Geometrical Interpretation of the Double Square Root Equation In Space-Time

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Abstract

The purpose of this article is a tutorial one. It is to facilitate an understanding of the double square root equation in space-time.

Geometric versus Algebraic Arguments

Consider a constant offset gather in the $(y-t)$ plane, with y the midpoint and t the travel-time. If this plane is seeded by an impulse at (y_o, t_o), then the travel-time for such a point scatterer is

$$t = \frac{1}{v}\left[\sqrt{(z^2 + (y - y_o + h)^2} + \sqrt{(z^2 + (y - y_o - h)^2}\right] \tag{1}$$

where $t_o = 2z/v$ and h is the half offset. Classically, what is now done is to square both sides of (1) twice to get a single square root equation relating z (or t_o) to y, t, h and v. Equation (1) can be thought of as an imaging equation for unstacked data (Stolt, SEP-20, p191). It is therefore instructive to get a geometric derivation of this resulting single square root equation, rather than that obtained by a simple algebraic manipulation of (1).

To do this we must appeal to the physics of the problem. The point is that in the $y-z$ plane, the reflector we seek is elliptical. Given this knowledge, it is easy to calculate some of the parameters of the ellipse:

a) The center of the ellipse is at y_o.

b) The foci are at $y_o \pm h$.

c) The semi-major axis is $\pm \frac{vt}{2}$.

d) The semi-minor axis is $\pm \left[\frac{v^2t^2}{4} - h^2 \right]^{1/2}$.

Now to draw the ellipse, we first construct two circles, centered at y_o, the smaller one of radius $\left[\frac{v^2t^2}{4} - h^2 \right]^{1/2}$ and the larger one of radius $\frac{vt}{2}$ (See figure 1).

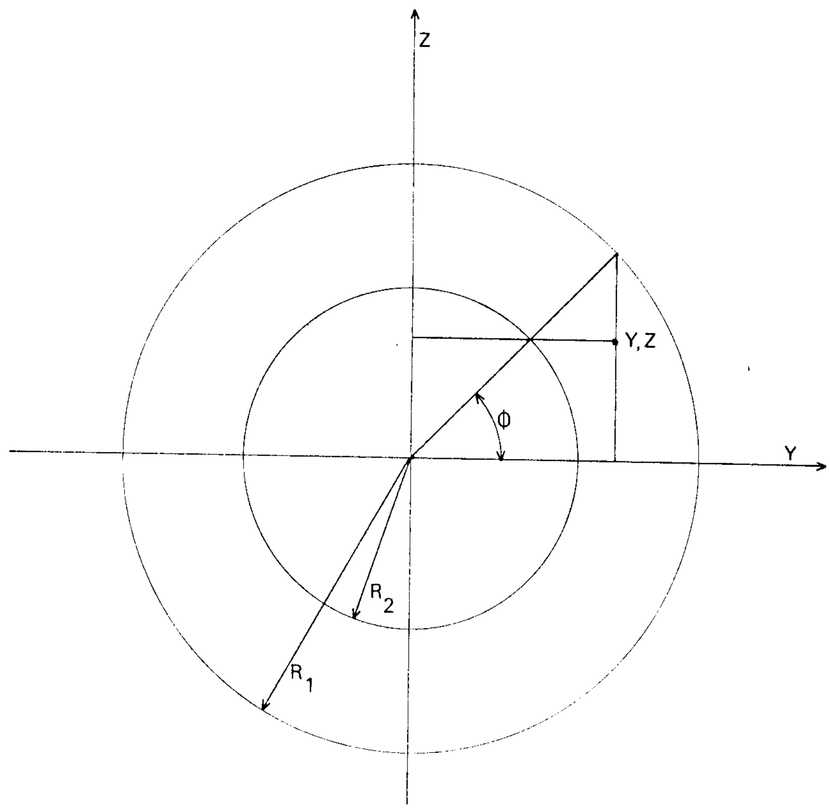

FIG. 1. This figure illustrates the geometric configuration for drawing the ellipse. The centers of the two circles are at y_o,0. The radius R_1 is $\frac{vt}{2}$, and R_2 is $(\frac{v^2t^2}{4} - h^2)^{1/2}$. Any point (y,z) on the ellipse can be simply described by the two radii, and the angle φ.

A point (y,z) on the elliptical reflector must lie between the two circles. By simple geometry of a circle,

$$y - y_o = \frac{vt}{2} \cos \varphi \tag{2}$$

$$z = \left(\frac{v^2t^2}{4} - h^2 \right)^{1/2} \sin \varphi \tag{3}$$

Equation (3) has a particularly nice geometric representation. When $\varphi = \frac{\pi}{2}$, $z = \left(\frac{v^2t^2}{4} - h^2 \right)^{1/2}$, the semi-minor axis of the ellipse. This corresponds to the ray from shot to geophone which would encounter a horizontal reflector tangent to the ellipse at y_o, the midpoint locating our point scatterer. For any other ray angle, z is simply a projection of this value onto the vertical z axis. From equation (2) we have that

$$\sin \varphi = \left(1 - \left[\frac{2(y - y_o)}{vt} \right]^2 \right)^{1/2} \tag{4}$$

Substitution of (4) into (3) results in

$$z = \left(\left[\frac{vt}{2} \right]^2 - h^2 \right)^{1/2} \left(1 - \left[\frac{2(y - y_o)}{vt} \right]^2 \right)^{1/2} . \tag{5}$$

That (5) is actually correct can be verified by substitution into the right hand side of (1) to yield an identity. To see that this construction works, a sample ellipse is drawn in figure 2, using the geometric algorithm described by equations (2) and (3).

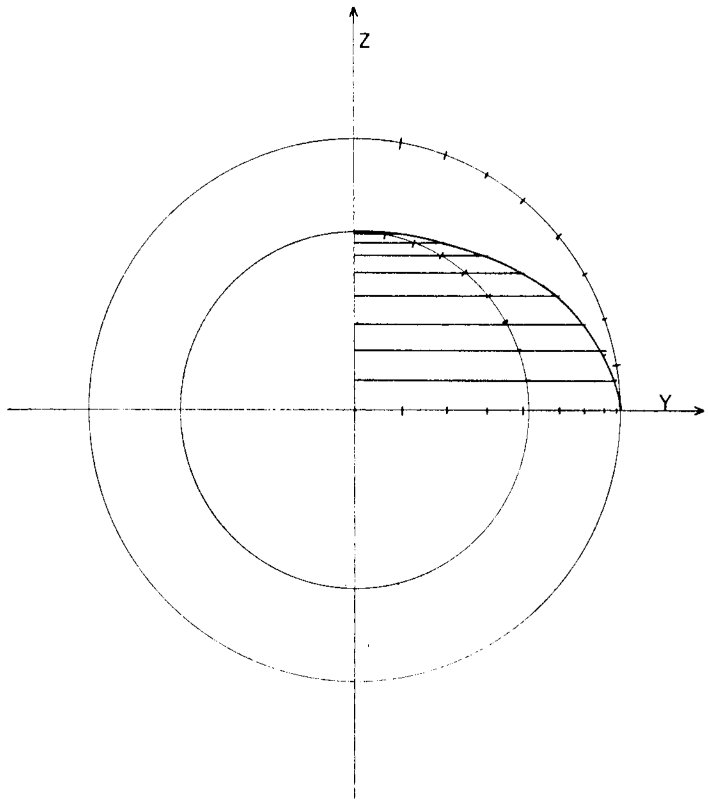

FIG. 2. One quadrant of the ellipse is drawn in this figure. The ellipse is constructed using the geometric algorithm suggested by equations (2) and (3).