Effect of Reflection Coefficients on Synthetic Seismograms
I. Theory

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Abstract

Our aim in this paper is to derive the equations to be used in generating a 2-D synthetic seismogram by wavefield extrapolation with the wave equation. We are going to work in the (ω,k_x) domain. The choice of this domain is motivated by the fact that it will be straightforward to include attenuation effects by specifying complex frequency-dependent elastic moduli. We derive the appropriate Green's function and an expression for the reflection coefficient at a liquid-solid interface.

1. Calculus of the Green's function for the wave equation.

Consider the wave equation when the driving force is a Dirac source at time $t = 0$ and at the point $(x=0, z=0)$. Its solution is the Green's function $G(x,z,t)$ and it is given by:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] G(x,z,t) = -2\pi\delta(x)\delta(z)\delta(t) \tag{1}$$

Taking the triple Fourier Transform of equation (1) using the definitions

$$F(k_x,k_z,\omega) = \int\int\int f(x,z,t)\, e^{ik_x x + ik_z z - i\omega t}\, dx\, dz\, dt$$

$$f(x,z,t) = \frac{1}{(2\pi)^3} \int\int\int F(k_x,k_z,\omega)\, e^{-ik_x x - ik_z z + i\omega t}\, dk_x\, dk_z\, d\omega$$

we obtain

$$\left[k_x^2 + k_z^2 - \frac{\omega^2}{v^2} \right] G(k_x,k_z,\omega) = 2\pi$$

which gives for $G(k_x, k_z, \omega)$

$$G(k_x,k_z,\omega) = \frac{2\pi}{k_z^2 - (\frac{\omega^2}{v^2} - k_x^2)}$$

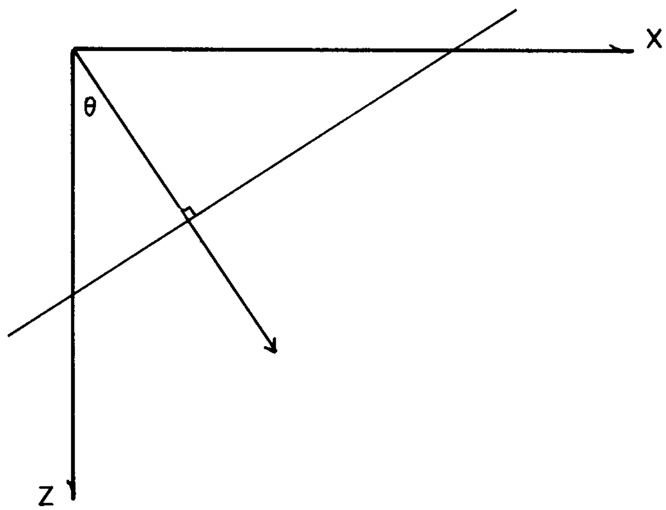

FIG. 1.

By definition of the Snell's parameter p (see figure 1), $p = \frac{k_x}{\omega} = \frac{\sin\vartheta}{v}$, we have $k_x = \frac{\omega}{v}\sin\vartheta$. Therefore

$$\frac{\omega^2}{v^2} - k_x^2 = \frac{\omega^2}{v^2}\cos^2\vartheta \geq 0$$

and

$$G(k_x,k_z,\omega) = \frac{2\pi}{\left[k_z - \left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2}\right]\left[k_z + \left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2}\right]}$$

This gives

$$G(k_x,\omega,z) = \int_{-\infty}^{+\infty} \frac{e^{-ik_z z}\,dk_z}{\left[k_z - \left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2}\right]\left[k_z + \left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2}\right]} \qquad (2)$$

As we want a causal function for $G(k_x,\omega,z)$, we are going to take the following contour of integration (Morse and Feshbach, 1953, p 850),

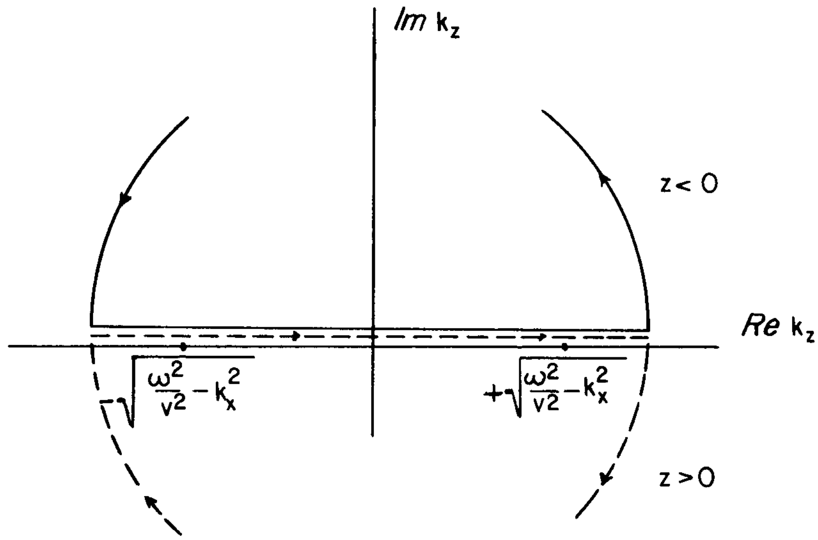

FIG. 2.

Using Cauchy's integral formula in equation (2), we get

$$G(k_x,\omega,z) = 0 \qquad z < 0 \qquad (3)$$

$$G(k_x,\omega,z) = i\pi \left[\frac{e^{-i\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2} z}}{\left(\frac{\omega^2}{v^2}-k_x^2\right)^{1/2}} - \frac{e^{i\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2} z}}{\left(\frac{\omega^2}{v^2}-k_x^2\right)^{1/2}} \right] \qquad z > 0$$

This formula contains the upcoming and downgoing waves. It is necessary to separate them for every ω in order to write $G(k_x,\omega,z)$ as a function of $\frac{\omega}{v}\left[1-\left(\frac{k_x v}{w}\right)^2\right]^{1/2}$.

As we have

$$\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2} = \frac{\omega}{v} sgn\,\omega \left[1-\left(\frac{k_x v}{w}\right)^2\right]^{1/2}$$

we can rewrite $G(k_x,\omega,z)$ as

$$G(k_x,\omega,z) = i\pi \left[\frac{e^{-i\frac{\omega}{v} sgn\,\omega \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} sgn\,\omega \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}} - \frac{e^{i\frac{\omega}{v} sgn\,\omega \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} sgn\,\omega \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}} \right]$$

which gives

$$G(k_x,\omega,z) = i\pi \left[\frac{e^{-i\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}} - \frac{e^{i\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}} \right] \quad \omega > 0$$

and

$$G(k_x,\omega,z) = G(k_x,-\omega,z) \tag{4}$$

(therefore $G(k_x,\omega,z)$ is an even function of ω).

The downgoing wave (DW) and the upcoming wave (UW) are represented by the following parts of the total Green's function,

$$DW: \qquad i\pi \frac{e^{-i\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}}$$

$$UW: \qquad -i\pi \frac{e^{i\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2} z}}{\frac{\omega}{v} \left[1-\left(\frac{vk_x}{\omega}\right)^2\right]^{1/2}}$$

Remark: We could have put the source at the point $(x=x_0, z=z_0, t=t_0)$ and the problem would not have been changed in principle. The Green's function solution of the full wave equation would have been instead

$$G(k_x,k_z,\omega) = 2\pi \frac{e^{ik_x x_0 + ik_z z_0 - i\omega t_0}}{k_z^2 - (\frac{\omega^2}{v^2} - k_x^2)}$$

and in the (ω,k_x,z) domain, the solution would have been

$$G(k_x,\omega,z) = 0 \qquad z < z_0$$

$$G(k_x,\omega,z) = i\pi \left[\frac{e^{-i\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2}(z-z_0)}}{\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2}} - \frac{e^{i\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2}(z-z_0)}}{\left[\frac{\omega^2}{v^2}-k_x^2\right]^{1/2}} \right] e^{ik_x x_0 - i\omega t_0}$$

$$z > z_0$$

2. Synthetic seismogram for a liquid-solid interface.

We can model the effects of the reflection coefficient for a liquid-solid interface with the following experiment (figure 3): put a source function $s(x,t)$ at the sea-surface, go to the sea-floor by using the Green's function derived previously, multiply by the reflectivity function and use wavefield extrapolation to the surface to get the synthetic seismogram. In our discussion we will consider the source function separately from the reflection coefficients.

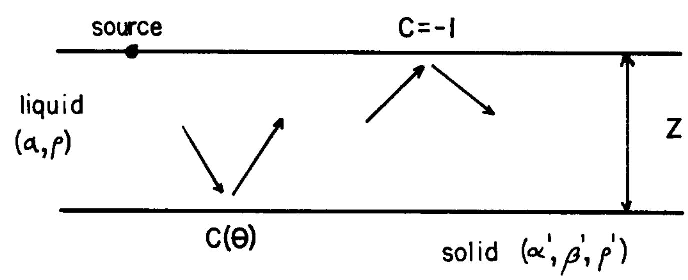

FIG. 3.

Adding the source function, the right-hand side of equation (1) becomes

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} \right] G(x,z,t) = -2\pi\delta(x)\delta(z)\delta(t) ** s(x,t)$$

where ** denotes convolution in space and time.

The total Green's function becomes

$$G_{total}(k_x,\omega,z) = S(k_x,\omega)\, i\pi \left[\frac{e^{-ik_z z}}{k_z} - \frac{e^{ik_z z}}{k_z} \right]$$

where $S(k_x,\omega)$ is the two-dimensional Fourier transform of the source function, and $k_z = \frac{\omega}{v}\left\{1 - \frac{v^2 k_x^2}{\omega^2}\right\}^{1/2}$

The Green's function for the downgoing wave is

$$G_{DW}(k_x,\omega,z) = i\pi\, S(k_x,\omega)\, \frac{e^{-ik_z z}}{k_z}$$

To come back to the surface, we must multiply by the well-known extrapolation function, which is

$$e^{-ik_z z}$$

the total wavefield will be a superposition of primary and multiple reflections. Since we know that the reflection coefficient at the air-water interface is very close to -1, we can write the total wavefield as

$$\left[1 - \frac{1}{1 + C(k_x,\omega)\, \frac{e^{-2ik_z \Delta z}}{k_z}} \right] S(k_x,\omega) =$$

$$= S(k_x,\omega) \left[C(k_x,\omega)\, \frac{e^{-2ik_z \Delta z}}{k_z} - C^2(k_x,\omega)\, \frac{e^{-4ik_z \Delta z}}{k_z} + \cdots \right] \qquad (5)$$

This series comes form the fact that the first sea-floor multiple has traveled a distance $2\Delta z$ while the primary has traveled only Δz. Their amplitudes are, at each bounce, multiplied by the reflection coefficient $C(\vartheta(k_x,\omega))$, where $\sin\vartheta = \frac{v k_x}{\omega}$.

The synthetic seismogram is obtained by taking the inverse Fourier transform of equation (5),

$$f(x,t) =$$

$$\frac{i}{2(2\pi)^2} \int\int \left[C(k_x,\omega)\, \frac{e^{-2ik_z \Delta z}}{k_z} - C^2(k_x,\omega)\, \frac{e^{-4ik_z \Delta z}}{k_z} + \cdots \right] S(k_x,\omega)\, e^{-ik_x x + i\omega t}\, dk_x\, d\omega$$

where $k_z = \frac{\omega}{v}\left[1 - \left(\frac{v k_x}{\omega}\right)^2\right]^{1/2}$

3. Reflection coefficient.

For a *liquid–solid* interface, the reflection coefficient is given in many books and articles (see for example Ewing *et al*, 1957). We give an outline of the derivation of this formula. The notation is given in figure 4.

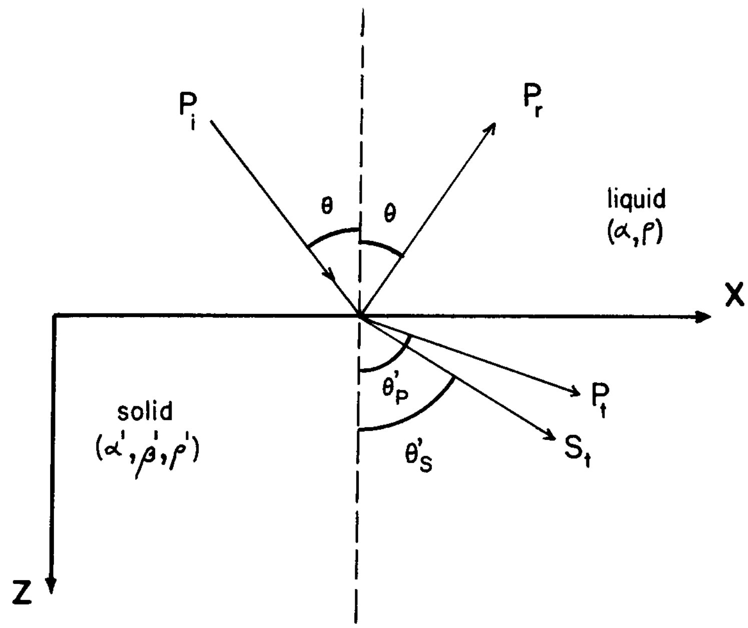

FIG. 4.

In terms of P and S wave potentials we have for the incident, reflected and transmitted waves

$$\Phi_i = A_1 e^{ik_\alpha(x \sin\vartheta + z \cos\vartheta) - i\omega t}$$

$$\Phi_r = A_2 e^{ik_\alpha(x \sin\vartheta - z \cos\vartheta) - i\omega t}$$

and $\Phi = \Phi_i + \Phi_r$, $\Psi = 0$

$$\Phi_t = A' e^{ik_{\alpha'}(x \sin\vartheta'_p + z \cos\vartheta'_p) - i\omega t}$$

$$\Psi_t = B' e^{ik_{\beta'}(x \sin\vartheta'_s + z \cos\vartheta'_s) - i\omega t}$$

and $\Phi' = \Phi_t$, $\Psi' = \Psi_t$.

$$k_\alpha = \frac{\omega}{\alpha} \quad k_{\alpha'} = \frac{\omega}{\alpha'} \quad k_{\beta'} = \frac{\omega}{\beta'}$$

Snell's law gives

$$\frac{\sin\vartheta}{\alpha} = \frac{\sin\vartheta'_p}{\alpha'} = \frac{\sin\vartheta'_s}{\beta'}$$

Continuity of stresses and displacements must be applied at the boundary, giving:

i) displacement: $u_z = u'_z$ or

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi'}{\partial z} + \frac{\partial\Psi'}{\partial x}$$

ii) stresses: $\sigma'_{zx} = 0$, $\sigma_{zz} = \sigma'_{zz}$ or

$$2\,\frac{\partial^2\Phi'}{\partial x\,\partial z} - \frac{\partial^2\Psi'}{\partial z^2} + \frac{\partial^2\Psi'}{\partial x^2} = 0$$

$$\lambda\,\Delta\Phi + 2\mu\left(\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Psi}{\partial x\,\partial z}\right) = \lambda'\,\Delta\Phi' + 2\mu'\left(\frac{\partial^2\Phi'}{\partial z^2} + \frac{\partial^2\Psi'}{\partial x\,\partial z}\right)$$

The reflection coefficient is obtained by solving the preceding system and this gives

$$C = \frac{\rho'\alpha'\cos\vartheta\left[(1-2\sin^2\vartheta'_s)^2 + \dfrac{4\beta'^3}{\alpha^2\alpha'}\sin^2\vartheta\cos\vartheta'_p\cos\vartheta'_s\right] - \rho\alpha\cos\vartheta'_p}{\rho'\alpha'\cos\vartheta\left[(1-2\sin^2\vartheta'_p)^2 + \dfrac{4\beta'^3}{\alpha^2\alpha'}\sin^2\vartheta\cos\vartheta'_p\cos\vartheta'_s\right] + \rho\alpha\cos\vartheta'_p} \tag{6}$$

or

$$C = \frac{A-B}{A+B}$$

where A and B can also be written:

$$A = \rho'\alpha'\cos\vartheta\left[(1-2\frac{\beta'^2}{\alpha^2}\sin^2\vartheta)^2 + \frac{4\beta'^3}{\alpha^2\alpha'}\sin^2\vartheta\left(1-\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta\right)^{1/2}\left(1-\frac{\beta'^2}{\beta^2}\sin^2\vartheta\right)^{1/2}\right]$$

$$B = \rho\alpha\left(1-\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta\right)^{1/2}$$

This formula is valid for the angle ϑ less than the smallest critical angle.

If $\alpha \geq \alpha'$, there is *no* critical angle and equation (3) holds everywhere.

If $\beta' \leq \alpha < \alpha'$, there is *one* critical angle defined by $\vartheta_c = \sin^{-1} \frac{\alpha}{\alpha'}$.

Formally, the mathematics can be derived in the same way for $\vartheta > \vartheta_c$. The difference is that $\cos\vartheta'_p$ is going to be pure imaginary, implying that the transmitted P energy will be evanescent in the z direction. As we do not want to increase the energy with propagation, we must have a *minus* sign in front of the square root. Therefore

$$"\cos\vartheta'_p" = i \left[\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta - 1 \right]^{1/2}$$

In this case the reflection coefficient is complex-valued, which means that there is a phase shift after critical angle. The expression for the reflection coefficient is thus given by

$$C(\vartheta) = \frac{D + i(E - F)}{D + i(E + F)} = C_1(\vartheta) + iC_2(\vartheta) \tag{7}$$

where

$$C_1(\vartheta) = \frac{D^2 + E^2 - F^2}{D^2 + (E + F)^2}$$

$$C_2(\vartheta) = \frac{-2DF}{D^2 + (E + F)^2}$$

and

$$D = \rho'\alpha'\cos\vartheta\,(1 - 2\frac{\beta'^2}{\alpha^2}\sin^2\vartheta)^2$$

$$E = \left[\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta - 1 \right]^{1/2} \rho'\alpha'\cos\vartheta\, \frac{4\beta'^3}{\alpha^2\alpha'}\sin^2\vartheta \left[1 - \frac{\beta'^2}{\beta^2}\sin^2\vartheta \right]^{1/2}$$

$$F = \rho\alpha \left[\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta - 1 \right]^{1/2}$$

If $\alpha < \beta' < \alpha'$, we have two critical angles

$$\vartheta_{c_1} = \sin^{-1}\frac{\alpha}{\alpha'}$$

$$\vartheta_{c_2} = \sin^{-1}\frac{\alpha}{\beta'}$$

For $\vartheta < \vartheta_{c_1}$, equation (6) holds for the reflection coefficient.

For $\vartheta_{c_1} < \vartheta < \vartheta_{c_2}$, equation (7) holds for the reflection coefficient.

For $\vartheta > \vartheta_{c_2}$, we have to consider the *pseudo−S−transmission* angle defined like the one for P waves by

$$"\cos\vartheta'_s" = i\left(\frac{\beta'^2}{\alpha^2}\sin^2\vartheta - 1\right)^{1/2}$$

accordingly we obtain for $C(\vartheta)$

$$C(\vartheta) = \frac{G - iH}{G + iH} = E_1(\vartheta) + iE_2(\vartheta) \tag{8}$$

where

$$E_1(\vartheta) = \frac{G^2 + H^2}{G^2 + H^2}$$

$$E_2(\vartheta) = \frac{-2HG}{G^2 + H^2}$$

and

$$G = \rho'\alpha'\cos\vartheta\left[\left(1 - 2\frac{\beta'^2}{\alpha^2}\sin^2\vartheta\right)^2 - \frac{4\beta'^3}{\alpha^2\alpha'}\sin^2\vartheta\left(\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta - 1\right)^{1/2}\left(\frac{\beta'^2}{\beta^2}\sin^2\vartheta - 1\right)^{1/2}\right]$$

$$H = \rho\alpha\left(\frac{\alpha'^2}{\alpha^2}\sin^2\vartheta - 1\right)^{1/2}$$

4. Reflection coefficient in the (ω,k_x) plane

To define the reflection coefficient in the (ω,k_x) plane, we must give its value in the four quadrants. It is possible to do that utilizing the properties of the 2D-inverse Fourier transform of the reflection coefficient, that is $c(t,x)$. This function must be *real*, *symmetric* in x and *causal*.

First condition: *Reality*.

A general function of 2 variables can be written in terms of its even and odd parts as:

$$C(\omega,k_x) = \mathrm{Re}\, E_{k_x}E_\omega + \mathrm{Re}\, E_{k_x}O_\omega + \mathrm{Re}\, O_{k_x}E_\omega + \mathrm{Re}\, O_{k_x}O_\omega$$
$$+\, i\,(\mathrm{Im}\, E_{k_x}E_\omega + \mathrm{Im}\, E_{k_x}O_\omega + \mathrm{Im}\, O_{k_x}E_\omega + \mathrm{Im}\, O_{k_x}O_\omega)$$

where Re refers to the *real* part and Im to the *imaginary* part, E_{k_x} refers to the *even* part of k_x, O_ω refers to the *odd* part of ω, and so on.

Since the Fourier transform of a real-even function is real-even, and of a real-odd function is imaginary-odd, to guarantee $c(x,t)$ to be real, $C(\omega,k_x)$ must be of the form

$$C(\omega,k_x) = \text{Re } E_{k_x}E_\omega + \text{Re } O_{k_x}O_\omega + i\,(\text{Im } E_{k_x}O_\omega + \text{Im } O_{k_x}E_\omega)$$

Second condition: *Symmetry* in x.

The condition of symmetry in the x direction implies that $C(\omega,k_x) = C(\omega,-k_x)$, thus we are left with

$$C(\omega,k_x) = \text{Re } E_k E_\omega + i \text{ Im } E_k O_\omega$$

if we call

$$C(|\omega|,|k_x|) = C_1(|\omega|,|k_x|) + i\, C_2(|\omega|,|k_x|)$$

then

$$C(\omega,k_x) = C_1(|\omega|,|k_x|) + i\, sgn\,\omega\, C_2(|\omega|,|k_x|)$$

Let us define now $C(|\omega|,|k_x|)$. We can use (6), (7) and (8) and replace $\sin\vartheta$ by $\frac{vk_x}{\omega}$. In fact it is better to write C as a function of $(-i\omega)$ to help inspection for causality later. (6), (7) and (8) become

$$C(|\omega|,|k_x|) = \frac{A^* - B^*}{A^* + B^*} = C_1(|\omega|,|k_x|) + i\, C_2(|\omega|,|k_x|) \tag{9}$$

where

$$A^* = \rho'\alpha' \left[(-i\omega)^2 + \alpha^2k_x^2\right]^{1/2}$$

$$\left\{\left[(-i\omega)^2 + 2\beta'^2k_x^2\right]^2 - \frac{4\beta'^3k_x^2}{\alpha'}\left[(-i\omega)^2 + \alpha'^2k_x^2\right]^{1/2}\left[(-i\omega)^2 + \beta'^2k_x^2\right]^{1/2}\right\}$$

$$B^* = \rho\alpha(-i\omega)^4\left[(-i\omega)^2 + \alpha'^2k_x^2\right]^{1/2}$$

with

$$\left[(-i\omega)^2 + v^2k_x^2\right]^{1/2} = \begin{cases} (-i\omega)\left[1 - \dfrac{v^2k_x^2}{\omega^2}\right]^{1/2} & \text{for } 1 - \dfrac{v^2k_x^2}{\omega^2} \geq 0 \\ \omega\left[\dfrac{v^2k_x^2}{\omega^2} - 1\right]^{1/2} & \text{for } 1 - \dfrac{v^2k_x^2}{\omega^2} < 0 \end{cases}$$

whence

$$C(\omega,k_x) = \frac{\text{Re}\,(A^* - B^*) + i\ sgn\,\omega\ \text{Im}\,(A^* - B^*)}{\text{Re}\,(A^* + B^*) + i\ sgn\,\omega\ \text{Im}\,(A^* + B^*)}$$

Third condition: *Causality*.

To verify causality of the reflection coefficient, we are going to apply Muir's rules: (Claerbout, 1979. SEP 16, p 141-134). Namely:

i) The sum of two causal operators is causal.

ii) The product of two causal operators is causal.

iii) The inverse of a causal operator is causal if it has a positive real part.

iv) The square of a causal operator is causal.

We know that $(-i\omega)$ is causal (Claerbout and Kjartansson, 1979. SEP 16, p 131-140).

Therefore $\left[(-i\omega)^2 + v^2k_x^2\right]^{1/2}$ and $\left[(-i\omega)^2 + 2\beta'^2k_x^2\right]^2$ are causal.

This implies that $A^* - B^*$ and $A^* + B^*$ are causal.

To check the causality of C we have to verify that

$$\text{Re}\left[Den \equiv A^* + B^*\right] \geq 0$$

To do that, it is important to write $-i\omega$ in the form $-i\omega + \varepsilon$, so that when we will look at the real part of Den it will be the true one and not the imaginary one. We will take the limit for $\varepsilon \to 0$ to determine the sign of the real part of Den.

We have to distinguish the three cases:

i) $\alpha \geq \alpha'$

For this case

$$Den = Den1\ (Den2 + Den3) + Den4$$

with

$$Den1 = \rho'\alpha'(-i\omega+\varepsilon)\left[1+\frac{\alpha^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

$$Den2 = \left[2\beta'^2 k_x^2 + (-i\omega+\varepsilon)^2\right]^2 \tag{10}$$

$$Den3 = \frac{-4\beta'^3}{\alpha'}k_x^2(-i\omega+\varepsilon)^2\left[1+\frac{\alpha'^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}\left[1+\frac{\beta'^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

$$Den4 = \rho\alpha(-i\omega+\varepsilon)^5\left[1+\frac{\alpha'^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

After calculus to the first order with respect to ε we find for Re *Den*

$$\rho'\alpha'\varepsilon\left[1-\frac{\alpha^2k_x^2}{\omega^2}\right]^{1/2}\left[(2\beta'^2k_x^2-\omega^2)^2+\frac{4\beta'^3}{\alpha'}k_x^2\omega^2\left(1-\frac{\alpha'^2k_x^2}{\omega^2}\right)^{1/2}\left(1-\frac{\beta'^2k_x^2}{\omega^2}\right)^{1/2}\right]$$
$$+\ \rho\alpha\varepsilon\omega^4\left[1-\frac{\alpha'^2k_x^2}{\omega^2}\right]^{1/2}$$

which is obviously positive.

Therefore $C(\omega,k_x)$ is causal.

ii) $\beta' \leq \alpha < \alpha'$

In this case

If $\vartheta \leq \vartheta_c = \sin^{-1}(\alpha/\alpha')$, then *Den* is the same as in (10) and the real part is positive.

If $\vartheta > \vartheta_c$, then for this case we can write also *Den* in the form

$$Den = Den1\,(Den2 + Den3) + Den4$$

but with

$$Den1 = \rho'\alpha'(-i\omega+\varepsilon)\left[1+\frac{\alpha^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

$$Den2 = \left[2\beta'^2 k_x^2 + (-i\omega+\varepsilon)^2\right]^2 \tag{11}$$

$$Den3 = \frac{-4\,i\,\beta'^3}{\alpha'}k_x^2(-i\omega+\varepsilon)^2\left[-\frac{\alpha'^2 k_x^2}{(-i\omega+\varepsilon)^2}-1\right]^{1/2}\left[1+\frac{\beta'^2 k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

$$Den4 \;=\; i\,\rho\alpha(-i\omega+\varepsilon)^5\left[-\frac{\alpha'^2k_x^2}{(-i\omega+\varepsilon)^2}-1\right]^{1/2}$$

After calculus to the first order with respect to ε, the real part of Den is given by

$$\rho'\alpha'\left[1-\frac{\alpha^2k_x^2}{\omega^2}\right]^{1/2}\frac{4\beta'^3}{\alpha'}k_x^2\omega^3\left[\frac{\alpha'^2k_x^2}{\omega^2}-1\right]^{1/2}\left[1-\frac{\beta'^2k_x^2}{\omega^2}\right]^{1/2}$$

$$+\;\rho\alpha\omega^5\left[\frac{\alpha'^2k_x^2}{\omega^2}-1\right]^{1/2}$$

which is positive and therefore $C(\omega,k_x)$ is causal.

iii) $\alpha < \beta' < \alpha'$

If $\vartheta \le \vartheta_{c_1} \equiv \sin^{-1}(\alpha/\alpha')$, then Den is the same as in (10).

If $\vartheta_{c_1} < \vartheta \le \vartheta_{c_2} \equiv \sin^{-1}(\alpha/\beta')$, then Den is the same as in (11).

If $\vartheta > \vartheta_{c_2}$, then for this case, if we write

$$Den \;=\; Den1\;(Den2 + Den3) + Den4$$

we have

$$Den1 \;=\; \rho'\alpha'(-i\omega+\varepsilon)\left[1+\frac{\alpha^2k_x^2}{(-i\omega+\varepsilon)^2}\right]^{1/2}$$

$$Den2 \;=\; \left[2\beta'^2k_x^2+(-i\omega+\varepsilon)^2\right]^2$$

$$Den3 \;=\; \frac{4\,\beta'^3}{\alpha'}k_x^2(-i\omega+\varepsilon)^2\left[-\frac{\alpha'^2k_x^2}{(-i\omega+\varepsilon)^2}-1\right]^{1/2}\left[-\frac{\beta'^2k_x^2}{(-i\omega+\varepsilon)^2}-1\right]^{1/2}$$

$$Den4 \;=\; i\,\rho\alpha(-i\omega+\varepsilon)^5\left[-\frac{\alpha'^2k_x^2}{(-i\omega+\varepsilon)^2}-1\right]^{1/2}$$

After calculus to the first order with respect to ε, we find for Re Den

$$\rho\alpha\omega^5\left[\frac{\alpha'^2k_x^2}{\omega^2}-1\right]^{1/2}$$

which is positive. Therefore $C(\omega,k_x)$ is causal.

In the next paper we discuss the implementation of these equations.

REFERENCES

Ewing W.M., W.S. Jardetzky, F. Press, 1957; Elastic Waves in Layered Media. McGraw-Hill Book Co, New York.

Morse P.M., H. Feshbach, 1953; Methods of Theoretical Physics. McGraw-Hill Book Co, New York.

Admiration: Our polite recognition of another's resemblance to ourselves.

"Friends, Romans, Hipsters,
Let me clue you in;
I come to put down Caeser, not to groove him.
The square kicks some cats are on stay with them;
The hip bits, like, go down under; so let it lay with Caeser. The cool Brutus
Gave you the message: Caeser had big eyes;
If that's the sound, someone's copping a plea,
And, like, old Caeser really set them straight.
Here, copacetic with Brutus and the studs, -- for Brutus is a real cool cat;
So are they all, all cool cats, --
Come I to make this gig at Caeser's laying down.

It is the business of the future to be dangerous.
-- Hawkwind

NOBODY EXPECTS THE SPANISH INQUISITION

"Either I'm dead or my watch has stopped."
-- Groucho Marx' last words

You might have mail

Albert Einstein, when asked to describe radio, replied:

You see, wire telegraph is a kind of a very, very long cat. You pull his tail in New York and his head is meowing in Los Angeles. Do you understand this? And radio operates exactly the same way: you send signals here, they receive them there. The only difference is that there is no cat.

Misfortune: The kind of fortune that never misses.

Artistic ventures highlighted. Rob a museum.

Main's Law:

For every action there is an equal and opposite government program.

It is said that the lonely eagle flies to the mountain peaks while the lowly ant crawls the ground, but cannot the soul of the ant soar as high as the eagle?