \tau}{2Q_0} - i\omega\tau - i\varphi(\tau,\omega)\right] \tag{2}$$

The impulse response of an earth with a single, horizontal reflector at depth $z = v\tau/2$ is given (neglecting multiples) by the inverse Fourier transform (*IFT*) of $B(\tau,\omega)$ with respect to ω:

$$b(\tau,t) \equiv IFT\left\{B(\tau,\omega)\right\}.$$

Now, $\varphi(\tau,\omega)$ is a linear function of τ since (1) the minimum-phase spectrum is the Hilbert transform of the logarithm of the amplitude spectrum and (2) the Hilbert transform is a linear transform. Therefore,

$$B(2\tau,\omega) = \exp\left[-\frac{|\omega| 2\tau}{2Q_0} - 2i\omega\tau - i2\varphi(\tau,\omega)\right]$$

$$= B(\tau,\omega) \cdot B(\tau,\omega)$$

and, by the convolution theorem,

$$b(2\tau,t) = b(\tau,t) * b(\tau,t)$$

where "*" denotes convolution with respect to t. In general, the impulse response to a reflector at a "depth" of $j\tau$ is

$$b(j\tau,t) = b^j(\tau,t)$$

where $b^j(\tau,t)$ denotes $j-1$ self-convolutions of $b(\tau,t)$. Now let τ denote the temporal sampling interval in our seismic experiment. Sampling $b^j(\tau,t)$,

$$b_i^j \equiv b^j(\tau, i\tau)$$

is the impulse response for a reflector at a "depth" of j sampling intervals. We note from the linear-phase term in equation (2) that the first j samples of b^j are zero, so we define a shifted version of b^j:

$$q_i^j \equiv b_{i+j}^j$$

where, as before, q^j denotes $j-1$ self-convolutions of q. q^1 is just the discretized version of

$$q(t) \equiv IFT\left\{\exp\left[-\frac{|\omega|\tau}{2Q_0} - i\varphi(\tau,\omega)\right]\right\} \tag{3}$$

and q^j, unlike b^j, is minimum-phase for all j $(= 1,2, \cdots)$.

Now let x_i $(i = 1,2, \cdots ,n)$ denote a sampled (at interval τ) sequence of n reflection coefficients. The impulse response y_i of the corresponding n-layered earth is found (again neglecting multiples) by superposing the responses to each of the individual reflectors:

$$y_i = x_1 b_i^1 + x_2 b_i^2 + \cdots + x_n b_i^n$$

or

$$y_i = \sum_{j=1}^{i} q_{i-j}^j x_j \tag{4}$$

Equation (4) looks like a convolution except for the superscript j on q. In fact, like convolution, equation (4) can be thought of as a matrix multiplication. An $n=4$ example is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} q_0^1 & 0 & 0 & 0 \\ q_1^1 & q_0^2 & 0 & 0 \\ q_2^1 & q_1^2 & q_0^3 & 0 \\ q_3^1 & q_2^2 & q_1^3 & q_0^4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5}$$

which we abbreviate as $y = Qx$. The "Q" matrix is lower-triangular because q^j is causal.

We can now see how "q" and "y" in Figure 1 were produced. First, q^1 was derived numerically using equation (3) with $Q_0 = 100$ and $\tau = 2$ msec. [Specifically, the minimum-phase q^1 was obtained from its zero-phase equivalent by spectral factorization via the Toeplitz method (Claerbout, 1976).] Then y was computed using equation (4). Since $q^j = q^{j-1} * q^1$, only q^j and q^1 need be held in memory at any stage in the matrix multiplication. q^j was output every 100 samples (200 msec), resulting in "q" of Figure 1.

Naturally, once we have the impulse response y of our lossy earth, we can then find the response z to any source waveform f by

$$z = f * y$$

or

$$z_i = \sum_j f_{i-j} y_j = \sum_j f_{i-j} \sum_{k \le j} q_{j-k}^k x_k$$

Thinking in terms of matrices,

$$z = Fy = FQx \tag{6}$$

where F is a Toeplitz matrix with elements $F_{ij} = f_{i-j}$.

Inverse-filtering

The deconvolution problem is to find x given z. From equation (6) we have

$$x = Q^{-1} y = Q^{-1} F^{-1} z$$

Notice the order in which these inverse matrices are multiplied. If F and, hence, F^{-1} are known, then this ordering presents no difficulty. But if F is unknown and we attempt to estimate F via statistical deconvolution, then we might hope that F^{-1} and Q^{-1} commute so we could estimate F from the stationary $Q^{-1}z$. F^{-1} and Q^{-1}, however, do not generally commute (just as F and Q do not generally commute), and we should at least be aware of the error in assuming that they do. If the inverse of the source waveform is short and q^1 is almost a delta function, then this error may be negligible.

In any case, we need Q^{-1}, whether we apply it before or after F^{-1}. Since Q is lower-triangular, we can solve equation (4) or (5) for x by back-substitution, computing $x_1, x_2, \cdots, x_n$ recursively; but this method turns out to be computationally less efficient than using Q^{-1} directly. As a first guess for Q^{-1}, we might expect it to contain p^1 where $p^1 * q^1 = \delta$; i.e., p^1 is the inverse of q^1 derived by sampling

$$p(t) \equiv IFT \left\{ \exp \left[\frac{+|\omega|\tau}{2Q_0} + i\varphi(\tau,\omega) \right] \right\} \tag{7}$$

Since q^j is minimum-phase for all j (= 1,2, $\cdots$), we know that p^i is minimum-phase for all i (= 1,2, $\cdots$); and we can define a lower-triangular matrix P with elements $P_{ij} = p_{i-j}^i$. Just as the *columns* of Q can be derived by the recursion $q^j = q^{j-1} * q^1$, so the *rows* of P can be derived by the recursion $p^i = p^{i-1} * p^1$. P is not, however, the inverse of Q as is perhaps best illustrated by the $n=4$ example:

$$\begin{bmatrix} p_0^1 & 0 & 0 & 0 \\ p_1^2 & p_0^2 & 0 & 0 \\ p_2^3 & p_1^3 & p_0^3 & 0 \\ p_3^4 & p_2^4 & p_1^4 & p_0^4 \end{bmatrix} \begin{bmatrix} q_0^1 & 0 & 0 & 0 \\ q_1^1 & q_0^2 & 0 & 0 \\ q_2^1 & q_1^2 & q_0^3 & 0 \\ q_3^1 & q_2^2 & q_1^3 & q_0^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_1^1 & 1 & 0 & 0 \\ p_2^2 & p_1^1 & 1 & 0 \\ p_3^3 & p_2^2 & p_1^1 & 1 \end{bmatrix} \neq I$$

For example, $(PQ)_{21} = (p^2 * q^1)_1 = (p^1 * p^1 * q^1)_1 = (p^1 * \delta)_1 = p_1^1 \neq 0$. Although $PQ \neq I$, it is Toeplitz and lower-triangular as is its inverse which we define as

$$S \equiv \begin{bmatrix} s_0 & 0 & 0 & 0 \\ s_1 & s_0 & 0 & 0 \\ s_2 & s_1 & s_0 & 0 \\ s_3 & s_2 & s_1 & s_0 \end{bmatrix}$$

where the s_i are given by the recursion

$$s_i = -\sum_{j=1}^{i} p_j^j s_{i-j} \quad ; \quad s_0 = 1$$

Equivalently, $s * (1, p_1^1, p_2^2, \cdots) = \delta$. (Remember that multiplying two Toeplitz, lower-triangular matrices is like convolving two time-invariant, causal filters.) So the desired inverse is $Q^{-1} = SP$; i.e.,

$$x = SPy$$

or

$$x_i = \sum_{j=0}^{i} s_{i-j} \sum_{k=0}^{j} p_{j-k}^j y_k$$

In practice, one may not wish to construct S, even though it is fairly inexpensive to do so. First, the off-diagonal elements of PQ will likely be small ($\ll 1$) for typical values of Q_0 ($\approx$100); and, secondly, if our goal is only to reduce non-stationarity, then we gain nothing in applying s, a time-invariant filter.

Approximate Inverse-filtering

Even with the recursion $p^i = p^{i-1} * p^1$, the computational expense in multiplying P by a seismic trace z is potentially great. In practice, however, the expense is comparable to that of ordinary convolution, the reasons being that (1) p^1 can be approximated by a very short operator and (2) p^i can, therefore, be truncated to be relatively short. Recall that p^1

is computed from equation (7) and that the exponent $|\omega|\tau/2Q_0$ in that equation ranges from zero to a maximum value of $\pi/2Q_0$ at the Nyquist frequency. For realistic earth values of Q_0 ($\approx$100), the amplitude spectrum will be almost unity for all ω between zero and Nyquist. p^1, therefore, is almost a delta function and can be approximated rather well by $p^1 \approx (1+\varepsilon,-\varepsilon)$, a two-term filter where $0<\varepsilon\ll 1$. Notice that the two terms of this approximate add to unity since equation (7) prescribes no amplification for zero-frequency.

The traces labeled "P_2q" and "P_2y" in Figure 2 illustrate the results of using this two-term approximate in inverse-filtering traces q and y of Figure 1. p^1 was computed by the Toeplitz method to be the best (in a least-square-error sense) two-term inverse to q^1 for $Q_0 = 100$. ε was found to be $\varepsilon \approx 0.0064$. Also shown in Figure 2 are the results, "$P_{10}q$" and "$P_{10}y$", of using a ten-term approximate p^1. The cost of computing p^i in these examples was reduced by restricting the maximum length of p^i to be forty samples. In both examples, the filter s was computed and applied; the errors in the inverse-filtered traces result solely from approximating p^1 with short operators.

The errors in using the two-term, approximate inverse are more clearly understood by comparing the Fourier amplitude spectrum of $(1+\varepsilon,-\varepsilon)$ with that of the exact p^1 given by equation (7). Figure 3 plots the *ratio* of these two amplitude spectra as a function of frequency. The small deviations from unity become magnified by repeated self-convolutions of p^1, so that the inverse-filter p^i becomes progressively worse at later traveltimes (large i). For seismic data, with a more limited bandwidth, the errors may not be so significant; and ε could, conceivably, be chosen to minimize the error in the "seismic band".

One further modification to p^1 may be desired. At late traveltimes, high-frequency components of seismic signals may have been so severely attenuated as to be submerged in ambient noise. And with Vibroseis data, we can be certain that frequencies below and above the sweep frequencies contain only noise. We can avoid excessive amplification of high-frequency noise by modifying the amplitude spectrum of p^1. Figure 4 illustrates the results of using a p^1 of length ten with an amplitude spectrum given by $A(\omega) = [\exp(-|\omega|\tau/2Q_0) + \alpha\omega^4]^{-1}$ where α was chosen so that $A(\omega_{Nyquist}) = 1$. At low frequencies the exponential term dominates, and at high frequencies (approaching Nyquist) the second term drives $A(\omega)$ back down to unity; the result is a "bandlimited" inversion of Q.

Speculation

Nothing has yet been said about the problem of determining Q_0; some reasonable estimate of Q_0 is needed to compute p^1. Methods do exist for determining a Q_0 [see, for example, Jacobson et al (1981)]; but, if the inverse-filtering process is fast enough, the best way of determining a p^1 might be to try several and then "pick" the best result. The inverse-filtering method described in this paper was, in fact, motivated by the following possible (being optimistic) application. The typical, statistical deconvolution problem is to determine, say, m coefficients of an filter which, when convolved with a seismic trace, maximizes (or minimizes) some objective function of the output trace. Why not ask for one more number, Q_0, or even more simply, ε in the two-term approximation to p^1? In a minimum-entropy approach, a fast line-search over ε might be performed to determine the value which maximizes "spikiness". A similar approach was used by Gray (1979) to reduce non-stationarity due to spherical divergence; he determined an exponential-gain constant that minimized spikiness. The feasibility of "minimum-entropy inverse-Q filtering", however, remains to be seen.

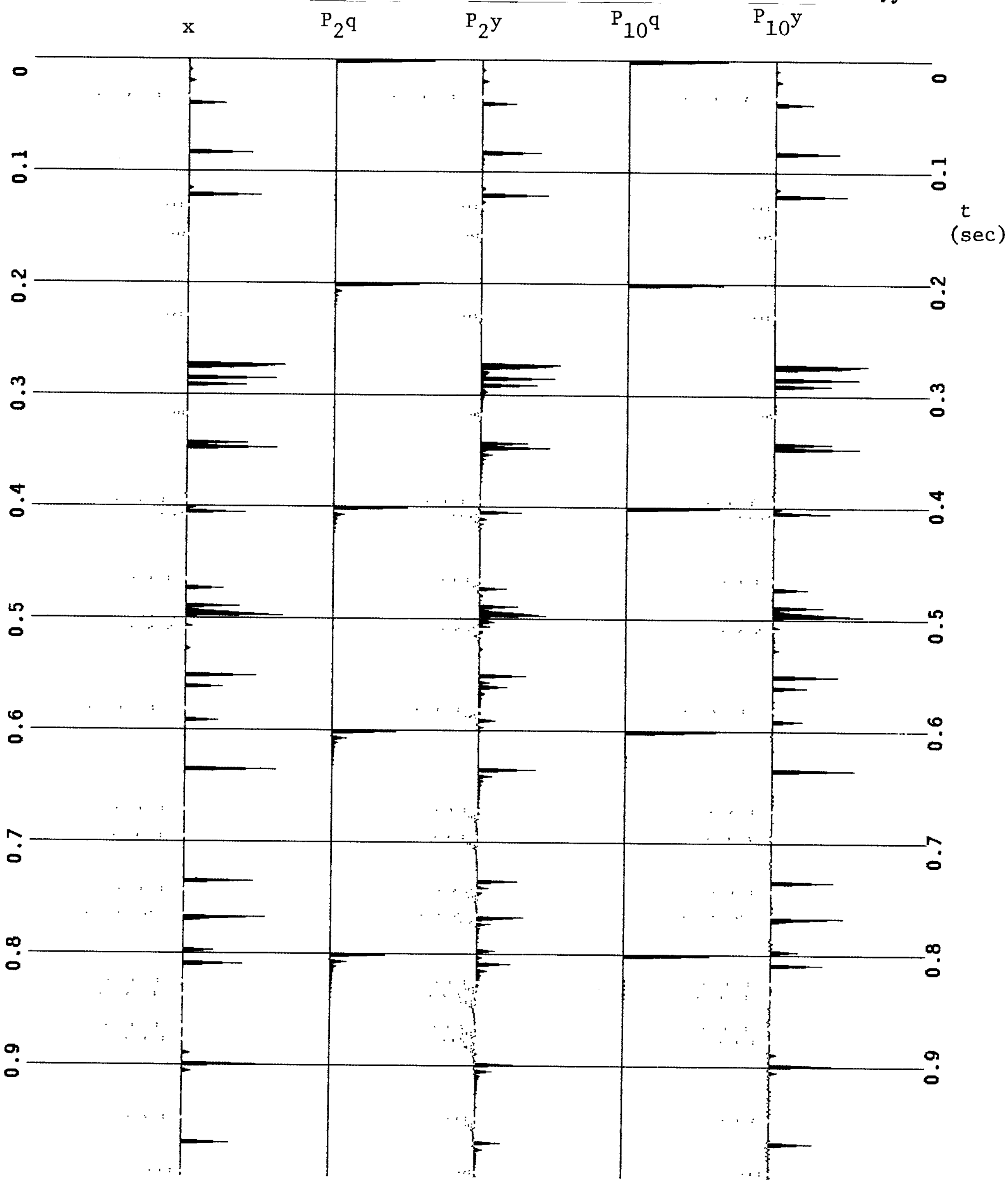

FIG. 2. Approximate inverse-Q filtering. x is the reflectivity series from Figure 1. P_2q and P_2y are the results of using a two-term p^1 in filtering q and y of Figure 1. $P_{10}q$ and $P_{10}y$ are the results for a p^1 with ten coefficients.

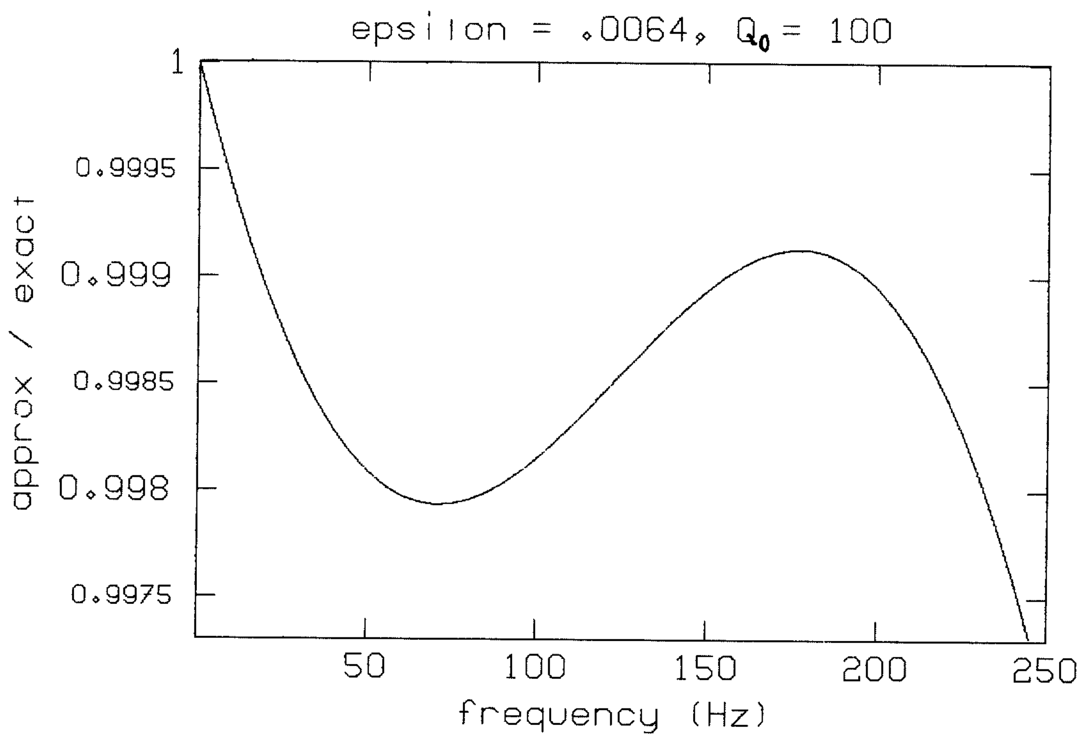

FIG. 3. Ratio of Fourier amplitude spectra for the two-term, approximate $(1+\varepsilon, -\varepsilon)$ and the exact p^1 as a function of frequency.

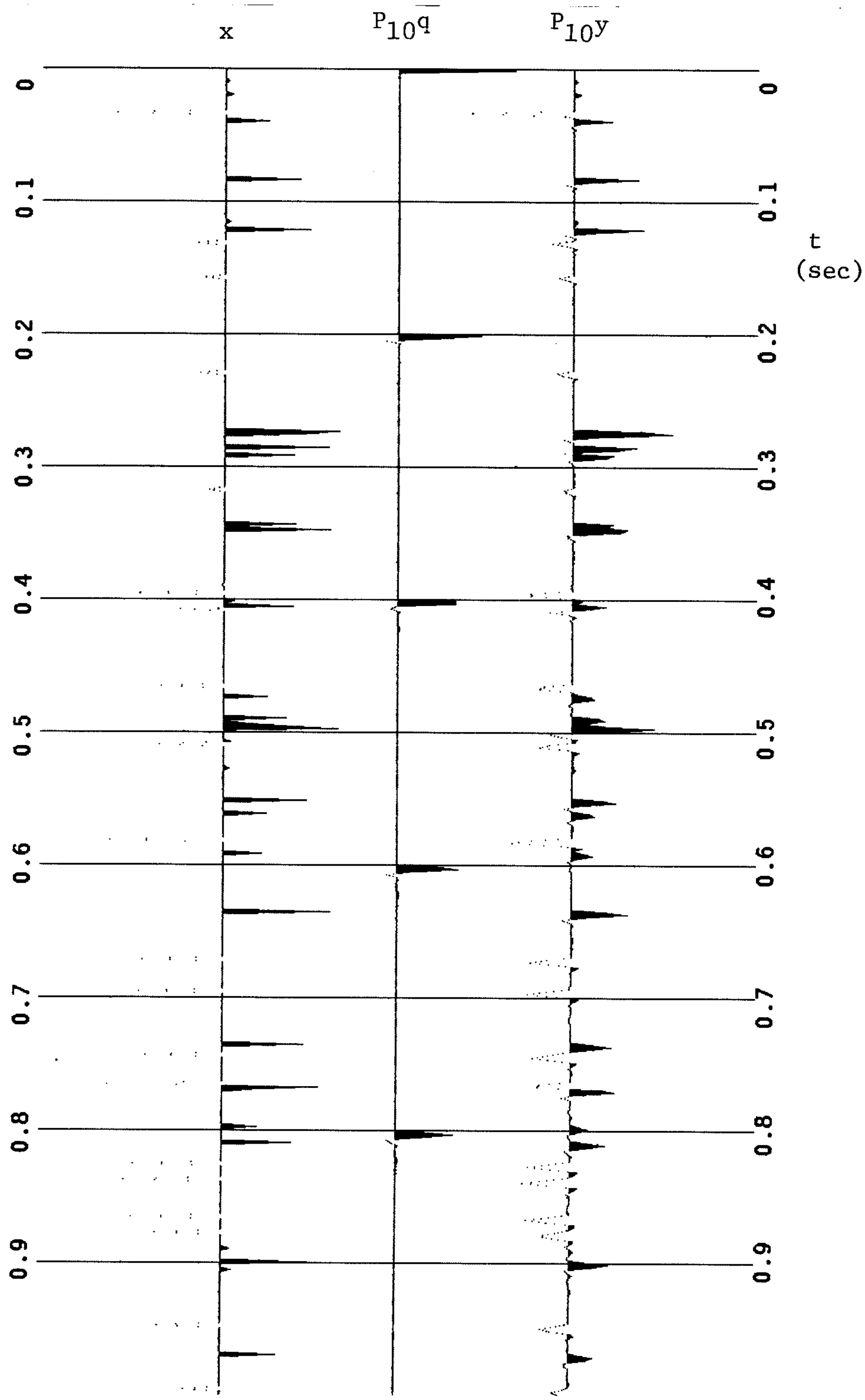

FIG. 4. Bandlimited inverse-Q filtering. x is the reflectivity series from Figure 1. $P_{10}q$ and $P_{10}y$ are the results of using a bandlimited, ten-term approximate to p^1.

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From "The Swords of Lankhmar", By "Fritz Leiber"

Pardo's First Postulate:
Anything good in life is either illegal, immoral, or fattening.

Arnold's Addendum:
Anything not fitting into these categories causes cancer in rats.

Drive defensively, buy a tank.

"We have met the enemy, and he is us."
-- Walt Kelly

What this country needs is a good five cent ANYTHING!

Hand: A singular instrument worn at the end of a human arm and commonly thrust into somebody's pocket.

George Orwell was an optimist.

We really don't have any enemies. It's just that some of our best friends are trying to kill us.

Time is nature's way of making sure that everything doesn't happen at once.

Today is a good day to bribe a high-ranking public official.

Death is nature's way of telling you to slow down

Deliberation: The act of examining one's bread to determine which side it is buttered on.

Certain old men prefer to rise at dawn, taking a cold bath and a long walk with an empty stomach and otherwise mortifying the flesh. They then point with pride to these practices as the cause of their sturdy health and ripe years; the truth being that they are hearty and old, not because of their habits, but in spite of them. The reason we find only robust persons doing this thing is that it has killed all the others who have tried it.