INVERSION IN AN INHOMOGENEOUS MEDIUM

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The Born-WKBJ inverse method outlined elsewhere in this report requires a background velocity that is laterally invariant. When the real earth does not favor that restriction, we have to generalize the method; although, as it turns out, we can retain the WKBJ concept of a locally constant velocity.

We define a background wave operator

$$\hat{L} \sim \nabla^2 + \frac{\omega^2}{v^2(\vec{x})} \tag{1a}$$

with the understanding that $v(\vec{x})$ varies slowly compared to the wavelength of the waves we wish to describe.

The WKBJ solutions P to the background wave equation

$$\hat{L}P \sim 0 \tag{1b}$$

can be found by writing $P = e^{i\varphi}$, yielding a non-linear partial differential equation for φ:

$$-\vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + i\,\vec{\nabla}^2\varphi + \frac{\omega^2}{v^2} = 0 \tag{2}$$

Express φ as a power series in ω^{-1}:

$$\varphi = \omega\,\varphi_0 + \varphi_1 + \omega^{-1}\varphi_2 + \cdots \tag{3}$$

and equate each power of ω in equation (2):

$$(\vec{\nabla}\varphi_0) = \frac{1}{v^2} \tag{4a}$$

$$2\vec{\nabla}\varphi_0 \cdot \vec{\nabla}\varphi_1 = i\vec{\nabla}^2\varphi_0 \tag{4b}$$

$$2\vec{\nabla}\varphi_0 \cdot \vec{\nabla}\varphi_2 = i\nabla^2\varphi_1 - (\vec{\nabla}\varphi_1)^2 \tag{4c}$$

etc.

The WKBJ approximation to φ is the high-frequency limit in which only φ_0 and φ_1 need be retained.

The nice thing about WKBJ solutions is that they are essentially one-way. They do not generate reflections and won't, without special effort, refract around turning points. This will allow us to retain the concept of up- and downgoing waves that is central to any migration scheme.[1]

WKBJ Green's functions satisfying

$$\left[\nabla^2 + \frac{\omega^2}{v^2(\vec{x})}\right] G_\pm(\omega\, \vec{x}\,|\,\vec{x}_0) = -\delta(\vec{x} - \vec{x}_0) \tag{5}$$

can also be constructed. As a function of $\vec{x}-\vec{x}_0$, these creatures take the form (in 3-D)

$$G_\pm^{3D}(\omega;\vec{x}\,|\,\vec{x}_0) = \frac{exp\left[\pm i\,\omega\,|\vec{x} - \vec{x}_0| / v(\vec{x}_0)\right]}{4\pi\,|\vec{x} - \vec{x}_0|} \tag{6a}$$

or, in 2-D

$$G_\pm^{2D}(\omega;\vec{x}\,|\,\vec{x}_0) = \frac{1}{\pm 4i}\left[\pm i Y_0\left[\frac{\omega\,|\vec{x} - \vec{x}_0|}{v(\vec{x}_0)}\right] + J_0\left[\frac{\omega\,|\vec{x} - \vec{x}_0|}{v(\vec{x}_0)}\right]\right] \tag{6b}$$

$$= \frac{1}{\pm 4i} H_0^{(\frac{1}{2})}\left[\frac{\omega\,|\vec{x} - \vec{x}_0|}{v(\vec{x}_0)}\right]$$

If v depends on x and z only, a Fourier transform over y of the 3-D Green's function yields the 2-D like form

$$G_\pm^{3D}(\omega,k_y;x,z\,|\,x_0,z_0) = \frac{\pm i}{4(2\pi)^{1/2}} H_0^{(\frac{1}{2})}\left[\frac{\omega}{v(\vec{x}_0)}\sqrt{1 - \frac{k_y^2 v(\vec{x}_0)^2}{2}}\;|x - x_0|\right] \tag{6c}$$

[1] For waves traveling nearly horizontally, the concepts of up- and downgoing may not be well defined. We are not really interested in waves traveling at such extreme angles, however, and so we will pretend they don't exist.

As $\vec{x}$ departs from $\vec{x}_0$, these expressions become invalid. However, equations (4a) and (4b) can be used to extrapolate the Green's function from the region of $\vec{x}_0$ to anywhere.[2] It is convenient, though perhaps not necessary, to make v so well behaved that φ_0 remains single-valued and continuous.

We are now ready for our fundamental assumption, namely that our WKBJ wave and Green's function accurately represent wave propagation in the real earth, neglecting only reflections, scattering, refraction turn-arounds, and the like. We can then model reflections using the Born approximation just as before.

The inversion scheme we hope to develop will look much like the earlier ones. We first migrate the data, obtaining a function of midpoint, offset, and depth $M(x_m, x_h, z)$. The total Fourier transform of this quantity, we hope, will be a linear combination of the double Fourier transforms of the two potential components a_1 and a_2 .

The first thing to do is to define a downward-continuation operation, which we will do via Green's theorem. Given a closed surface S enclosing a volume V, and a wave function P which obeys

$$\hat{L}\, P \sim \left[\nabla^2 + \frac{\omega^2}{v^2(\vec{x})}\right] P \;=\; 0 \tag{7}$$

Then, for any point $\vec{x}$ in V, we can define a surface integral over S the value of which is P at $\vec{x}$:

$$\begin{aligned} I_\pm(x,\omega) &= -\int_V ds'\, [P(x',\omega)\, \frac{\partial}{\partial n'}\, G_\pm(\omega,\vec{x}\,|\,\vec{x}\,') - G_\pm(\omega,\vec{x}\,|\,\vec{x}\,')\frac{\partial}{\partial n'}\, P(x',\omega)] \qquad (8) \\ &= -\int_V dV'\, [P(\vec{x}\,',\omega)\, L' G_\pm(\omega,\vec{x}\,|\,\vec{x}\,') - G_\pm(\omega,\vec{x}\,|\,\vec{x}\,')\, L' P(\vec{x}\,',\omega)] \\ &= P(\vec{x},\omega) \end{aligned}$$

Note that this integral gives the same result regardless of whether G_+ or G_- is used.

Suppose, now, that S can be divided into two parts, the "upper" part S_U located above $\vec{x}$ and the "lower" part S_L beneath $\vec{x}$. We then *define* P to be *upgoing* at $\vec{x}$ if

$$I_-^{(L)}\, (\vec{x},\omega) \;=\; -\int_{S_L} dS' \left[P\, \frac{\partial G_-}{\partial n'} - G_-\, \frac{\partial P}{\partial n'}\right] \;=\; 0 \tag{9}$$

[2] I am not necessarily advocating this procedure as the only or even a desirable method for constructing the one-way Green's functions. The important point in all this is that "one-way" Green's functions should exist which can, if necessary, be fabricated one way or another.

This definition should make good intuitive sense, since G_- can only map backwards in time, which means downwards in space, if a wave is upgoing.

Hence, for an upward-traveling wave,

$$P(\vec{x},\omega) = I_-^{(U)}(\vec{x},\omega) = -\int_{S_U} dS' \left[P \frac{\partial G_-}{\partial n'} - G_- \frac{\partial P}{\partial n'} \right] \tag{10}$$

Equation (10) represents a formula [3] for downward continuing P; that is, reconstructing P at $\vec{x}$ given values of P along a surface *above* $\vec{x}$.

A downward-traveling wave may be defined in a corresponding manner. We also expect equations complimentary to (9) and (10) to hold when the exploding Green's functions G_+ is used.

Equations (8) through (10) were developed assuming P to be source-free inside V. If, more generally,[4]

$$P = P_0 + P_+ = P_0 + \int_V dV' \, G_+ S \tag{11}$$

so that

$$\hat{L}P = -S \tag{12}$$

then equation (8) becomes (if P_0 is an upgoing wave)

$$I_+(\vec{x},\omega) = P_0(\vec{x},\omega) \tag{13a}$$

and

$$I_-(\vec{x},\omega) = P(\vec{x},\omega) - \int_V dV \, G_- S = P - P_- \tag{13b}$$

Since the source term P_+ should be more or less upgoing on S_U and downgoing on S_L , it is reasonable that it should be invisible to the I_+ integral (13a). That is not true for the I_- integral, however; P_+ is visible to I_- on both S_U and S_L . Consequently, both $I_-^{(U)}$ and $I_-^{(L)} \neq 0$.

[3] I don't necessarily advocate this to be the actual formula to use in downward continuing P, since perfectly good finite-difference methods have been developed which can do the job. Care should be taken, however, that whatever method is used, the result should be rendered equivalent, in phase and amplitude, to equation (10). This could be a minor bother, since finite-difference algorithms tend to concentrate on phase and ignore amplitudes.

[4] I seem to be using the same symbol (S) to denote surfaces [e.g. as in equation (10)] and source distributions [as in (11)]. I hope this doesn't confuse anyone any more than they are already.

However, it does not follow that P cannot be downward continued in this case. Suppose that the source S goes off at time = 0. Then P_+ is located purely in positive time and, conversely, P_- is located only in negative time. Thus, for t > 0,

$$I_-(\vec{x},t) = P(\vec{x},t) \tag{14}$$

We expect $I_-^{(L)}$ to "see" only those sources which are above the observation point $\vec{x}$ (hence producing a downgoing wave). The integral $I_-^{(U)}$ over the upper surface will register the sources below $\vec{x}$, so, for t > 0.

$$I_-^{(U)}(\vec{x},t) = P_0(\vec{x},t) + \int_{z'>z} dv'\, G_+(\vec{x},t\,|\,\vec{x}',t' = 0)\, S(\vec{x}') \tag{15}$$

This expectation can be strengthened by applying Green's theorem to a volume whose upper boundary is S_U , and whose lower boundary S_z is a plane just barely above the observation depth z. Since the observation point ($\vec{x}$) is not in this volume, we have

$$I_-^{(U)}(\vec{x},\omega) + I_-^{(z)}(\vec{x},\omega) = - \int_{z'<z} dv' G_-(\omega;\vec{x}\,|\,\vec{x}')\, S(\vec{x}') \tag{16}$$

The right-hand side of this equation involves only sources above z, and exists purely in negative time. Thus the surface integral $I_-^{(U)}(\vec{x},t)$ is, for t > 0 identical to the surface integral $-I_-^{(z)}(\vec{x},t)$ (the sign change occurs because the direction of the normal derivatives changes), taken along a plane infinitesimally above z. Since, for t > 0, all the waves from sources above z are downgoing, they make no contribution to $I_-^{(z)}$. All the rest of P, including that from sources below z, are upgoing[5] at z, hence register on $I_-^{(z)}$. Since they would not register on a boundary infinitesimally below z, equation (15) follows.

All of this brings us up to the point where Claerboutians have been (intuitively) for years. A "downward-continued" wave recreates, in the absence of sources or reflectors, the real wave. If at t = 0 a source exists, the real wave is duplicated down to the source for positive time. If we downward-continue past the source, its contributions to the downward-continued wave are the time-reversal of its contribution to the real wave. So, what does all this have to do with inversion? Suppose that the reflected wave measured at the earth's surface is

$$D(\vec{x}_g\,|\,\vec{x}_s;\omega) = \int dV\, G_+(\vec{x}\,|\,\vec{x};\omega)\, V(\vec{x},\omega)\, G_+(\vec{x}\,|\,\vec{x}_s;\omega) \tag{17}$$

with $\vec{x}_g$ and $\vec{x}_s$ measured on the earth's surface S_E.

[5] Okay, okay. It's possible that the arrival from a source sightly below z but a long way away may pick up enough curvature to arrive at z a (slightly) downgoing wave. So what.

Define a downward-continued wave (downward continuing both sources and receivers)

$$W^{(E)}(\vec{x}_g \mid \vec{x}_s;\omega) = \int_{S_E} dS_s' \int_{S_E} dS_g' G_-(\vec{x}_g \mid \vec{x}_g';\omega) T(\vec{x}_g') D(\vec{x}_g' \mid \vec{x}_s';\omega) T(\vec{x}_s') G_-(\vec{x}_s' \mid \vec{x}_s;\omega) \quad (18)$$

where T is a shorthand for the differential operator

$$T(\vec{x}) = \frac{\partial}{\partial \underset{\leftarrow}{n}} - \frac{\partial}{\partial \underset{\rightarrow}{n}} \quad (19)$$

and $\partial/\partial n$ is the derivative normal to the earth's surface. ($\partial/\partial \underset{\leftarrow}{n}$ is understood to operate on the function to its left.) W, then, is just a double candidate for Green's theorem. We expect for $W^{(E)}$ the following generalization of equation (15) to hold: for source and receiver coordinates $\vec{x}_s$ and $\vec{x}_g$ on the same plane $z_s = z_g = z$, then

$$W^{(E)}(x_g, z \mid x_s, z) = \int_{z'>z} d\vec{x}' \, G_+(\vec{x}_g \mid \vec{x}') \, V(\vec{x}') \, G_+(\vec{x}' \mid \vec{x}_s) + \int_{z'<z} d\vec{x}'$$

$$G_-(\vec{x}_g \mid \vec{x}') \, V(\vec{x}') \, G_-(\vec{x}' \mid \vec{x}_s) \quad (20)$$

(All the quantities in this equation are functions of frequency ω' which, for the sake of brevity, has not been explicitly included.) By interpretation, $W^{(E)}$ should include reflections from points below the source-observation plane $z_g = z_s = z$, plus the time reversal of reflections from points above z.

To support this expectation, form a closed surface S whose upper boundary is the earth's surface S_E and whose lower boundary is a plane $z' = z - \varepsilon$ located just above z. Define the surface integral

$$W(x_g, z \mid x_s, z) = \int_S dS_g' \int_S DS_s' \, G_-(\vec{x}_g \mid \vec{x}_g') \, T(\vec{x}_s') \int d\vec{x}'' \cdot$$

$$\cdot G_+(\vec{x}_g' \mid \vec{x}'') \, V(\vec{x}'') \, G_x(\vec{x}'' \mid \vec{x}_s') \, T(\vec{x}_s') \, G_0(\vec{x}_s' \mid \vec{x}_s) \quad (21)$$

The portion of W with both $\vec{x}_g'$ and $\vec{x}_s'$ are on S_E is just $W^{(E)}$. W also has three other parts:

$$W = W^{(E)} + W^{(E,z)} + W^{(z,E)} + W^{(z)} \quad (22)$$

$W^{(E,z)}$ corresponds to $\vec{x}_g'$ on S_E but $z_s' = z - \varepsilon$. $W^{(z,E)}$ corresponds to $z_g' = z - \varepsilon$ and $\vec{x}_s'$ on S_E. $W^{(z)}$ has both $\vec{x}_g'$ and $\vec{x}_s'$ on the lower surface.

W can be converted to a volume integral via Green's theorem. We have

$$W(x_g, z \mid x_s, z) = \int_{z_g'<z-\varepsilon} d\vec{x}_g' \int_{z_s'<z-\varepsilon} d\vec{x}_s' \, G_-(\underset{\leftarrow}{\hat{L}} - \underset{\rightarrow}{\hat{L}}) \int d\vec{x}'' \, G_+ \, V G_+ (\underset{\leftarrow}{\hat{L}} - \underset{\rightarrow}{\hat{L}}) \, G_-$$

Now, since the points $\vec{x}_g$ and $\vec{x}_s$ are outside the volumes spanned by $\vec{x}_g{}'$ and $\vec{x}_s{}'$, we have

$$G_- \underset{\leftarrow}{\hat{L}} - \underset{\rightarrow}{\hat{L}} G_- = 0$$

More over,

$$\underset{\rightarrow}{\hat{L}} \int d\vec{x}'' \, G_+ V G_+ \underset{\leftarrow}{\hat{L}} = V(\vec{x}_g{}') \, \delta(\vec{x}_g{}' - \vec{x}_s{}')$$

so

$$W(x_g,z \,|\, x_s,z) = \int_{z'<z-\varepsilon} d\vec{x}' \, G_-(\vec{x}_g \,|\, \vec{x}') \, V(\vec{x}') \, G_-(\vec{x}' \,|\, \vec{x}_s) \tag{23}$$

As $\varepsilon \to 0$, the double integral W around the closed surface S becomes the time reversal of reflections from above the plane $z_g = z_s = z$. It is in fact the second term in the expected form (20) for $W^{(E)}$.

We now argue that of the four sub-integrals $W^{(E)}, W^{(E,z)}, W^{(E,z)}, W^{(z,E)}$, and $W^{(z)}$ in W, only $W^{(E)}$ can depend on reflections above z. $W^{(z)}$ and $W^{(z,E)}$ cannot because reflections from above $z - \varepsilon$ are downgoing at $z - \varepsilon$, hence an attempt to extrapolate them downward to z using G_- must simply yield zero. To see that $W^{(E,z)}$ is zero too, we need only exploit the symmetry of the problem. Thus, by default, the time-reversed reflections from above z must be included as part of $W^{(E)}$. Thus,

$$W^{(E)}(x_g,z \,|\, x_s,z) = W(x_g,z \,|\, x_s,z) + \int_{S_E} dS_g{}' \int_{S_E} dS_s{}' \cdot$$

$$\cdot G_-(\vec{x}_g \,|\, \vec{x}_g{}') \, T(x_g{}') \int_{z''>z} d\vec{x}'' \, G_+(\vec{x}_g{}' \,|\, x'') \, V(\vec{x}'') \, G_+(\vec{x}'' \,|\, \vec{x}_s{}') \, T(\vec{x}_s{}') \, G_-(\vec{x}_s{}' \,|\, \vec{x}_s) \tag{24}$$

To complete our argument that the correct form for $W^{(E)}$ is (20), we need only inquire about the fate of reflections from below z. We now assert that such reflections are necessarily upgoing at z. Hence, we can use equation (10) to eliminate the $dS_g{}'$ integral in (24):

$$W^{(E)}(x_g,z \,|\, x_s,z) = W(x_g,z \,|\, x_s,z) + \int_{S_E} dS_s{}' \int_{z''>z} d\vec{x}'' \cdot$$

$$\cdot G_+(x_g,z \,|\, \vec{x}'') \, V(\vec{x}'') \, G_+(\vec{x}'' \,|\, \vec{x}_s{}') \, T(\vec{x}_s{}') \, G_-(\vec{x}_s{}' \,|\, x_s,z)$$

Symmetry allows us to remove the integral over $dS_s{}'$ too, leaving us with equation (20).

Let us now generalize the definition of D given in equation (17) to be

$$D(x_g,z \,|\, x_s,z) = \int_{z'>z} d\vec{x}' \, G_+(x_g,z \,|\, x',z') \, V(x',z') \, G_+(x',z' \,|\, x_s,z) \tag{25}$$

that is, D is the reflection response from points *below* the source-observation plane $z_g = z_s = z$. Then equation (20) can be written as

$$W^{(E)} = W + D$$

We now assert that the other surface sub-integral $W^{(z)}$ of W is equal to D. This follows from the definition of $W^{(z)}$ and the same argument that gets from equation (24) to (25). Thus

$$W^{(E)}(x_g,z\,|\,x_s,z) = W(x_g,z\,|\,x_s,z) + W^{(z)}(x_g,z\,|\,x_s,z) \tag{26}$$

If we were to take a Fourier transform over ω of equation (26), we would discover, since W is nonzero only for negative time, that

$$W^{(E)}(x_g,z\,|\,x_s,z;t) = W^{(z)}(x_g,z\,|\,x_s,z;t) \;\; if \;\; t>0 \tag{27}$$

Equation (27) will allow us to proceed from migration to inversion of reflection data.

We now define a "migrated" field M to be the limit of $W^{(E)}$ as time approaches zero through positive values:

$$M(x_m,x_h,z) = \lim_{t\to 0} \int d\omega\, e^{-i\omega t}\; W^{(E)}(x_g,z\,|\,x_s,z;\omega) \tag{28}$$

M has been expressed in terms of midpoint and offset coordinates $x_m = (x_g + x_s)/2$, $x_h = (x_g - x_s)/2$, and a single depth $z = z_g = z_s$. In 3-D, x_m and x_h are two-vectors.

Because of equation (27), $W^{(E)}$ can be replaced in (28) by $W^{(z)}$ which, according to definition, is just (bringing back the small positive parameter ε

$$W^{(z)}(x_g,z\,|\,x_s,z;\omega) = \int dx_g{}' \int dx_s{}'\; G_-(x_g,z\,|\,x_g,z-\varepsilon;\omega)\; T(z-\varepsilon)\cdot$$

$$\cdot D(x_g{}',z-\varepsilon\,|\,x_s{}',z-\varepsilon;\omega)\; T(z-\varepsilon)\; G_-(x_s{}',z-\varepsilon\,|\,x_s,z;\omega) \tag{29}$$

The only values of $x_g{}'$ and $x_s{}'$ in the integral (29) that can contribute to the migrated field M are those very close to x_g and x_s (which must also be close together if M is to be nonzero). Under these conditions, G_- and D assume constant velocity forms, the relevant velocity being $v(x_m,z)$. Denoting these limiting forms to be $G_-^{[v(x_m,z)]}$,$D^{[v(x_m,z)]}$ we have

$$M(x_m,x_h,z) = \lim_{t\to 0} \int d\omega\, e^{-i\omega t} \int dx_g{}' \int dx_s{}'\; G_-^{[v(x_m,z)]}\,((x_g - x_g{}'),\varepsilon;\omega)\cdot$$

$$\cdot T\; D^{[v(x_m,z)]}\,(x_g{}',z-\varepsilon\,|\,x_s{}',z-\varepsilon;\omega)\; T\; G_-^{[v(x_m,z)]}\,((x_s{}'-x_s),\varepsilon;\omega) \tag{30}$$

Using the expression

$$G_-^{[v]}(x,z;\omega) = \int dk_x\, e^{ik_x x}\, \frac{e^{-iqz}}{2iq}\,, \quad q = \frac{\omega}{v}\sqrt{1 - \frac{k_x^2 v^2}{\omega^2}} \tag{31}$$

equation (30) becomes (after doing the T-derivatives)

$$M(x_m,x_h,z) = -\int d\omega \int dk_g \int dk_s\, \exp i[k_g x_g - k_s x_s - \varepsilon(q_g + q_s)] \cdot D^{[v(x_m,z)]}(k_g, z-\varepsilon \,|\, k_s, z-\varepsilon;\omega) \tag{32}$$

We now need an expression for D. We have

$$D^{[v]}(k_g, z-\varepsilon \,|\, k_s, z-\varepsilon, \omega) = -\int_{z-\varepsilon}^{\infty} dz'\, e^{i(\varepsilon - z + z')(q_g + q_s)} \cdot \sum_{n=1}^{2} a_n(k_g - k_s, z')\, \alpha_n(k_g, k_s, \omega, x_m, z) \tag{33}$$

where

$$\alpha_1(k_g,k_s,\omega,x_m,z) = \frac{1}{4 q_g q_s}\, \frac{\omega^2}{v^2} \tag{34a}$$

and

$$\alpha_2(k_g,k_s,\omega,x_m,z) = \frac{1}{4 q_g q_s}\,(q_g q_s - k_g k_s) \tag{34b}$$

Putting (33) into (32), we get

$$M(x_m,x_h,z) \approx \int dk_g \int dk_s \int_{z-\varepsilon}^{\infty} dz' \int d\omega \sum_{x=1}^{2} \alpha_n(k_g,k_s,\omega,x_m,z) \cdot e^{i[k_g x_g - k_s x_s + (q_g + q_s)(z' - z)]}\, a_n(k_g - k_s, z') \tag{35}$$

Now, change some integration variables. We define

$$k_m = k_g - k_s \tag{36a}$$

$$k_h = k_g + k_s \tag{36b}$$

$$k_z = q_g + q_s \tag{36c}$$

and note that

$$\frac{d\omega}{dk_z} \cdot \frac{\omega^2}{v^2} \cdot \frac{1}{q_g q_s} = \frac{v}{2}\sqrt{\left[1 + \frac{k_m^2}{k_z}\right]\left[1 + \frac{k_h^2}{k_z^2}\right]} \tag{36d}$$

So, in terms of these new variables, (35) becomes

$$M(x_m,x_h,z) \propto \int_{z-\varepsilon}^{\infty} dz' \int dk_m \sum_{n=1}^{2} a_n(k_m,z')\, e^{ik_m x_m} \cdot$$

$$\cdot \int dk_z\, e^{ik_z(z'-z)} \int dk_h\, e^{ik_h x_h}\, \gamma_n(k_m,k_n,k_z;x_m,z) \tag{37}$$

where

$$\gamma_1(k_m,k_n,k_z,x_m,z) = v \sqrt{\left[1 + \frac{k_m^2}{k_z^2}\right]\left[1 + \frac{k_h^2}{k_z^2}\right]} \tag{38a}$$

and

$$\gamma_2(k_m,k_h,k_z,x_m,z) = \gamma_1(k_m,k_h,k_z,x_m,z) \left[\frac{k_z^2 - k_h^2}{k_z^2 + k_h^2}\right] \tag{38b}$$

We now note that the functions γ_n are real, even, very slowly varying functions of k_z. Therefore, the k_z integral in (37) is nonzero only for z' very close to z (we actually knew this already). This means that the lower bound of the z' integral can be changed from $z - \varepsilon$ to $-\infty$ without significantly affecting the result. The z' integral is then recognizable as a Fourier transform of a_n, leaving us with

$$M(x_m,x_h,z) \propto v(x_m,z) \int dk_m \int dk_h \int dk_z\, e^{i(k_m x_m + k_h x_h - k_z z)} \cdot$$

$$\cdot v \sqrt{\left[1 + \frac{k_m^2}{k_z^2}\right] + \left[1 + \frac{k_h^2}{k_z^2}\right]} \left[a_1(k_m,-k_z) + \frac{k_z^2 - k_h^2}{k_z^2 + k_h^2}\, a_2(k_m,-k_z)\right] \tag{39}$$

That is, the three-dimensional Fourier transforms of $M(x_m,x_n,z)/v(x_m,z)$ is proportional to a linear combination of the two-dimensional Fourier transforms of a_1 and a_2.