Analysis of Focusing in Retarded Snell Coordinates

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Abstract

In previous reports (see González and Claerbout; p181, SEP16), the retarded Snell coordinate frame wad developed and then applied to the problem of velocity analysis. A new coordinate frame was established, for which energy was focused to the tops of skewed hyperboloids. The elegant advantage of this coordinate system is that, after downward continuation, and application of the appropriate imaging condition, the velocity is easily determined in the new offset-time coordinate system. The retarded Snell coordinate system inspired the authors to investigate the quality of focusing using a ray-tracing approach, and then comparing the result with wave equation techniques.

Introduction

Given a gather in offset-time, (h,t) space, it is clear that in the process of downward continuation to zero time and zero offset, energy must move along the group velocity lines. For the case of constant velocity, these lines will be straight. These straight lines can of course be determined by application of the formal ray-tracing equation (Cerveny *et al* 1977, Yedlin 1978), such equations are valid in the case when there is anisotropy, and the direction of the phase velocity vector is not the same as the group velocity vector. Another advantage of direct application of the ray-tracing equations is that they can be transformed into any coordinate system. In what follows, the ray equation will be derived, and then applied to the problem of focusing. The ray tracing results will be compared with those obtained using the wave equation. For the ray tracing, the amplitudes are not calculated, but these can be qualitatively determined by looking at the instantaneous ray density.

Theory

The general ray tracing equations, which will be derived for a 2-dimensional constant velocity medium, can be best obtained by using the dispersion relation for the particular wave equation under consideration. Let us consider a general dispersion relation of the form:

$$F(p,q) = 0$$

where $p = k_x/\omega$ and $q = k_z/\omega$, and $k_x =$ horizontal wavenumber, $k_z =$ vertical wavenumber.

The frequency is scaled out of the dispersion relation, as we do not want our ray trace equations to have any explicit frequency dependence. Also it is convenient to work in slowness coordinates, which are the duals of the displacement coordinates.

Now consider a ray whose coordinates (x,z), are parameterized by the time t along the ray. Then the components of the tangent vector to the ray are proportional to the group velocity. The group velocity in turn is proportional to the normal derivatives of the dispersion relation, $F(p,q) = 0$. Therefore,

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \lambda \begin{bmatrix} \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial q} \end{bmatrix} \tag{1}$$

For most cases, it is easy to calculate $\frac{\partial F}{\partial p}$ and $\frac{\partial F}{\partial q}$. What remains is to determine λ.

The parameter λ can be evaluated by looking at the definition of a wavefront. A wavefront is defined to be the locus of points such that at a particular $t = \tau(x,z)$ defines the location of the wavefront. Differentiating the above relation as a function of t, and using the chain-rule, we get

$$1 = \frac{\partial \tau}{\partial x}\frac{dx}{dt} + \frac{\partial \tau}{\partial z}\frac{dz}{dt} \tag{2}$$

The quantities $\frac{\partial \tau}{\partial x}$ and $\frac{\partial \tau}{\partial z}$ are the wavefront normals p and q. Substitution of equation (1) into (2) results in:

$$\lambda = \left[p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \right]^{-1} \tag{3}$$

The formalism of equations (1) (2) (3) is convenient, in that is can be applied to any dispersion relation. There is no need to get involved in complicated geometric projections, as in the case if the dispersion relation departs from a circle. At this point a series of examples will be given below for different dispersion relations.

Examples

In the next examples, the following basic geometry will be considered (figure 1)

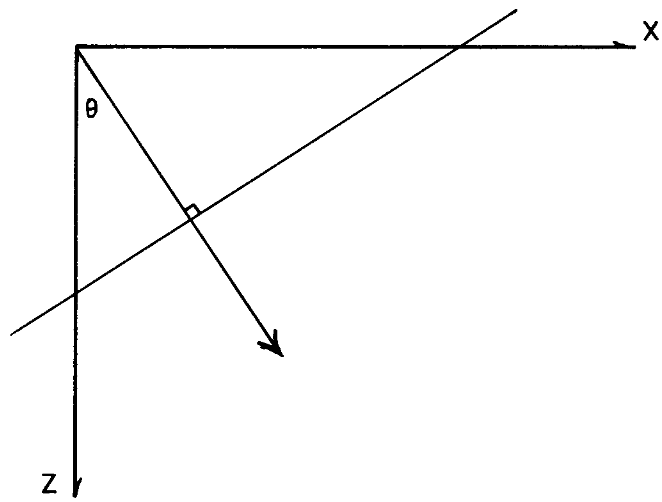


FIG. 1.

A. Acoustic equation

The dispersion relation for this case is given by

$$F(p,q) = p^2 + q^2 - \frac{1}{v^2} = 0$$

differentiating F with respect to p and q we get

$$\begin{bmatrix} \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial q} \end{bmatrix} = \begin{bmatrix} 2p \\ 2q \end{bmatrix}$$

and

$$\lambda = (2p^2 + 2q^2)^{-1} = \frac{1}{v^2}$$

using (1) we get the ray equations

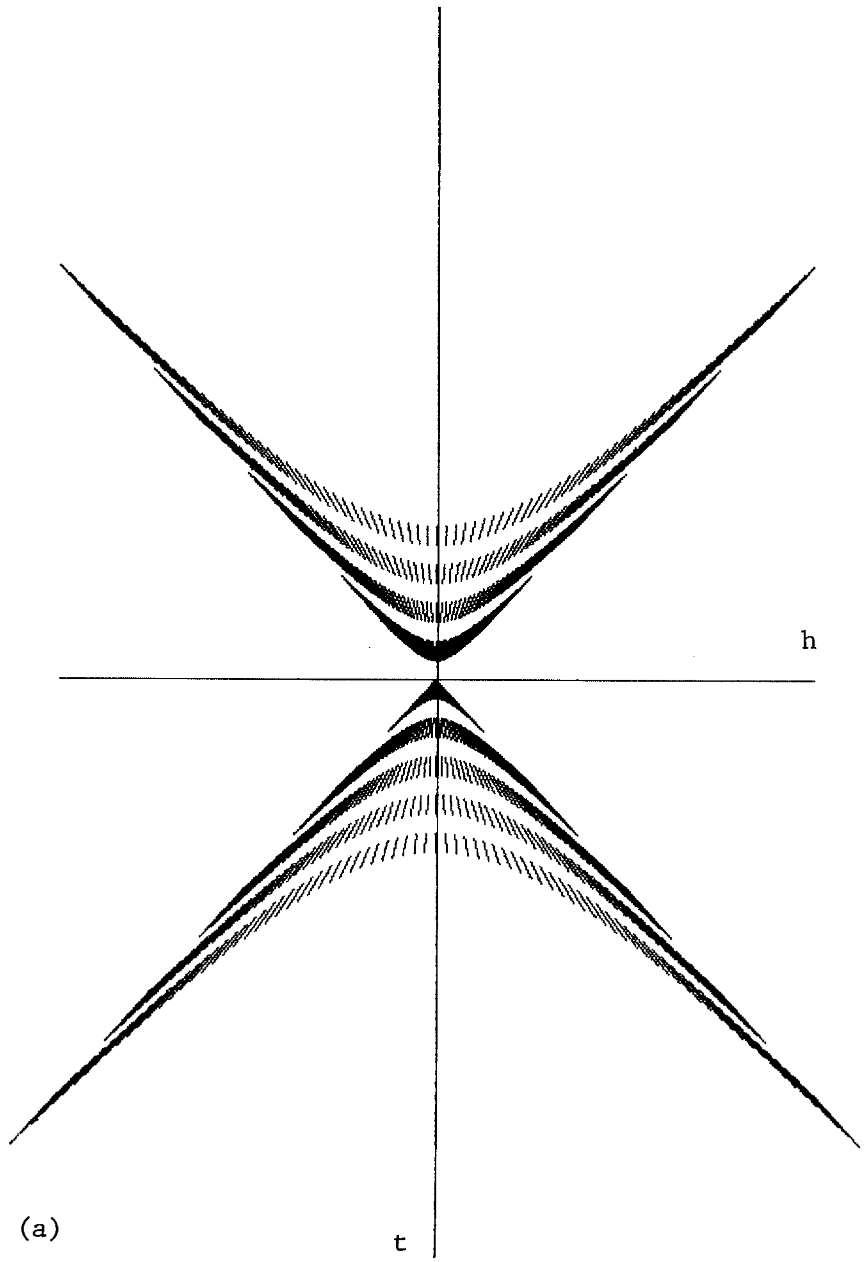
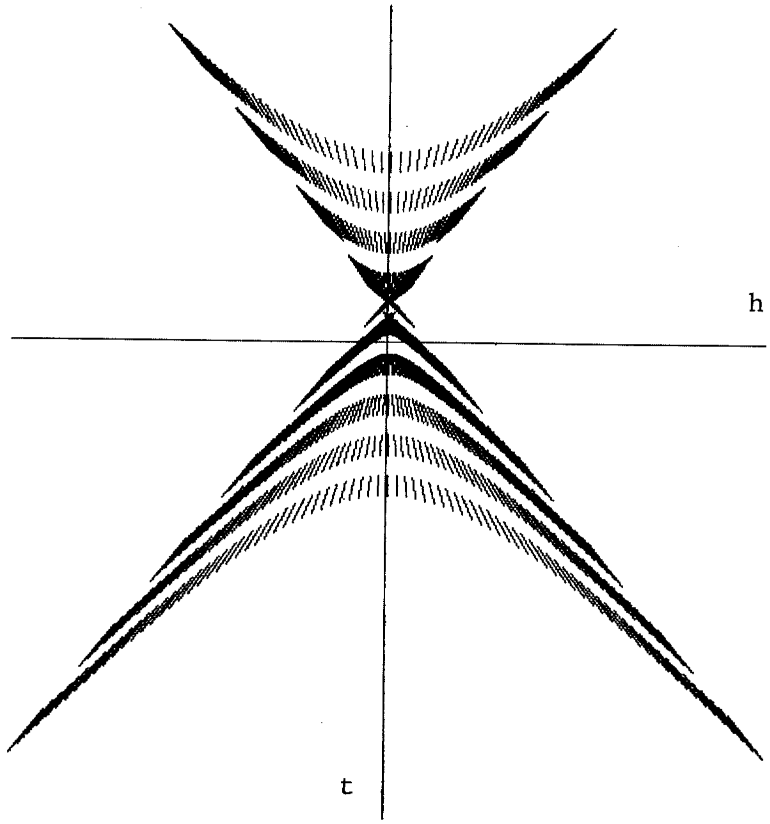
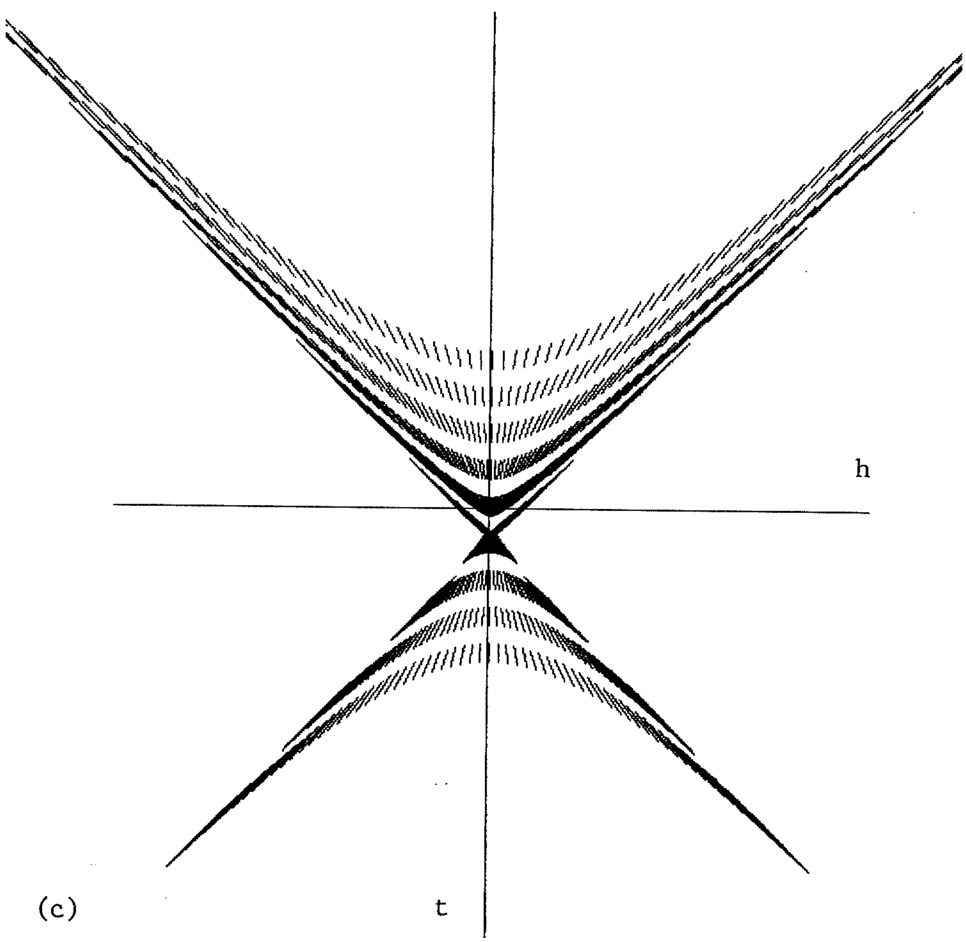


FIG. 2. This figure illustrates the way energy moves in the (x,t) plane as we extrapolate with the 90 *degree* wave equation. The figure was computed starting with some hyperbolic event for a fixed depth, subsequently extrapolating the wavefront using the ray equations. The process was done at fixed increments of *depth*, represented as slashed lines in the figure. In (a) the extrapolation was done with the same velocity as the velocity used to generate the hyperbolic event, we get a perfect focus at $t = 0$ since all the rays are traveling at the correct speed for all angles. In (b) we used a 5 % lower velocity in the extrapolation, now the energy is focusing at $t < 0$, however now the speed at which the wavefront moves has become p-dependent, and we no longer achieve a perfect focus. In (c) the extrapolation velocity was 5% higher, now the focus is at $t > 0$, and again the wavefront velocity is dependent on p.



(b)



(c)

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} v^2 p \\ v^2 q \end{bmatrix} \tag{4}$$

Figure 2 shows an example of using these ray-tracing equations.

B. Fifteen degree version of A.

For the fifteen degree equation the dispersion relation is given by

$$F(p,q) = q + \frac{p^2 v}{2} - \frac{1}{v} = 0$$

differentiating F with respect to p and q we get

$$\begin{bmatrix} \frac{\partial F}{\partial p} \\ \frac{\partial F}{\partial q} \end{bmatrix} = \begin{bmatrix} pv \\ 1 \end{bmatrix}$$

solving for λ we get

$$\lambda = \left[pv^2 + q \right]^{-1} = \frac{2v}{p^2 v^2 + 2}$$

and the ray equations are

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \frac{2pv^2}{p^2v^2 + 2} \\ \frac{2v}{p^2v^2 + 2} \end{bmatrix} \tag{5}$$

Note that as $p \to 0$ $\frac{dx}{dt} = pv^2$ and $\frac{dz}{dt} = v$, this is of course just the paraxial approximation to (4), since $qv^2 = v\sqrt{1 - p^2v^2}$ becomes v as p approaches zero.

Figure 3 illustrates the behavior of these new ray tracing equations.

C. Forty-five degree version of A.

For the forty-five degree equation the dispersion relation is given by

$$F(p,q) = q - \frac{1}{v}\,\frac{1 - \frac{3}{4}p^2v^2}{1 - \frac{1}{4}p^2v^2}$$

differentiating F with respect to p and q we get

$$\begin{bmatrix} \dfrac{\partial F}{\partial p} \\ \\ \dfrac{\partial F}{\partial q} \end{bmatrix} = \begin{bmatrix} \dfrac{pv}{(1 - \frac{1}{4}p^2v^2)^2} \\ \\ 1 \end{bmatrix}$$

for λ we obtain

$$\lambda = \frac{1 + \frac{3}{16}p^4v^4}{v\,(1 - \frac{1}{4}p^2v^2)^2}$$

the ray equations become

$$\frac{d}{dt}\begin{bmatrix} x \\ \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{pv^2}{1 + \frac{3}{16}p^4v^4} \\ \\ \dfrac{v\,(1 - \frac{1}{4}p^2v^2)^2}{1 + \frac{1}{16}p^4v^4} \end{bmatrix} \tag{6}$$

We can check that these ray equations have also the correct asymptotic behavior for ray parameters p close to zero. Figure 4 shows an example using these equations.

D. Retarded Snell Coordinates.

Before proceeding with the analysis we must first redefine the geometry (figure 5).

The coordinates are defined as follows:

$$\begin{bmatrix} t' \\ \\ x' \\ \\ z' \end{bmatrix} = \begin{bmatrix} t - p_0\,x + \dfrac{z\,\cos\vartheta_0}{v} \\ x + z\,\tan\vartheta_0 \\ \\ z \end{bmatrix} \tag{7}$$

The utility of the above coordinate derives form the fact that, for the particular p_0 chosen, as we downward continue, x' is fixed. Also, the imaging condition for t' is simply $t' = \dfrac{z\cos\vartheta_0}{v}$. Thus, applying the transformation (7) to the original data, and then migrating,

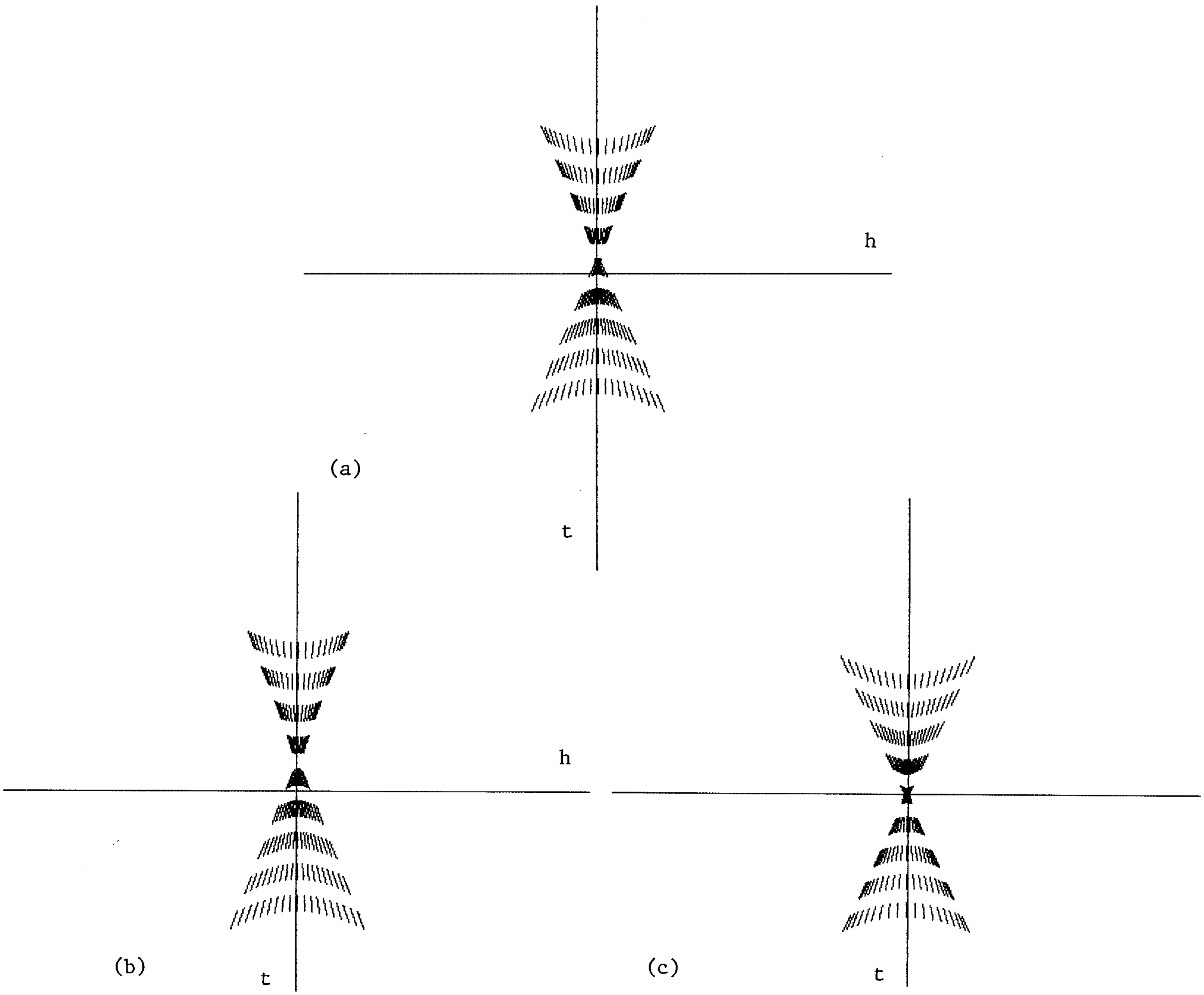


FIG. 3. In this figure we are extrapolating the wavefront with the 15 *degree* equation. The figure was computed using angles $\vartheta < 30^0$. We are again extrapolating the wavefield at fixed increments of depth, represented by broken lines in the figure. In (a) the extrapolation velocity equals the event velocity. We can check that for angles close to 0^0 the energy arrives very close to $t = 0$, however for wider angles the energy is traveling slower than required for correct focusing. In (b) the extrapolation velocity is 5 % slower, so energy focuses at $t < 0$. In (c) the extrapolation velocity is 5 % faster. Note that even thou the rays close to 0^0 focus earlier in time, since the 15 degree equation moves energy slower than required for wider angles, the apparent best focusing occurs with this higher velocity than with the exact velocity. This is just the very well known fact that overmigration gives better results with the 15 degree equation.

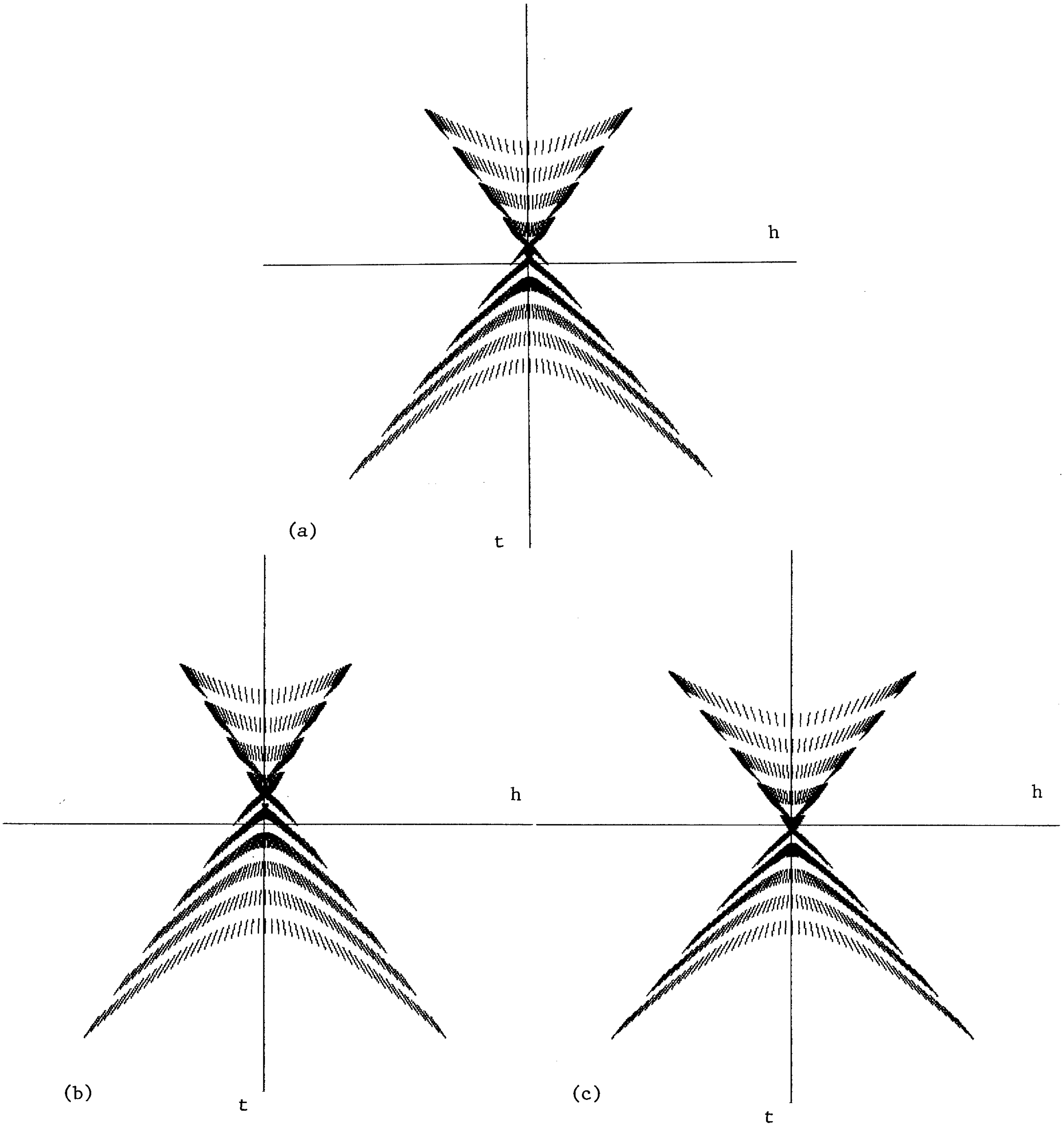


FIG. 4. In this figure we are extrapolating the wavefront with the 45 *degree* equation. The figure was computed using angles $\vartheta < 60^0$. See figure 3 for details.

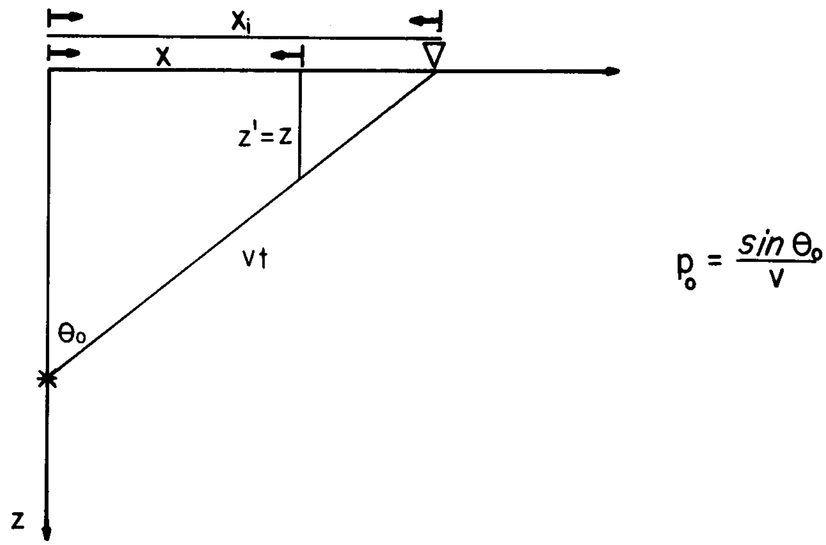


FIG. 5.

results in the top of the new skewed hyperbola remaining fixed. This result is useful in velocity analysis (see González and Claerbout; p190, SEP16).

To find the appropriate ray equations, we need to compute $\frac{dx'}{dt'}$ and $\frac{dz'}{dt'}$. Application of the chain rule to x' in (7) results in

$$\frac{d}{dt}\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} + \frac{dz}{dt}\tan\vartheta_0 \\ \frac{dz}{dt} \end{bmatrix} \tag{8}$$

But

$$\begin{bmatrix} \frac{dx'}{dt'} \\ \frac{dz'}{dt'} \end{bmatrix} = \begin{bmatrix} \frac{dx'}{dt'}\frac{dt'}{dt} \\ \frac{dz'}{dt'}\frac{dt'}{dt} \end{bmatrix}$$

Therefore

$$\frac{d}{dt'}\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} + \frac{dz}{dt}\tan\vartheta_0 \\ \frac{dz}{dt} \end{bmatrix} \left(\frac{dt'}{dt}\right)^{-1} \tag{9}$$

What remains are simple calculations which incorporate previous results. However, a word is in order here about sign convention. From figure 5, we notice that as we project the ray from the geophone back to the source, as both dt and dx are negative, $\frac{dx}{dt}$ is positive. Similarly, as t is decreasing, z' is increasing. Therefore,

$$sgn\left(\frac{dx}{dt}\right) = -sgn\left(\frac{dz}{dt}\right)$$

With the sign convention determined, the rest is straightforward. From (7) we obtain

$$\frac{dt'}{dt} = 1 - p_0\frac{dx}{dt} + \frac{\cos\vartheta_0}{v}\frac{dz}{dt}$$

and form (4) we already have

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} v^2 p \\ v^2 q \end{bmatrix}$$

Substituting these equations into (9), and using the correct sign convention, we have the final result for the ray equations:

$$\frac{d}{dt'}\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \dfrac{pv^2 - \dfrac{p_0}{q_0}qv^2}{1 - p_0pv^2 - q_0qv^2} \\ \\ \dfrac{qv^2}{1 - p_0pv^2 - q_0qv^2} \end{bmatrix} \tag{10}$$

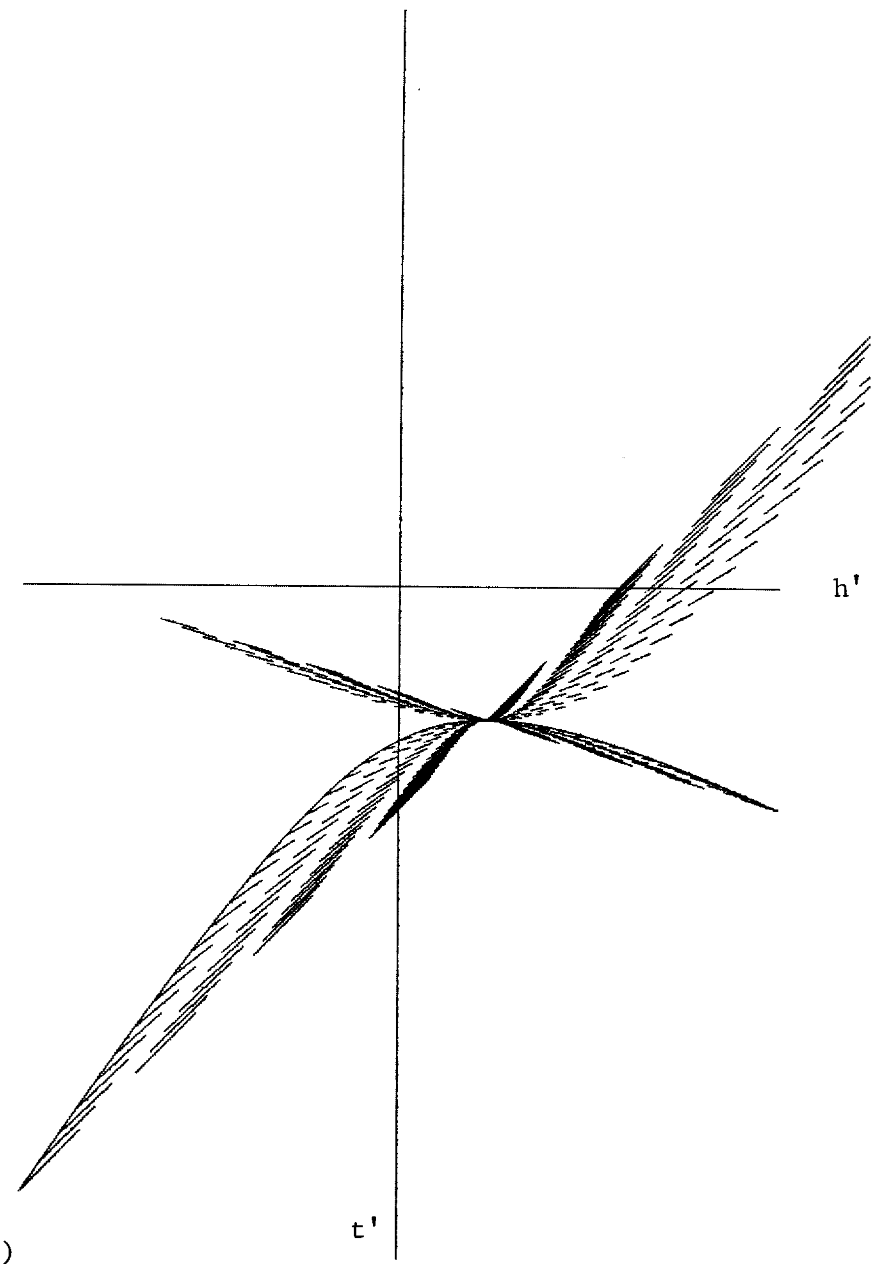
where $q_0 = \frac{\cos\vartheta_0}{v}$.

We notice, however that the p and q in (10) are those which correspond to the standard coordinate frame. To find the p' and q' in the new coordinate system we use (7) and find that

$$\frac{\partial t'}{\partial x'} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial x'} - p_0$$

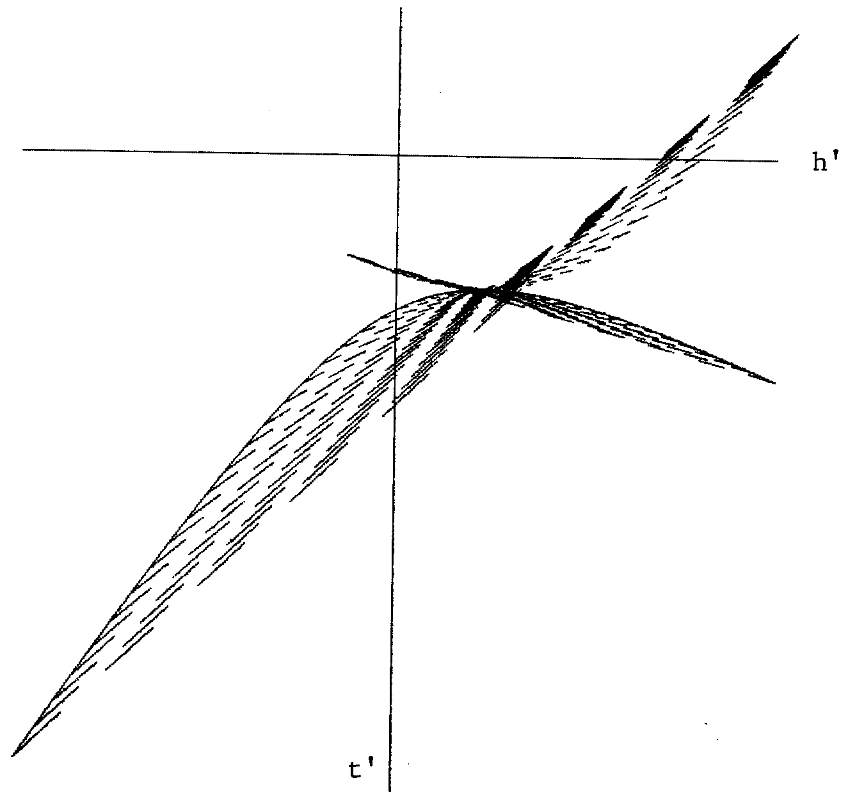
(for z fixed $\frac{\partial x}{\partial x'} = 1$). Or

$$\frac{\partial t}{\partial x} = \frac{\partial t'}{\partial x'} + p_0$$



(a)

FIG. 6. This figure shows the wavefront extrapolation in retarded Snell midpoint coordinates. At the surface the new coordinates require to apply a linear moveout correction, we used pv = 0.5 for this correction. In (a) we used the correct extrapolation velocity, so all rays arrive to the focusing point in phase. In (b) the extrapolation velocity is 5 % slower. In (c) we are using a 5 % faster velocity. As we downward continue the wavefront, the energy moves towards the top of the skewed hyperboloid, independently of the extrapolation velocity as can be appreciated in the figures.



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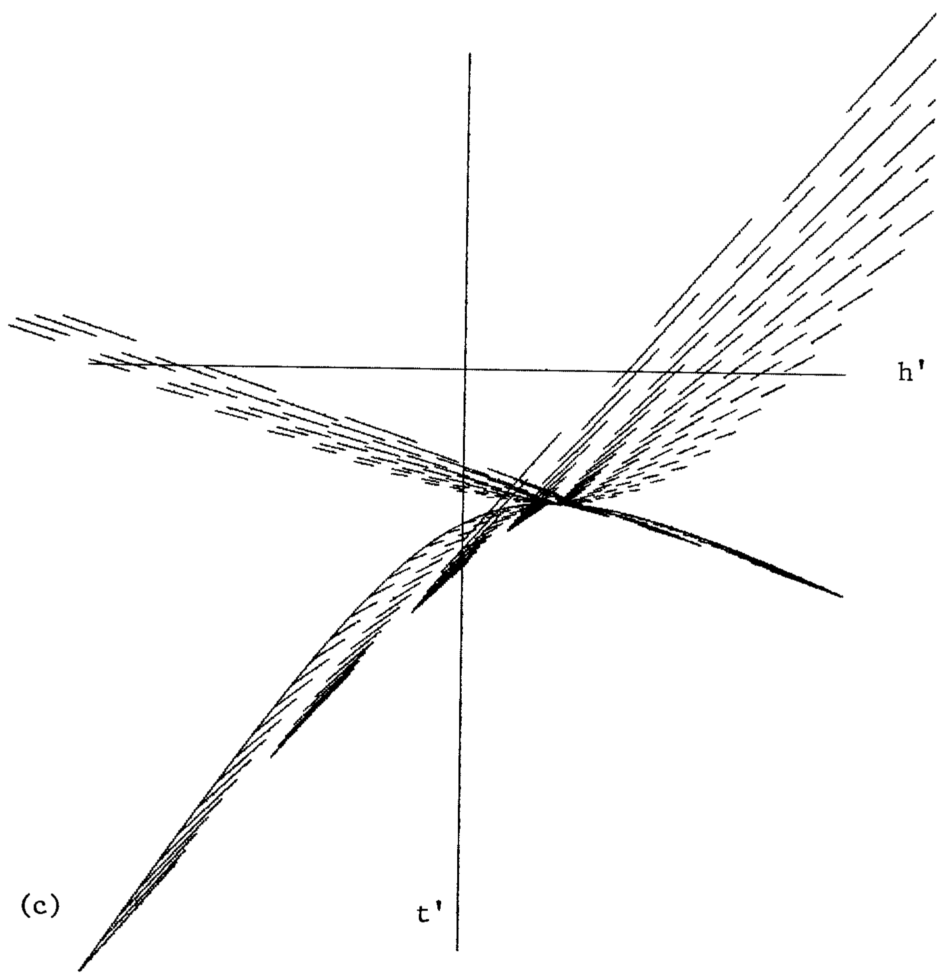

(c)

Now $\frac{\partial t}{\partial x}$ is of course p. Therefore, if all variables are seen in the slanted coordinate system, we simply replace p in (10) by $p' + p_0$, where p' is the measured $\frac{\partial t'}{\partial x'}$. Accordingly we find for q:

$$q = \frac{1}{v}\left[1 - v^2 (p' + p_0)^2\right]$$

The final result is

$$\frac{dx'}{dt'} = \frac{q_0 (p' + p_0) v^2 - p_0 v \left[1 - v^2 (p' + p_0)^2\right]^{1/2}}{q_0 \left[1 - p_0 (p' + p_0) v^2 - q_0 v \left[1 - v^2 (p' + p_0)^2\right]^{1/2}\right.} \qquad (11a)$$

$$\frac{dz'}{dt'} = \frac{v \left[1 - v^2 (p' + p_0)^2\right]^{1/2}}{\left[1 - p_0 (p' + p_0) v^2 - q_0 v \left[1 - v^2 (p' + p_0)^2\right]^{1/2}\right.} \qquad (11b)$$

In (11) all variables are in the slanted and retarded coordinate system. It is (11) that describes the propagation of energy in the (x', t', z') volume. (Figure 6).

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