Uniform Asymptotic Expansion of the Green's Function for the Two-dimensional Acoustic Equation

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Summary

A uniform asymptotic expansion in the frequency domain is derived for the Green's function of the two-dimensional acoustic equation. The expansion is uniform in that it is valid near the source region. It is not valid for caustics, which can arise due to rapid changes in the gradients of the material parameters - the density and bulk modulus. The Green's function which is obtained describes only the body wave arrivals in a smoothly varying whole space. Other wave types, such as surface waves or head waves are not included in this expansion.

Introduction

There has been extensive literature dealing with asymptotic ray series solutions to the wave equation (Babich 1964; Karal & Keller 1962; Cerveny, Molotkov & Psencik 1977). In this paper, a technique is developed for the calculation of the Green's function for the two-dimensional acoustic equation with material parameters depending both on x and z. The usual geometric optics assumptions, that the wavelength, λ, is much less than any material property divided by its gradient, is used in the derivation of the Green's function. It is the purpose of this communication to demonstrate how the classical method of computing the Green's function (Courant & Hilbert 1962) can be combined with an appropriate asymptotic series expansion in inverse powers of frequency. The Green's function may then be calculated by the solution of a system of ordinary differential equations which define the arrival time and amplitude of the wave disturbance. For body wave arrivals, only the first term of the expansion need be included (Aki & Richards 1980).

Solution of the Canonical Problem

The two-dimensional time-transformed acoustic equation for pressure is

$$(\frac{\omega^2}{\kappa} + \nabla\cdot\frac{1}{\rho}\nabla)\, P = \delta(x-x')\delta(z-z') \tag{1}$$

where ρ is the density, κ is the bulk modulus and P is the pressure. The parameters ρ and κ depend on both x and z, so standard Fourier separation methods may not be used. The right hand side of (1) indicates that the pressure field, P, is driven by a line source. The task now is to construct the Green's function, or impulse response for the pressure field, with the constraining assumption of geometrical optics described previously.

A first attempt might be to use an asymptotic expansion of the form

$$P(x,z,\omega) = \frac{e^{i\omega\tau}}{(\omega\tau)^{1/2}} \sum_{n=0}^{\infty} \frac{A_n(x,z)}{(i\omega)^n} \tag{2}$$

where τ is the arrival time of the disturbance, and $A_n(x,z)$ are a set of amplitude coefficients. The factor $(\omega\tau)^{-1/2}$ in front of the summation indicates that the outgoing wave is cylindrical. A direct substitution of (2) into (1), with the requirement that the coefficients of the powers of ω vanish, yields τ and $A_n(x,z)$. Such a direct substitution leads to two fundamental problems.

Firstly, the delta function term on the right hand side of (1) appears only in the third term of the asymptotic expansion. Secondly, as τ approaches zero, the expansion becomes invalid since $(\omega\tau)^{-1/2}$ becomes infinite.

The first problem is easily solvable by recognizing that it is the Green's function which is being computed. Hence, the homogeneous equation is solved under the appropriate boundary and initial conditions. Then, to compute the strength of the source, both sides of (1) are integrated over a disk surrounding the source. Gauss' theorem is applied in two dimensions, and the strength of the source is evaluated.

To solve the second problem, a new expansion is introduced. Instead of $\frac{e^{i\omega\tau}}{(\omega\tau)^{1/2}}$, the factor in front of the summation in (2) is chosen to be the Hankel function, which represents travelling cylindrical waves. Therefore,

$$P(x,z,\omega) = H_0^{(\frac{1}{2})}(\omega\Theta) \sum_{n=0}^{\infty} \frac{A_n(x,z)}{(i\omega)^n} \tag{3}$$

where Θ, like τ, is a phase function to be determined. The kind of Hankel function chosen in (3) depends on the sign of the inverse Fourier transform over frequency. In this paper, the inverse Fourier transform kernel will be $e^{i\omega t}$. Thus, the Hankel function of the second kind will be chosen, since that choice represents waves travelling away from the source at (x',z').

Substitution of the expansion (3) into the two-dimensional homogeneous acoustic equation results in a complete hierarchy of differential equations. Each equation is obtained by setting the coefficient of ω to zero. The result of these substitutions is given by the following equation:

$$\frac{1}{\rho}\nabla^2\left[H_o^{(2)}(\omega\Theta)\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\right]+\nabla\frac{1}{\rho}\cdot\nabla\left[H_o^{(2)}(\omega\Theta)\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\right]$$

$$+\frac{\omega^2}{\kappa}H_o^{(2)}(\omega\Theta)\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}=0 \tag{4}$$

Use of the expressions

$$\nabla\left(H_o^{(2)}(\omega\Theta)\right)=\omega\,\frac{dH_o^{(2)}(\omega\Theta)}{d\alpha}\,\nabla\Theta$$

and

$$\frac{d^2H_o^{(2)}(\omega\Theta)}{d\alpha^2}=-H_o^{(2)}(\omega\Theta)-\frac{1}{\alpha}\frac{dH_o^{(2)}(\omega\Theta)}{d\alpha}$$

with $\alpha=\omega\Theta$, $\Theta_x=p=\frac{\partial\Theta}{\partial x}$ and $\Theta_z=q=\frac{\partial\Theta}{\partial z}$ results in

$$\omega^2H_o^{(2)}(\omega\Theta)\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\left[-\frac{(\nabla\Theta)^2}{\rho}+\frac{1}{\kappa}\right]$$

$$+\omega\frac{dH_o^{(2)}(\omega\Theta)}{d\alpha}\left[\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\left[\nabla\frac{1}{\rho}\cdot\nabla\Theta+\frac{\nabla^2\Theta}{\rho}-\frac{(\nabla\Theta)^2}{\rho\Theta}\right]+\frac{2}{\rho}\nabla\Theta\cdot\nabla\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\right]$$

$$+H_o^{(2)}(\omega\Theta)\left[\nabla\frac{1}{\rho}\cdot\nabla\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}+\frac{1}{\rho}\nabla^2\sum_{n=o}^{\infty}\frac{A_n(x,z)}{(i\omega)^n}\right]=0 \tag{5}$$

In equation (5) the coefficients of ω must vanish separately. The coefficients of ω^2 and ω^1 which vanish yield the differential equations for the phase Θ and the leading amplitude term, A_o. By setting the coefficient of ω^2 to zero, the eikonal equation is obtained:

$$(\nabla\Theta)^2=\frac{\rho}{\kappa} \tag{6}$$

The right hand side of (6) is the slowness squared (v^{-2}), and the components of $\nabla\Theta$ are proportional to the components of the wave front normal. Equation (6) can be viewed as the frequency-normalized dispersion relation associated with (1), when the medium is homogeneous. Similarly, corresponding to the coefficient of ω^1, the following equation is derived:

$$\left[\frac{1}{\rho}\nabla^2\Theta - \frac{(\nabla\Theta)^2}{\rho\Theta} + \nabla\frac{1}{\rho}\cdot\nabla\Theta\right]A_o + \frac{2}{\rho}\nabla\Theta\cdot\nabla A_o = 0 \tag{7}$$

or

$$\left[\frac{1}{\rho}\nabla^2\Theta - \frac{1}{\kappa\Theta} + \nabla\frac{1}{\rho}\cdot\nabla\Theta\right]A_o + \frac{2}{\rho}\nabla\Theta\cdot\nabla A_o = 0$$

Solution of the Transport Equation

Whereas Θ , the phase function appearing in (3), can be determined by solving the following four ray equations,

$$\frac{dx}{d\Theta} = pv^2 \qquad \frac{dz}{d\Theta} = qv^2$$

$$\frac{dp}{d\Theta} = -\frac{\partial \ln v}{\partial x} \qquad \frac{dq}{d\Theta} = -\frac{\partial \ln v}{\partial z},$$

the transport equation for A_o is a more difficult one to solve. The equation for A_o can be solved along a ray by converting it to an ordinary differential equation for A_o. This can be accomplished by using the fact that, from the eikonal equation,

$$\nabla\Theta = \frac{\hat{s}}{v}$$

where $\hat{s}$ is a unit tangent vector along a ray. Substitution of $\nabla\Theta$ into (7) results in

$$\frac{2}{\rho v}\frac{dA_o}{ds} = -\left[\frac{1}{\rho}\nabla^2\Theta - \frac{1}{\kappa\Theta} + \nabla\frac{1}{\rho}\cdot\nabla\Theta\right]A_o \tag{8}$$

or

$$A_o = C\,\tilde{A}_o = Ce^{-\frac{1}{2}\int_{s_1}^{s_2} v\nabla^2\Theta - \frac{1}{v\Theta} + \rho v\nabla\frac{1}{\rho}\cdot\nabla\Theta \; ds} \tag{8a}$$

with s chosen as the arc length along a ray and C a constant to be determined. It is worthwhile examining (8a), in the neighborhood of the source, under the assumption that ρ and κ are essentially constant.

With these assumptions in view, s simply becomes r, the distance from the source to the observation point. Also, all gradients of ρ vanish. The phase Θ is simply $\frac{r}{v}$ and $\nabla^2\Theta = \nabla\cdot\left[\frac{\hat{r}}{v}\right] = \frac{1}{rv}$. Therefore,

$$\tilde{A}_o = e^{-\frac{1}{2}\int_{r_1}^{r_2} \frac{1}{r} - \frac{1}{r}\, dr} \tag{9}$$

With r_1 and r_2 approaching zero, $\tilde{A}_o$ is identically 1 , and A_o becomes C. It is important to note that if $H_o^{(2)}(\omega\Theta)$ was not used in the original expansion, then A_o would have been given by $C \left[\frac{r_1}{r_2}\right]^{1/2}$. Thus, with r_1 and r_2 approaching zero, the value for A_o in the neighborhood of the source, would have become indeterminate.

Now all that is left to do is to find C in (8a). This is done by writing the pressure field in the neighborhood of the source as

$$P(x,z,\omega) = CH_o^{(2)}(\omega\Theta)$$

where $\Theta = \frac{r}{v}$. Integration of (1) around the source, and use of the fact that P and not ∇P is continuous in the source neighborhood results in

$$\lim_{r\to 0} \int_0^{2\pi}\int_0^r \frac{\nabla^2 P}{\rho}\, r\, dr\, d\Theta = \lim_{r\to 0} \int_0^{2\pi}\int_0^r \frac{\delta(r)}{2\pi r}\, r\, dr\, d\Theta = 1, \tag{10}$$

where Θ is the polar angle, and $\frac{\delta(r)}{2\pi r}$ is the circularly symmetric two-dimensional delta function. Equation (10) may be further simplified by use of Gauss' theorem and replacing P by $CH_o^{(2)}(\omega\frac{r}{v})$.

The application of Gauss' theorem to the integral on the left hand side of (10) results in

$$\lim_{r\to 0} \int_0^{2\pi}\int_0^r \frac{\nabla^2 P}{\rho}\, r\, dr\, d\Theta = \lim_{r\to 0} \frac{1}{\rho(x',z')} \int_0^{2\pi} r\, \frac{\partial P}{\partial r}\, d\Theta = 1 \tag{11}$$

where $\rho(x',z')$ is the value of the density at the source point. Now as r approaches zero, $H_o^{(2)}(\omega\frac{r}{v})$ becomes $-\frac{2i}{\pi}\ln(\omega\frac{r}{v})$. Thus, $\frac{\partial P}{\partial r}$ becomes $-C\,\frac{2i}{\pi r}$. Substitution of this asymptotic result in (11) results in

$$C = -\frac{\rho(x',z')}{4i} \tag{12}$$

Therefore, with C determined, the leading term of the the Green's function for the two-dimensional acoustic equation is given by

$$P = -\frac{\rho(x',z')}{4i} H_0^{(2)}(\omega\Theta)\,\tilde{A}_0(x,z) \tag{13}$$

where Θ satisfies (6), and $\tilde{A}_0(x,z)$ satisfies (7). Implicit in (13) is the dependence of P on source coordinates (x',z') and observation coordinates, as is the case in the usual description of the Green's function.

Conclusions

A combination of the method of asymptotic expansions with the standard method of computing the Green's function has been used for the case of the two-dimensional acoustic equation. The method is valid for body wave arrivals, and is not valid when a caustic is encountered. Such will not be the case if the material parameters ρ and κ vary smoothly enough over a wavelength. The expansion is uniform in the sense that the singularity at the source is correctly accounted for.

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