Non-existence of a Gelfand-Levitan Coordinate System for the Wave Equation

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Abstract

The Gelfand-Levitan inversion procedure can be extended to multidimensional problems when the scattering potential is both local in one of the spatial variables and is frequency independent. Unfortunately, there is, in general, no coordinate transformation of the spatial variables which converts the pressure wave equation into a Schrodinger equation of the desired form. Thus, the procedure is probably not applicable to the pressure wave equation when the propagation medium is laterally and vertically heterogeneous.

Introduction

Gelfand-Levitan techniques were applied to the multidimensional Schrodinger equation early in the last decade.[1] Any partial differential equation which is obtainable from the Schrodinger equation via coordinate transformations is therefore invertible. Two of the partial differential equations that govern this coordinate transformation for the pressure field wave equation are those that govern ray tracing. A third seems to do nothing except frustrate attempts at inversion.

Our goal is to invert, using Gelfand-Levitan type methods, seismic data recorded near the surface. The inversion will be done in an as yet undefined coordinate system. In fact, it will turn out that when a coordinate system for the job exists it is determined by the output of the inversion step. What defines an appropriate coordinate system, when such a system can be found, and what to do with it are the questions that need to be answered

Multidimensional Gelfand-Levitan Inversion

The Gelfand-Levitan procedure inverts for the potential operator $V(\vec{x},\vec{x}\,')$ in the Schroedinger equation

$$\left[\nabla^2 + E - \hat{V}(\vec{x},\vec{x}\,')\right]u(\vec{x},E) = 0 \tag{1}$$

given reflection data. On the basis of Kay and Moses's paper (1955),[2] we conclude that a sufficient condition for a Schrodinger equation to be invertible is that its potential operator $\hat{V}(\vec{x},\vec{x}\,')$ be local in some direction. We identify the direction by a unit vector $\hat{\tau}$. In practice $\hat{V}$ will only be non-local through derivatives. In this case, it is convenient to think of V as a function of neighboring points $\vec{x}$ and $\vec{x}\,'$ separated by an infinitesimally small distance.

Suppose we have in our possession the plane wave response $R(x_s,\tau,x_g,\tau)$ in terms of the shot and geophone positions, x_s and x_g, and time τ. Letting $\vec{x}$ denote the vector with first component x_g (or x_s, when appropriate) and second component τ, we can plug R into the Gelfand-Levitan integral equation

$$K(\vec{x},\vec{x}\,') + R(\vec{x},\vec{x}\,') + \int_{((\vec{x}\,''-\vec{x})\cdot\hat{\tau}>0)} d\vec{x}\,''\, K(\vec{x},\vec{x}\,'')\, R(\vec{x}\,'',\vec{x}\,') = 0 \tag{2}$$

which can be solved by suitable mathemagic for the triangular (in τ) operator K. The potential can be recovered from K through the equation

$$V(\vec{x},\vec{x}\,') = -2\,\delta((\vec{x}-\vec{x}\,')\cdot\hat{\tau})\,\frac{\partial}{\partial\tau}K(\vec{x},\vec{x}) \tag{3}$$

Readers interested in the derivations of these equations are referred to Faddeev's (1976) paper for the gory details.[1]

Coordinate Transformations

Since the Schroedinger equation is invertible, one way to get a "nice" potential for the wave equation is to first convert it into Schroedinger form. In a layered medium there are several ways in which to accomplish this transformation.[3] We now consider a medium with laterally varying bulk modulus and density. The wave equation for the pressure field in this medium is

$$\left[\frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial}{\partial z} + \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial x} + \frac{\omega^2}{\kappa}\right]u(x,z,\omega) = 0 \tag{4}$$

To obtain the desired form of equation (1) from equation (4) we change from the Cartesian $(x,z) = (x^1,x^2)$ to curvilinear $(\bar{x}^1,\bar{x}^2)$. Superscripts appear instead of of subscripts because coordinates transform contravariently (see reference 4 or the appendix).

Denoting the covariant metric tensor and its determinant in the new coordinate frame by $[\bar{g}_{ij}]$ and $\bar{g}$, respectively, the wave equation changes to

$$\left[\frac{1}{|\bar{g}|^{1/2}} \sum_i \frac{\partial}{\partial \bar{x}^i} |\bar{g}|^{1/2} \frac{1}{\rho} \sum_j \bar{g}^{ij} \frac{\partial}{\partial \bar{x}^j} + \frac{\omega^2}{\kappa} \right] u(\bar{x}^1,\bar{x}^2,\omega) = 0 \tag{5}$$

where $\bar{g}^{ij}$ is the ij component of the inverse of the matrix $[\bar{g}_{lm}]$.

$$\bar{g}^{ij} = ([\bar{g}_{lm}]^{-1})^{ij} \tag{6}$$

One of the restrictions on an invertible potential is that it be frequency independent, so the κ under the ω^2 must go. Thus, a redefinition of wave variable and a similarity transformation must be done. Defining

$$\varphi(\bar{x}^1,\bar{x}^2,\omega) = \frac{1}{\kappa^{1/2}} u(\bar{x}^1,\bar{x}^2,\omega) \tag{7}$$

it is found that φ obeys the equation

$$\kappa^{1/2} \frac{1}{|\bar{g}|^{1/2}} \sum_{ik} \frac{\partial}{\partial \bar{x}^i} |\bar{g}|^{1/2} \bar{g}^{ik} \frac{1}{\rho} \frac{\partial}{\partial \bar{x}^k} \kappa^{1/2} \varphi + \omega^2 \varphi = 0 \tag{8}$$

The potential in Equation (8) is frequency independent but is not yet in the desired form since it is not yet diagonal in some direction. For this to hold, some restrictions must be placed on the metric tensor. We will try to get a potential which is diagonal in the $\bar{x}^2$ direction.

A step in the right direction is to require that the coefficient of $(\partial\varphi)/(\partial\bar{x}^1\partial\bar{x}^2)$ vanish. Since the material properties κ and ρ are both positive and since the determinant of the metric tensor dare not disappear we are forced to set

$$\bar{g}^{12} = 0 \tag{9}$$

To get the coefficient of $(\partial^2\varphi)/[(\partial\bar{x}^2)^2]$ to equal 1 we require that

$$v^2\bar{g}^{22} = 1 \tag{10}$$

The most complex of the restrictions is obtained by requiring that the coefficient of $(\partial\varphi)/(\partial\bar{x}^2)$ vanish. This is the last condition needed to get a x^2-diagonal form from the wave equation via coordinate transformations. The resulting differential equation is

$$\frac{\partial}{\partial \bar{x}^2} \left(|\bar{g}|^{1/2} \frac{1}{\rho v^2} \partial \kappa^{1/2} \right) + |\bar{g}|^{1/2} \frac{1}{\rho v^2} \frac{\partial \kappa^{1/2}}{\partial \bar{x}^2} = 0$$

which has solutions of the form

$$|\bar{g}| = F(\bar{x}^1)$$

where F is an as yet arbitrary function of $\bar{x}^1$. Since $|\bar{g}|$ is equal to $|\bar{g}^{11}-\bar{g}^{22}|$ we can get an equation relating F to $\bar{g}^{11}$.

$$|\bar{g}^{11}| = v^2 F(\bar{x}^1) \tag{11}$$

Equations (9), (10), and (11) are the required restriction on the coordinate transformation metric. Each of these relations is a partial differential equation for $\bar{x}^1,\bar{x}^2$ in terms of x^1,x^2. For instance, equation (9) is equivalent to

$$\frac{\partial \bar{x}^1}{\partial x^2}\frac{\partial \bar{x}^2}{\partial x^2} + \frac{\partial \bar{x}^1}{\partial x^1}\frac{\partial \bar{x}^2}{\partial x^1} = 0 \tag{12}$$

while equation (10) can be written as

$$\left(\frac{\partial \bar{x}^2}{\partial x^1}\right)^2 + \left(\frac{\partial \bar{x}^2}{\partial x^2}\right)^2 = \frac{1}{v^2} \tag{13}$$

These two equations are familiar and almost expected. Equation (13) is the eikonal equation and equation (12) is the requirement that the wavefronts and raypaths remain orthogonal. So far, we conclude (prematurely) that the the right coordinate system in which to do inversion is that defined by ray tracing procedures. Now, let's look at equation (11).

$$\left(\frac{\partial \bar{x}^1}{\partial x^1}\right)^2 + \left(\frac{\partial \bar{x}^1}{\partial x^2}\right)^2 = v^2\, F(\bar{x}^1(x^1,x^2)) \tag{14}$$

If we solve the eikonal equation for increments in $\bar{x}^2$ and feed the result into the ray equation to get increments in $\bar{x}^1$, we may be able to determine $F(\bar{x}^1(x^1,x^2))$ by equation (14). For this to work, $\bar{g}^{11}$ must have the same sort of behavior in the $\bar{x}^2$ direction as v^2. This is extremely unlikely, so it is safe to conclude that for media that lack symmetry the wave equation cannot be turned into a Schrodinger equation by coordinate transformations.

Appendix on Curvilinear Coordinates and Tensors

To get the wave equation in curvilinear coordinates it is necessary to consider how vectors and vector operators behave under coordinate transformations. Tensor calculus is a requisite in problems like these. Eventually expressions for the divergence and gradient operators in nearly arbitrary coordinate systems will be found.

Given unit base vectors $\hat{e}_k$ we can get to new base vectors $\vec{E}_M$ by forming constant coefficient linear combinations

$$\vec{E}_M = \sum_k \alpha_M^k \hat{e}_k \tag{A.1}$$

where the α_M^k are constants and do not depend on the coordinates x^k. We can form a matrix of the α's and take this matrix's inverse. The k,M element of the inverse matrix of the transformation matrix $[\alpha_M^k]$ is given by

$$\beta_k^M = \frac{cofactor\ of\ \alpha_M^k}{\det[\alpha_M^k]} = \frac{A_M^k}{\Delta} \tag{A.2}$$

$$\Delta = \det[\alpha_M^k]$$

Since $[\beta_k^M]$ is the inverse matrix for $[\alpha_M^k]$

$$\sum_M \alpha_M^k \beta_j^M = \delta_j^k \qquad \sum_k \alpha_M^k \beta_k^L = \delta_M^L \tag{A.3}$$

$$\hat{e}_k = \sum_L \beta_k^L \vec{E}_L \tag{A.4}$$

So far we have only considered transformations on unit vectors. In general a covariant vector $\vec{a}$ with components $(a_1, a_2, \ldots, a_r)$ transforms to a covariant vector with components $(A_1, A_2, \ldots, A_r)$ according to a rule like that followed by the basis vectors

$$A_M = \sum_{k=1}^{r} \alpha_M^k a_k \qquad a_k = \sum_{L=1}^{r} \beta_k^L A_L \tag{A.5}$$

The vectors which we commonly use are not covariant but contravariant. If $\vec{E}$ is a contravariant vector with components $(x^1, x^2, \ldots, x^r)$ then it transforms to a contravariant vector with components $(X^1, X^2, \ldots, X^r)$ according to the rule

$$\vec{E} = \sum_M X^M \vec{E}_M = \sum_M \sum_k X^M \alpha_M^k \hat{e}_k = \sum_k x^k \hat{e}_k$$

Hence, a change of basis vectors according to a covariant rule is equivalent to a change of coordinate systems according to a contravariant rule

$$X^M = \sum_k \beta_k^M x^k \qquad x^k = \sum_M \alpha_M^k X^M \tag{A.6}$$

In general, covariant indices will be written as subscripts, while contravariant indices will appear as superscripts.

Equations (A.5) and (A.6) tell us how to make transform vectors under a constant coefficient, linear coordinate transformation. The equations for an even higher rank tensor in curvilinear coordinate transformations are not much more difficult. The new coordinates X^M in can be expressed a in terms of old coordinates x^j by a function of the form

$$X^M = F^M(x^1,...,x^r)$$

Then an increment of X_M is given by

$$\delta X^M = \sum_k \frac{\partial F^M}{\partial x^k} \delta x^k = \sum_k \beta_k^M \delta x^k \tag{A.7}$$

which serves to define the coefficients $\alpha_M^k(x^1,x^{2,...,x^r})$ and $\beta_k^M(x^1,x^2,...,x^r)$.

$$\alpha_M^k = \frac{\partial x^k}{\partial X^M} \qquad \beta_k^M = \frac{\partial X^M}{\partial x^k} \tag{A.8}$$

Tensors are those quantities that transform according to

$$t_{i_1 \cdots i_n}^{k_1 \cdots k_m} = \sum_{J_1 \cdots L_m} \alpha_{L_1}^{k_1} \cdots \alpha_{L_m}^{k_m} \beta_{i_1}^{J_1} \cdots \beta_{i_n}^{J_n} T_{J_1 \cdots J_n}^{L_1 \cdots L_m}$$

$$T_{J_1 \cdots J_n}^{L_1 \cdots L_m} = \sum_{i_1 \cdots k_m} \beta_{k_1}^{L_1} \cdots \beta_{k_m}^{L_m} \alpha_{J_1}^{i_1} \cdots \alpha_{J_n}^{i_n} t_{i_1 \cdots i_n}^{k_1 \cdots k_m} \tag{A.9}$$

and similarly for the inverse transformation relation.

One of the most important tensors are the metric tensors, g_{ij} and $\bar{g}_{ij}$. If ds^2 is an incremental distance, the square of the length of an infinitesimally small vector with components dx^i at a point $P(x^1,x^2)$ defines covariant tensors g_{ik} and $\bar{g}_{ik}$ in the Cartesian and curvilinear coordinate systems, respectively, where

$$ds^2 = \sum_{ik} g_{ik} \, dx^i dx^k = \sum_{ik} \delta_{ik} \, dx^i dx^k$$

$$= \sum_{lm} \bar{g}_{lm} \, d\bar{x}^l d\bar{x}^m = \sum_{ij} \sum_{lm} \bar{g}_{lm} \beta_i^l \beta_j^m dx^i dx^j \tag{A.10}$$

By equating coefficients we find that

$$g_{ij} = \sum_{lm} \bar{g}_{lm} \beta_i^l \beta_j^m \qquad \bar{g}_{ij} = \sum_{lm} g_{lm} \alpha_i^l \alpha_j^m \tag{A.11}$$

which is just the condition that g_{ij} represent the components of a covariant tensor. In Cartesian coordinates, x^k, the metric tensor takes the form $g_{ij} = \delta_{ij}$. Using the fact that $[\alpha]$ is the inverse of $[\beta]$

$$\bar{g}_{ij} = \sum_{lm} g_{lm} \alpha_i^l \alpha_j^m = \sum_l \alpha_i^l \alpha_j^m = \sum_l \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \tag{A.12}$$

$$\bar{g}^{ij} = \sum_l \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial \bar{x}^j}{\partial x^l}$$

The metric tensors can be used to define the covariant components of a vector [u] given its contravariant components according to

$$u_i = \sum_k g_{ik} u^k$$

$$u^k = \sum_j g^{jk} u_j$$

where g^{jk} is the jk component of the inverse of the matrix $[g_{ij}]$. We can also use the g_{ij} to define the inner product of two vectors [u] and [v] in terms of the contravariant components of these vectors

$$([u],[v]) = \sum_i u_i v^i = \sum_{ik} g_{ik} u^i v^k = \sum_k u^k v_k = \sum_{ik} g^{ik} u_i v_k$$

It is easily proved that these expressions define a scalar which has a numerical value independent of the coordinate system in which the mathematics is expressed.

The tensor coefficients $\bar{g}_{ij}$ can be placed in a matrix $[\bar{g}]$. Let $\bar{g}$ denote the determinant of this matrix. The divergence operator in curvilinear coordinates can be defined by

$$\nabla \cdot [\bar{a}^i] = \frac{1}{|\bar{g}|^{1/2}} \sum_i \frac{\partial}{\partial \bar{x}^i} |\bar{g}|^{1/2} \bar{a}^i \tag{A.13}$$

Similarly the gradient of a scalar V is a covariant vector with components $\bar{a}_i$, where

$$\bar{a}_i = \frac{\partial V}{\partial \bar{x}^i} \tag{A.14}$$

The vector $[\bar{a}]$ with covariant components $\bar{a}_i$ has contravariant components $\bar{a}^i$. The relation connecting these two sets of components is

$$\bar{a}^i = \sum_j \bar{g}^{ij} \bar{a}_i \tag{A.15}$$

$$\bar{g}^{ij} = ([\bar{g}_{lm}]^{-1})^{ij} \tag{A.16}$$

Divergence and grad are two operators needed to express the wave equation in curvilinear coordinates. Collecting appropriate expressions

$$\nabla \cdot \frac{1}{\rho} \nabla V = \frac{1}{|\bar{g}|^{1/2}} \sum_i \frac{\partial}{\partial \bar{x}^i} |\bar{g}|^{1/2} \frac{1}{\rho} \sum_j \bar{g}^{ij} \frac{\partial V}{\partial \bar{x}^j} \tag{A.17}$$

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