Convergence of the Continued Fraction for the Square Root Function

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Abstract

The square root function can be approximated with a continued fraction for use in wave equation migration algorithms. This continued fraction can be generated by a recursion. If no dissipation other than dip filtering is employed then the recursion converges in the propagating region and on the boundary between the propagating and evanescent regimes. The recursion diverges in the evanescent zone and at zero temporal frequency. Depending on the starting point convergence may also occur along a pair of radial lines in the fk-plane.

The evanescent zone disappears when the wave operator is causal and has a spectrum with a strictly-positive real part. The continued fraction in this case will converge essentially everywhere.

Introduction

The one-way wave-equation governing the modeling of downgoing waves in a laterally homogeneous medium has a single square root operating on its derivative operators. If this equation is split into diffracting and shifting parts, then the diffracting part takes the form

$$D_z\, P \;=\; -\left[-\Lambda D_t + (\Lambda^2 D_t^2 + |D_x|^2)^{1/2}\right] P \tag{1}$$

where D_z and D_t are causal differentiators. The derivative with respect to time, D_t, may be implemented in the frequency rather than the time domain. The operator $|D_x|^2$ is a positive semi-definite operator which is a discrete implementation of the negative of a second differentiator with respect to the lateral spatial variable x. The reciprocal of the acoustic velocity is represented by Λ.

Usually equation (1) is discretized with respect to z with the Crank-Nicholson approximation. The problem which now occurs is that some representation for $-\Lambda D_t + (\Lambda^2 D_t^2 + |D_x|^2)^{1/2}$ needs to be found. This operator is a special case of the class of operators of the form $-L1 + (L_1^2 + L_2)^{1/2}$, where L_1 is an operator with a non-negative real part and L_2 is non-negative definite. The square root is defined so that it is a non-negative real operator.

Computer Experiments

The continued fraction representation for the square root function can be generated by the recursion. Setting $L_1 = i$ then

$$S_{j+1} = \frac{L_2}{2i + S_j}$$

where S_j is jth member of the recursion.

Propagating waves correspond to values of L_2 less than 1. Consider, for example, $L_2 = 0.95$. Starting with $S_0 = 1 - i$, we find that $S_1 = 0.4750 - 0.4750i$, $S_2 = 0.1769 - 0.5679i$, and so on. After 25 iterations, S_j has more or less converged to $0.0000 - 0.7764i$.

The boundary between the propagating and evanescent zones is at $L_2 = 1.00$. This time $S_0 = 1 + i$, $S_1 = 0.50 - 0.50i$, $S_2 = 0.20 - 0.60i$, ... Convergence is slower in this case than for the case $L_2 < 1$. For instance, $S_{40} = 0.006 - 0.9750i$, $S_{41} = 0.006 - 0.9756i$, $S_{42} = 0.006 - 0.9762i$, and so on.

In the evanescent region the continued fraction often diverges. With $L_2 = 1.05$ and $S_0 = 1 - i$ the approximants oscillate about without settling on any one real number. As evidence of this, the first 14 approximants are displayed.

$S_1 = 0.5250 - 0.5250i$	$S_8 = 0.0517 - 1.0406i$
$S_2 = 0.2249 - 0.6318i$	$S_9 = 0.0588 - 1.0912i$
$S_3 = 0.1228 - 0.7473i$	$S_{10} = 0.0745 - 1.1506i$
$S_4 = 0.0814 - 0.8302i$	$S_{11} = 0.1076 - 1.2267i$
$S_5 = 0.0622 - 0.8932i$	$S_{12} = 0.1854 - 1.3321i$
$S_6 = 0.0531 - 0.9457i$	$S_{13} = 0.4051 - 1.4596i$
$S_7 = 0.0500 - 0.9934i$	$S_{14} = 0.9325 - 1.2441i$

At L_2 = 2.00, still within the evanescent region, a different sort of analytic behavior is observed. With $S_0 = 1 - i$ the sequence of approximants converges in a single step to $1 - i$. There are good reasons for believing that divergence is the rule in the evanescent region and that one-step conversion is exceptional. Those reasons are the proofs and discussions which comprise the remainder of this paper.

Convergence Properties

Truncating a continued fraction after successively larger partial denominators yields infinite sequence of approximants, numerators, and denominators. The convergence of many continued fractions is very difficult to prove. This is not true of the fractional representation for the operator

$$-L_1 + (L_1^2 + L_2)^{1/2} \qquad (2)$$

because most of the necessary work is already to be found in Wall's book on continued fractions. The continued fraction representation for operator (2) is

$$\cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cdots}}} \qquad (3)$$

Truncation of this continued fraction generates a sequence of approximants. For instance, the first three approximants are

$$\frac{L_2}{2L_1} \; , \quad \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1}} \; , \quad \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1}}} \qquad (4)$$

The convergence of sequence in (4) is examined in Appendix A.

We are more interested in a sequence which is derivable from the sequence in (4) by altering the "foot" of each approximant. This is the preferred form for a "diffraction" operator when dip filtering and some sort of phase correction is included. The first three terms in the altered sequence are given by

$$\frac{L_2}{2L_1c} \; , \quad \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1c}} \; , \quad \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1c}}} \qquad (5)$$

where c is a complex constant. The limiting behavior, assuming that none of the denominators vanishes, of sequence (5) is examined in Appendix B. For this sequence:

	Case	Limit
1.	$\lvert L_1 + (L_1^2 + L_2)^{1/2}\rvert < \lvert L_1 - (L_1^2 + L_2)^{1/2}\rvert$	$-L_1 - (L_1^2 + L_2)^{1/2}$
2.	$\lvert L_1 + (L_1^2 + L_2)^{1/2}\rvert > \lvert L_1 - (L_1^2 + L_2)^{1/2}\rvert$	$-L_1 + (L_1^2 + L_2)^{1/2}$
3.	$L_1 + (L_1^2 + L_2)^{1/2} = L_1 - (L_1^2 + L_2)^{1/2}$ *and* $L_2 > 0$	$-L_1$
4.	$2L_1c = \left[L_1 + (L_1^2 + L_2)^{1/2}\right]$	$-L_1 + (L_1^2 + L_2)^{1/2}$
5.	$2L_1c = \left[L_1 - (L_1^2 + L_2)^{1/2}\right]$	$-L_1 - (L_1^2 + L_2)^{1/2}$

Divergence will certainly occur in any case other than these five. For instance, if $L_1 + (L_1^2 + L_2)^{1/2}$ and $L_1 - (L_1^2 + L_2)^{1/2}$ are not equal but have the same modulus, then the sequence of terms will diverge. Finally, it should be noted that equation (4) can be obtained from equation (5) by setting $c = 1$ so that their convergence properties need not be considered separately. When $c = 1$, cases 3,4, and 5 are all equivalent to one another.

The Visco-acoustic Assumption

In practice we are only interested in operators of the form of equation (2) which are also causal non-negative real (CPR when non-negative is changed to the stronger positive)

$$
\begin{array}{ll}
\mathrm{Re}L_1 \ge 0 & \mathrm{Im}L_1 \ge 0 \\
\mathrm{Re}L_1 c \ge 0 & c \neq 0 \\
\mathrm{Re}L_2 \ge 0 & \mathrm{Im}L_2 = 0
\end{array}
\tag{6}
$$

The inequalities and equalities in (6) are a set of working hypotheses which will be referred to as the "visco-acoustic" assumption. This is because they are necessary properties for modeling many visco-acoustic wave equations. The assumption that $\mathrm{Im}L_1 \ge 0$ follows from the representation of D_t as $i\omega$ and reflects the fact that our wavefield is real (in the time domain) and only non-negative frequencies need be considered since the negative frequencies can be obtained by Hermitian symmetry. The restriction that $c \neq 0$ is necessary for the first term in (5) to make sense.

In the following, we will retain the convention that the square root of a quantity with a positive real part have a positive real part itself. This implies a branch cut along the negative real axis. The square root is defined so that it purely and positive imaginary along this cut. This choice reflects our desire to set

$$L_1 = (L_1^2)^{1/2}$$

For simplicity, we restrict L_1 and $L_1 c$, temporarily, so that they have strictly positive real parts and examine the different cases for convergence and their limits. Case 1, referring to the convergence table of the previous section, is equivalent to the inequality $\mathrm{Re}[L_1^*(L_1^2 + L_2)^{1/2}] < 0$. Since L_2 is real and non-negative, the imaginary part of the square root and the imaginary part of L_1^* have opposite signs. The real part of the square root and the real part of L_1^* are both positive. Consequently, the real part of their product $L_1^*(L_1^2 + L_2)^{1/2}$ is non-negative. In other words, case 1 never occurs. Similarly, it follows from the fact that $\mathrm{Re}(L_1^2 + L_2)^{1/2}>0$ that cases 3 and 5 never occur.

Next consider a situation in which $\mathrm{Re}L_1 > 0$ and $\mathrm{Re}(L_1 c) = 0$. Cases 1 and 3 never occur for the same reason they didn't occur when both of these operators were strictly positive real. It turns out that cases 4 and 5 are different. For case 4,

$$\mathrm{Re}[L_1(2c - 1)] = -\mathrm{Re}L_1 \leq 0$$

while $\mathrm{Re}(L_1^2 + L_2)^{1/2} > 0$. It follows that case 4 never occurs, either. Finally, we examine case 5. When it holds (if ever)

$$-\mathrm{Re}L_1 = -\mathrm{Re}(L_1^2 + L_2)^{1/2}$$

implying that $L_2 = 0$. When $L_2 = 0$ under case 5, c must equal 0, which can't be.

Now for $\mathrm{Re}L_1 = 0$ and $\mathrm{Re}(L_1 c) > 0$. Once again, case 1 can never occur. Cases 2, 3, and 4 can. Equation 5 is harder but not very hard. For $-(L_1^2 + L_2)^{1/2}$ is non-negative real and

$$\mathrm{Re}[L_1(2c - 1)] = 2\mathrm{Re}[L_1 c] > 0$$

when $\mathrm{Re}L_1 = 0$. Consequently, case 5 never occurs.

The situation that is the hardest to analyze is the one which lacks any damping, i.e.

$$\mathrm{Re}L_1 = 0 \qquad \mathrm{Re}[L_1 c] = 0 \tag{7}$$

Though it is a boring conclusion by now, case 1 just plain never happens. Cases 2 and 3 do. Cases 4 and 5, as usual, need to be looked at more closely. A first step is to note that for equations (7) to hold c must be pure real. A second is to combine cases 4 and 5 in the single statement

$$(2c - 1)L_1 = \pm(L_1^2 + L_2)^{1/2} \textit{ implies the limit } -L_1 \pm (L_1^2 + L_2)^{1/2}$$

Since L_1 is pure imaginary and c is pure real, $(L_1^2 + L_2)^{1/2}$ must be pure imaginary. By the branch cut convention, $\text{Im}(L_1^2 + L_2)^{1/2}$ has the same positive sign as $\text{Im}L_1$. This means that the size of c determines the convergence of the sequence. When $c \geq \frac{1}{2}$, the sequence converges to $-L_1 + (L_1^2 + L_2)^{1/2}$. However, when $c < \frac{1}{2}$, the sequence (5) converges to the wrong limit for migration purposes, namely, $-L_1 - (L_1^2 + L_2)^{1/2}$.

These results can be summarized in another table.

	$\text{Re}L_1>0$ $\text{Re}L_1c>0$	$\text{Re}L_1>0$ $\text{Re}L_1c=0$	$\text{Re}L_1=0$ $\text{Re}L_1c>0$	$\text{Re}L_1=0$ $\text{Re}L_1c=0$
$\text{Re}L_1^*(L_1^2 + L_2)^{1/2}<0$	never	never	never	never
$\text{Re}L_1^*(L_1^2 + L_2)^{1/2}>0$	$-L_1+\surd$	$-L_1+\surd$	$-L_1+\surd$	$-L_1+\surd$
$L_1^2+L_2=0,\ L_1\neq 0$	never	never	- L1	- L1
$L_1(2c-1)=(L_1^2 + L_2)^{1/2}$	$-L_1+\surd$	never	$-L_1+\surd$	$-L_1+\surd$
$L_1(2c-1)=-(L_1^2 + L_2)^{1/2}$	never	never	never	$-L_1-\surd$

A table entry of "never" in the ith row and jth column means that case number i cannot occur under the conditions listed at the top of the jth column. Otherwise, the limit is indicated. Divergence occurs for all cases not listed at the left-hand side of the table.

We can now conclude that if one of L_1, L_1c has a strictly positive real part and Muir's recurrence converges then the limiting value is the correct one for wave-equation work. When both L_1 and L_1c are pure imaginary then we must have $c \geq 1/2$ for correct convergence. If $c < 1/2$ then there will be convergence to an incorrect square root along a line in the fk-plane. Interestingly enough, this anomolous convergence takes place in the interior of the propagating region of the one-way wave equation.

The Visco-acoustic Assumption and Non-Vanishing Denominators

The continued fraction convergence criteria which have been offered for (4) or (5) assume that all of the denominators of the successive terms are non-zero. This will be true when the "visco-acoustic" conditions are met and either L_1 or L_1c is strictly positive. It will also be true if both of these quantities are pure imaginary (and non-zero) as long as either $c \geq 1/2$ or c is irrational.

The proofs in the appendices work from the top down. This time, we work from the bottom up. Let S_p denote the pth term, where

$$S_0 = \frac{L_2}{2L_1c}$$

$$S_p = \frac{L_2}{2L_1 + S_{p-1}} = \frac{N_p}{D_p} \tag{8}$$

When $L_1 c$ is non-zero and non-negative real, as is the case under the conditions of the visco-acoustic assumption, S_0 is well-defined and non-negative real. The next term,

$$S_1 = \frac{L_2}{2L_1 + \frac{L_2}{2L_1 c}}$$

is well-defined and (strictly) positive real when $L_2 > 0$. It is equal to zero when $L_2 = 0$. For an inductive proof, assume that either $S_1, S_2, \cdots, S_{p-1}$ are all well-defined, all strictly positive, and that $L_2 > 0$ or that $S_1, S_2, \cdots, S_{p-1}$ are all zero and $L_2 = 0$. In the first case,

$$S_p = \frac{L_2}{2L_1 + S_{p-1}}$$

has a positive real denominator and a real and positive numerator. Hence, S_p is well-defined and positive real. In the second case, the denominator still has a positive real part, but the numerator is zero. Hence, S_p is equal to zero. By mathematical induction, S_p is well-defined for all p.

If both Re$L_1 = 0$ and Re$[L_1 c] = 0$ one of the S_p's might well be ill-defined, even in a case listed as convergent in the table of the previous section. For example, if $c = 1/4$, $L_1 = i$, and $L_2 = 1$, then $S_0 = -2i$, and

$$S_1 = \frac{1}{2i - S_0}$$

has infinite modulus. Continuing by setting the next term, S_2, to 0, then $S_3 = -i/2$, $S_4 = -2i/3$, $S_5 = -3i/4$, ... , $S_p = -i(p-2)/(p-1)$. This sequence converges, as predicted, to $-i$.

We now proceed to the proof that divergence by exploding term will not be a bother for the case ReL_1 = Re$[L_1 c] = 0$ if $c \neq 1/2(1 - 1/p)$ for some positive integer p. From Appendix B, the denominator of the pth term in the recurrence under examination will vanish when

$$cx^{p+1} + (c-1)x^p - (c-1)x - c = 0 \tag{9}$$

$$x = \frac{L_1 - (L_1^2 + L_2)^{1/2}}{L_1 + (L_1^2 + L_2)^{1/2}}$$

$$u = L_1 + (L_1^2 + L_2)^{1/2} \qquad v = L_1 - (L_1^2 + L_2)^{1/2}$$

Since $x \neq 0$ when $L_1 \neq 0$, we can show that $1/x$ is a root of polynomial (9),too. This means that all roots of equation (9) lie on the unit circle. In the language of Appendices A and B, this is equivalent to the statement that $|u| = |v|$. Divergence by oscillation occurs whenever $|u| = |v|$ and $u \neq v$, so the only case that needs to be considered is that in which $u = v = L_1$. This is the case on the boundary between the evanescent and propagating regions of the fk-plane. Replacing u with L_1 and u/v with 1 in equation (B.2), the pth term of the recurrence is found to be

$$-L_1 + L_1 \frac{(2c - 1)}{(2c - 1)p + 1}$$

This expression will be well defined for all natural numbers p as long as $c \neq 1/2(1 - 1/m)$ for some positive integer m.

Special Case: $L_1 = i\omega\Lambda$, $L_2 = |D_x|^2$

Set L_1 equal to $i\omega\Lambda$, where ω is the temporal wavenumber, Λ is the medium's acoustic slowness. Set $L_2 = |D_x|^2$, an approximation of the operator which takes the negative of a second derivative. This is the usual diffraction operator. The resulting recursion is

$$S_1 = \frac{|D_x|^2}{2i\omega\Lambda c} \qquad S_{j+1} = \frac{|D_x|^2}{2i\omega\Lambda + S_j} \quad (j > 1)$$

If $|D_x|^2 < \omega^2\Lambda^2$ then the wave is in the propagation region. Case 2 in which Re$L_1^*(L_1^2 + L_2)^{1/2}$>0 holds, so S_∞ exists and is equal to $-i\omega\Lambda + (-\omega^2\Lambda^2 + |D_x|^2)^{1/2}$.

If $|D_x|^2 = \omega^2\Lambda^2$ then the wave is on the boundary between the propagation and evanescent zones. Case 3 in which $L_1^2 + L_2 = 0$ holds, so S_∞ exists and is equal to $-i\omega\Lambda$.

If $|D_x|^2 < \omega^2\Lambda^2$ then the wave is in the evanescent region of the fk-plane. This time, the sequence of S_j's does not (in general) converge. Exceptions may occur for certain choices of c.

If the right choice of c is made, there will be additional lines at which the sequence of S_j's converges. These occur whenever

$$2c\,\omega\Lambda = \pm|k_x|$$

If $c \geq 1/2$ then this convergence takes place within the evanescent zone. The limit is the correct, positive real diffracting operator.

Appendix A - Convergence of Approximants

The continued fraction of interest is a representation of the operator $-L_1 + (L_1^2 + L_2)^{1/2}$. To investigate the convergence of fraction (3) to this operator, consider the two roots u and v of the quadratic $x^2 - 2L_1 x - R = 0$.

$$L_2 = -uv \qquad 2L_1 = u + v$$

$$u = L_1 + (L_1^2 + L_2)^{1/2} \qquad v = L_1 - (L_1^2 + L_2)^{1/2}$$

Let Q take the place of the continued fraction (3) so that

$$Q = \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cfrac{L_2}{2L_1 + \cdots}}} \tag{A.1}$$

Substituting for L_2 and L_1 and dividing by u changes this expression to

$$\frac{1}{u}Q = -\cfrac{v}{u + v - \cfrac{uv}{u + v - \cfrac{uv}{u + v - \cdots}}} \tag{A.2}$$

Dividing above and below the first fraction bar by u transforms this last expression into the equation

$$\frac{1}{u}Q = -\cfrac{\frac{v}{u}}{1 + \frac{v}{u} - \cfrac{v}{u + v - \cfrac{uv}{u + v - \cdots}}} \tag{A.3}$$

Equations (A.2) and (A.3) have the same sequence of approximants. Similarly, we can divide above and below the second fraction bar by u without changing the sequence of approximants. In fact, this division can be done above and below all of the fraction bars without changing the sequence of approximants. The result of this equivalence transformation is

$$\frac{1}{u}Q = -\cfrac{\frac{v}{u}}{1 + \frac{v}{u} - \cfrac{\frac{v}{u}}{1 + \frac{v}{u} - \cfrac{\frac{v}{u}}{1 + \frac{v}{u} - \cdots}}} \tag{A.4}$$

In dividing by u we have lost some generality, so we will consider the case $u = 0$ later.

The convergence of the approximants of equations (A.1) and (A.4) is studied in terms of the convergence of still another continued fraction. Let $x = \frac{v}{u}$ and consider

$$\frac{1}{1 + \frac{1}{u} Q} = \cfrac{1}{1 - \cfrac{x}{1 + x - \cfrac{x}{1 + x - \cdots}}} \tag{A.5}$$

The numerator and denominator of the approximants of a continued fraction satisfy a pair of recursions. If we write the pth approximant of (A.5) by a_p / b_p, the recursions for a_p and b_p are

$$a_{-1} = 1 \qquad a_0 = 0 \qquad a_1 = 1$$

$$a_{p+1} = (1+x)\, a_p - x\, a_{p-1} \qquad (p \geq 1)$$

$$b_{-1} = 0 \qquad b_0 = 1 \qquad b_1 = 1$$

$$b_{p+1} = (1+x)\, b_p - x\, b_{p-1} \qquad (p \geq 1) \tag{A.6}$$

Equations (A.6) and mathematical induction imply that

$$a_p = \sum_{j=0}^{p-1} x^j \qquad (p \geq 1)$$

$$b_p = 1 \qquad (p \geq 1) \tag{A.7}$$

The simple form of the b_p's is the reason for introducing fraction (A.5).

If the pth approximant of fraction (A.2) is given by Q_p then the pth approximant of fraction (A.5) is

$$\frac{1}{1 + \frac{1}{u} Q_{p-1}} = \frac{a_p}{b_p}$$

so that

$$Q_p = \frac{u}{\sum_{j=0}^{p} \left(\frac{v}{u} \right)^j} - u \tag{A.8}$$

As $p \to \infty$, Q_p converges to $-v$ when $|v| < |u|$, converges to $-u$ when $|v| > |u|$, converges to $-u$ when $u = v$, and diverges otherwise.

We have assumed that $u \neq 0$. If, instead, $u = 0$ then $L_2 = 0$. As a consequence every approximant of the fraction in (A.1) vanishes. When $u = 0$, $|v| > |u|$ or

$v = u = 0$ so the convergence rules of the previous paragraph still hold.

Appendix B - Altered Feet

The terms of the sequence (5) can be obtained from the approximants of fraction (A.5). If Q_n is the pth approximant of the continued fraction (A.1) and if a_p and b_p are the pth numerator and denominator, respectively, of the continued fraction (A.5) then

$$\frac{1}{1+\frac{1}{u}Q_{p-1}} = \frac{a_p}{b_p} \tag{B.1}$$

The pth terms of the sequence (5) can be had by altering the pth numerator and denominator of the continued fraction in (A.5). In terms of the usual recurrence formulae for numerators and denominators of approximants (see Appendix C)

$$a_{p+1}' = (1+\frac{v}{u})\, c\; a_p - \frac{v}{u} a_{p-1} \qquad (p \geq 1)$$

$$b_{p+1}' = (1+\frac{v}{u})\, c\; b_p - \frac{v}{u} a_{p-1} \qquad (p \geq 1)$$

where the a_p' and b_p' are the pth numerator and denominator, respectively, of the sequence of equation (5). Substituting (A.7) into these last two expressions shows that

$$a_{p+1}' = (1+\frac{v}{u})\, c \sum_{j=0}^{p-1} \left[\frac{v}{u}\right]^j - \frac{v}{u} \sum_{j=0}^{p-2} \left[\frac{v}{u}\right]^j$$

$$b_{p+1}' = (1+\frac{v}{u})\, c - \frac{v}{u}$$

A little algebra is needed to combine the last few expressions and demonstrate convergence or divergence. If Q_p' denotes the pth term of the sequence under investigation then

$$Q_p' = -u + u \frac{c + (c-1)\frac{v}{u}}{c(1+\frac{v}{u})\sum_{j=0}^{p-1}\left[\frac{v}{u}\right]^j - \frac{v}{u}\sum_{j=0}^{p-2}\left[\frac{v}{u}\right]^j} \tag{B.2}$$

For finite p, this is equivalent to

$$Q_p' = -u + u \frac{\left[c + (c-1)\frac{v}{u}\right]\left[1 - \frac{v}{u}\right]}{\left[c + (c-1)\frac{v}{u}\right] - \left[\frac{v}{u}\right]^p \left[(c-1) + c\left[\frac{v}{u}\right]\right]} \tag{B.3}$$

As long as the denominators do not vanish, Q_p converges to $-v$ when $|v| < |u|$, converges to $-u$ when $|u| < |v|$. Convergence when $u = v$ is easier to see in equation (B.2). When $u/v = 1$, $Q_p = -u + (2c-1)u/[(2c-1)p + 1]$, so convergence occurs and the limiting value is $-u = -v$.

Special choices for c will also make the right side of equation (B.3) converge. For instance, if $c = v/(u+v)$ then the numerator and the first term in the denominator will both vanish. Q_p' will then equal $-u$ as long as the other term in the denominator does not vanish (under the constraint that $c = v/(u+v)$ this will happen only when $u = v$, a case which has already been covered). Alternatively, we might set $c = u/(u+v)$ and get the second term in the denominator to vanish. In this case, Q_p' will equal $-v$.

Appendix C - Continued Fraction Recurrences

Truncation of a continued fraction at successively lower "feet" yields a sequence of approximants, each of which has a numerator A_n and a denominator B_n. Following Wall, any continued fraction can be written as an infinite sequence of transformations $t_0, t_1, t_2, \cdots$ defined by

$$t_0(w) = b_0 + w \qquad t_p = \frac{a_p}{b_p + w}$$

where the a_p's and b_p's are complex numbers and w is a complex variable. The continued fraction can then be written as

$$\lim_{n \to \infty} t_0 t_1 \cdots t_n(0) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

The nth approximant of this fraction can be written

$$t_0 t_1 \cdots t_n(0) = \frac{A_n}{B_n}$$

The numerators and denominators satisfy a recurrence equation pair heavily used in this paper:

$$A_{-1} = 1 \qquad B_{-1} = 0 \qquad A_0 = b_0 \qquad B_0 = 1$$

$$A_{p+1} = b_{p+1} A_p + a_{p+1} A_{p-1} \qquad (p = 0,1,2, \cdots)$$

$$B_{p+1} = b_{p+1} B_p + a_{p+1} B_{p-1} \qquad (p = 0,1,2, \cdots)$$

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Good day for a change of scene. Repaper the bedroom wall.

A gleekzorp without a tornpee is like a quop without a fertsneet (sort of).

Tonights the night: Sleep in a eucalyptus trees.

"How doth the little crocodile
   Improve his shining tail,
And pour the waters of the Nile
   On every golden scale!

"How cheerfully he seems to grin,
   How neatly spreads his claws,
And welcomes little fishes in,
   With gently smiling jaws!"

What use is magic if it can't save a unicorn?
-- Peter S. Beagle

A diplomat is someone who can tell you to go to hell in such a way that you will look forward to the trip.

Things will be bright in P.M. A cop will shine a light in your face.

Dimensions will always be expressed in the least usable term. Velocity, for example, will be expressed in furlongs per fortnight.

What the hell, go ahead and put all your eggs in one basket.

Law of Procrastination:
Procrastination avoids boredom; one never has the feeling that there is nothing important to do.

I'd give my right arm to be ambidextrous.

Churchill's Commentary on Man:
Man will occasionally stumble over the truth, but most of the time he will pick himself up and continue on.

Never be led astray onto the path of virtue.