# AN APPROACH TO THE INVERSE SEISMIC PROBLEM

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**Abstract**

A formal method for inverting multi-dimensional seismic data is developed, borrowing freely from the concepts of quantum scattering theory. It is shown that the measured quantity in the seismic experiment is essentially the on-shell $T$ matrix of that theory. An expansion of the $T$ matrix in powers of inverse source amplitude yields successive approximations to the seismic parameters. The first approximation is essentially a migration. Each successive approximation requires calculation of a forward problem and a migration. The forward calculation involves volume integrals and appears in general to be computationally prohibitive. However, under circumstances where higher order terms are due mainly to surface multiples, the computational magnitude drops substantially.

**Introduction**

Since the 1977 paper by Cohen and Bleistein, linearized inversion schemes for seismic data have become fairly common. (See, for example, Bleistein and Cohen (1979); Cohen and Bleistein (1979); Raz (1980); Clayton and Stolt (1980)). These schemes ignore multiple reflections and solve only for small perturbations of earth parameters from an initial estimate. Also in the literature are Gelfand-Levitan inversions - starting with Ware and Aki (1969); Berryman and Green (1980); Jacobs and Stolt (1980) - which, if they work at all, require a layered medium and a perfectly deconvolved source.

We outline below a theory of seismic inversion based on ideas from quantum scattering theory. In common with the Gelfand-Levitan approach, multiple reflections are accounted for and a close initial estimate of all earth parameters is not required.

In common with the linearized inversion methods, spatial variations of earth parameters in all directions are allowed, and the source amplitude and waveform need not be perfectly known. Much of the following was presented in SEP 24.

A full inversion with this method appears computationally formidable, probably prohibitive. Used as a surface multiple remover it appears relatively attractive.

A simple 2-D acoustic version of the method is presented in some detail. However, the emphasis of this paper is on theory rather than the presentation of a practical algorithm, which, in fact, is yet to be accomplished.

**Theory**

The physical quantity to be measured we will call $\varphi$. If $\varphi$ is a scalar quantity (e.g. pressure) it will be a function of all spatial coordinates and time or frequency. If it is a vector quantity (e.g. displacement) it will also be a function of a discrete 3-valued parameter which indicates direction. In what follows the collection of spatial coordinates plus (if applicable) the discrete direction parameter will loosely be referred to as the vector $\vec{x}$. Since the wave equations dealt with here will be invariant under time translations, frequency can be treated as a constant parameter. In the frequency domain, for fixed frequency $\omega$, the collection of conceivable $\varphi(\omega,\vec{x})$ reside in a complex vector space on which the wave equation may be defined as a linear (differential) operator equation:

$$\hat{L}(\omega)\varphi(\omega) = 0 \tag{1}$$

In this equation a $\hat{}$ has been placed over the $L$ to indicate that it is an operator rather than a vector or scalar quantity.[1] By writing $\varphi(\omega)$ without its argument $\vec{x}$ we indicate the vector whose component or element at $\vec{x}$ is $\varphi(\omega,\vec{x})$.

The seismic experiment will involve generation of an impulse response which may be considered to be a Green's operator $\hat{G}(\omega)$ satisfying

$$\hat{L}(\omega)\hat{G}(\omega) = -\hat{1} \tag{2}$$

In this equation $\hat{1}$ is the unit operator ($\hat{1}\varphi = \varphi$). Since $\hat{G}$ is an operator rather than a vector, its elements depend on two sets of coordinates. We write $\hat{G}(\omega) \sim G(\omega;\vec{x}_g\,|\,\vec{x}_s) \equiv \langle\vec{x}_g\,|\,\hat{G}(\omega)\,|\,\vec{x}_s\rangle$, the last representation being in so-called Dirac

[1] In what follows we will assume the reader to be at least marginally familiar with linear operator theory and with the use of Dirac notation in the integral representations of operators. If you aren't, but still want to read this paper, Taylor, 1972, Chapter 1, gives an excellent introduction to the subject.

notation. The coordinates on the left refer to an observation point, those on the right a source location. For a fixed source point $\vec{x}_s$, $G$ is a function of $\vec{x}_g$ and may be treated as a vector much as, in a finite dimensional space, the individual columns of a matrix may be considered vectors.

The wave operator $\hat{L}$ is a functional of a number of earth parameters $a_m(\vec{x})$, $m = 1,2,...$ such as density, bulk modulus, and so on. In the inverse problem these parameters (or at least their rapid variations) are initially unknown. We attempt to determine them from a partial measurement of the impulse response $\hat{G}$. We assume that the known quantity is

$$\hat{D}(\omega) = s(\omega)\hat{\Lambda}_g \hat{G}(\omega)\hat{\Lambda}_s \tag{3}$$

where, in the simplest case, $s(\omega)$ is the source amplitude, and $\hat{\Lambda}_g$ and $\hat{\Lambda}_s$ are projection operators onto the surfaces (e.g. the plane at $z = 0$) where the geophones and sources are located, respectively. In actual practice the source and geophone surfaces are finite and source and geophone arrays are employed which put some spatial averaging filters into $\hat{\Lambda}_g$ and $\hat{\Lambda}_s$. Moreover, the geophone surface is a function of source location, which actually implies a slightly more general form than (3). However, in what follows we will use (3) as a convenient form, with the understanding that something slightly different may be used in practice.

One inversion scheme will require the user to choose a base or background wave operator $\hat{L}_0$, hopefully close enough to $\hat{L}$ to allow convergence, but not necessarily a close copy. Indeed, a close copy may not even be possible, because first of all we don't initially know $\hat{L}_0$ very well, and secondly we will have to be able (and willing) to solve the forward and imaging problems for $\hat{L}_0$ (about which more anon). $\hat{L}_0$ may be, but isn't necessarily, the explicitly invertible constant parameter wave operator.

We define a Green's operator $\hat{G}_0$ for $\hat{L}_0$, which satisfies impulse response boundary conditions and the equation

$$\hat{L}_0(\omega)\hat{G}_0(\omega) = -\hat{1} \tag{4}$$

$\hat{L}_0$ must be such that this equation can be inverted, one way or another, to find $\hat{G}_0$. If this can be done, the forward problem is considered solvable.

The difference between the actual and the background wave operators we define as a "potential" $\hat{V}$:

$$\hat{V}(\omega) = \hat{L}(\omega) - \hat{L}_0(\omega) \tag{5}$$

$\hat{V}$ is in general a frequency dependent differential operator. Concealed within it are

the parameters we are trying to find, however, and these parameters $a_m(\vec{x})$ are frequency independent functions of $\vec{x}$ only.

$\hat{G}$ and $\hat{G}_0$ are related by the Lippmann-Schwinger equation.[2]

$$\hat{G} - \hat{G}_0 = \hat{G}\hat{V}\hat{G}_0 = \hat{G}_0\hat{V}\hat{G} \tag{6}$$

All quantities are actually functions of the parameter $\omega$, which has been suppressed in this equation. It should be appreciated that equation (5) is an operator equation, which becomes, if we look at individual elements, an integral equation:

$$G(\omega;\vec{x}_g\,|\,\vec{x}_s) = G_0(\omega;\vec{x}_g\,|\,\vec{x}_s) + \int d\vec{x}\; G(\omega;\vec{x}_g\,|\,\vec{x})\; V(\omega,\vec{x})\; G_0(\omega;\vec{x}\,|\,\vec{x}_s) \tag{7}$$

If $\hat{V}$ were a general operator, two sets of integrals over $\vec{x}$ and $\vec{x}'$ would be required. As it is, we have to remember that $V$ in equation (7) is not a simple function of $\vec{x}$, but rather is a differential operator.

We wish to use the Lippmann-Schwinger equation (6) to find $V$, given the known data field $\hat{D}$ and the background impulse response $\hat{G}_0$. In the linearized inversion schemes now extant, equation (6) is replaced by the Born Approximation

$$\hat{G} - \hat{G}_0 \approx \hat{G}_0\hat{V}\hat{G}_0 \tag{8}$$

Defining a modified data field to be $\hat{D}$ less the surface projection of the known $G_0$:

$$\hat{D}_0(\omega) \approx s(\omega)\hat{\Lambda}_g\hat{G}(\omega)\hat{\Lambda}_s - s(\omega)\hat{\Lambda}_g\hat{G}_0(\omega)\hat{\Lambda}_s \tag{9}$$

we get the equation

$$\hat{D}_0(\omega) \approx s(\omega)\hat{\Lambda}_g\hat{G}_0\hat{V}\hat{G}_0\hat{\Lambda}_s \tag{10}$$

which must be inverted to find $\hat{V}$. Actually, this is impossible to do using data at only one frequency. However, it can be done by using data at all frequencies and by making use of the fact that the unknown parameters $a_m$ are frequency independent and local. The actual algorithm amounts to little more than "migration". The details of this having been discussed in other papers (e.g. Clayton and Stolt, 1980), we will only say here that $\hat{L}_0$ must be chosen so that $\hat{V}$ is determinable from (10).

Our present goal is to improve upon the Born estimate of $\hat{V}$. There are lots of ways to do this, and it is not too clear at this point which is the best. Perhaps the most straightforward is to iteratively improve the estimate of $\hat{V}$ within the confines of the Born approximation by updating $\hat{L}$. If $\hat{L}_n$ is the $nth$ estimate of $\hat{L}$, define a potential $\hat{V}_n$

[2] See Taylor, 1972, p. 133; Clayton and Stolt, 1980, or better still, derive it yourself.

and a Green's function $\hat{G}_n$ as

$$\hat{V}_n = \hat{L} - \hat{L}_n \tag{11}$$

$$\hat{L}_n \hat{G}_n = -\hat{1} \tag{12}$$

The modified data field becomes

$$\hat{D}_n = s\hat{\Lambda}_g(\hat{G} - \hat{G}_n)\hat{\Lambda}_s \tag{13}$$

and the equation to solve for $\hat{V}_n$ is

$$\hat{D}_n = s\hat{\Lambda}_g \hat{G}_n \hat{V}_n \hat{G}_n \hat{\Lambda}_s \tag{14}$$

This may in fact be the best way to do it. The problems with this approach are first, the forward problem for $\hat{G}_n$ must be solved, and second, the "migration" problem of obtaining $\hat{V}_n$ from $\hat{D}_n$ must be done, both at each iteration. As $\hat{L}_n$ takes on the high-frequency parameter variations of $\hat{L}$, $\hat{G}_n$ becomes a complicated function containing not just a direct arrival but multiple reflections as well. After the first iteration $\hat{G}_n$ is likely to be difficult to generate, and equation (14) rather difficult to invert.

There are alternative approaches which allow the parameters in the base wave operator to remain slowly varying and the corresponding Green's function to be multiple free. We will outline here a conceptually simple approach based on the $T$ matrix of quantum scattering theory. (See Taylor, p. 134-141)

The motivation is to make the Lippmann-Schwinger equation (6) look like its Born Approximation (8). To this end, we define an operator $\hat{T}(\omega)$ as

$$\hat{T} = \hat{V} + \hat{V}\hat{G}\hat{V} \tag{15}$$

If $\hat{V}$ is "small", $\hat{T}$ is, to first approximation $\hat{V}$.

By post-multiplying $\hat{T}$ by $\hat{G}_0$, we get

$$\hat{T}\hat{G}_0 = \hat{V}(\hat{G}_0 + \hat{G}\hat{V}\hat{G}_0) = \hat{V}\hat{G} \tag{16.a}$$

and similarly

$$\hat{G}_0\hat{T} = \hat{G}\hat{V} \tag{16.b}$$

By replacing $\hat{G}\hat{V}$ by $\hat{G}_0\hat{T}$ in the Lippmann-Schwinger equation (6), we get the Born Approximation like form

$$\hat{G} - \hat{G}_0 = \hat{G}_0\hat{T}\hat{G}_0 \tag{17}$$

The approximate equation (10) relating the data to the potential generalizes to the exact equation

$$\hat{D}_0(\omega) = s(\omega)\hat{\Lambda}_g \hat{G}_0\hat{T}\hat{G}_0\hat{\Lambda}_s \tag{18}$$

which now must be solved for $\hat{T}$. Given $\hat{T}$, the defining equation (15) must be solved for $\hat{V}$. Equation (15) looks nonlinear in $\hat{V}$, but the substitution $\hat{V}\hat{G} = \hat{T}\hat{G}_0$ yields

$$\hat{T} = (\hat{1} + \hat{T}\hat{G}_0)\hat{V} \tag{19}$$

This equation, sometimes called the Lippmann-Schwinger equation for $\hat{T}$ (see Taylor, 1972, p. 135), can in principle be inverted to obtain $\hat{V}$ given $\hat{T}$.

Both (18) and (19) pose some practical difficulties, however. What makes equation (10) reasonably tractable is the quasi-local[3] character of $\hat{V}$. $\hat{T}$, though it has no more degrees of freedom than $\hat{V}$, is structurally more complex and solving (18) would appear to be no easy matter. Equation (19) looks like a real mess, too.

A way out of these difficulties is to try a successive approximation scheme in which (18)[4] is used only to estimate $\hat{V}$ and (19) to estimate $\hat{T}$. A simple way to do this is to expand $\hat{T}$ and $\hat{V}$ in powers of $s(\omega)^{-1}$:

$$\hat{V} = \sum_{m=1}^{\infty} s^{-m}\,\hat{V}_m \quad ; \quad \hat{T} = \sum_{m=1}^{\infty} s^{-m}\,\hat{T}_m \tag{20}$$

Then, equating coefficients of powers of $s^{-1}$ in (19) gives

$$\hat{T}_1 = \hat{V}_1 \tag{21.a}$$

$$\hat{T}_2 = \hat{V}_2 + \hat{T}_1\hat{G}_0\hat{V}_1 = \hat{V}_2 + \hat{V}_1\hat{G}_0\hat{V}_1 \tag{21.b}$$

$$\hat{T}_3 = \hat{V}_3 + \hat{T}_1\hat{G}_0\hat{V}_2 + \hat{T}_2\hat{G}_0\hat{V}_1$$

$$\hat{V}_3 + \hat{V}_1\hat{G}_0\hat{V}_2 + \hat{V}_2\hat{G}_0\hat{V}_1 + \hat{V}_1\hat{G}_0\hat{V}_1\hat{G}_0\hat{V}_1 \tag{21.c}$$

etc., and in general

$$\hat{T}_m = \hat{V}_m + \sum_{m'=1}^{m-1} \hat{T}_{m'}\hat{G}_0\hat{V}_{m-m'} \tag{21.d}$$

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[3] A "local" or "diagonal" operator is one which merely multiplies functions of $\vec{x}$ by another function of $\vec{x}$. A differential operator, though strictly speaking not local, is essentially so in that it does not require two sets of coordinates to represent it.

[4] You might ask how (18) can be used to estimate $\hat{V}$ when $\hat{V}$ does not appear in (18). The answer is that the relation (19) between $\hat{T}$ and $\hat{V}$ must come into play here too, as will shortly be seen (we hope).

Equations (21) allow computation of $\hat{T}_m$ given $\hat{V}_m$ and previous terms in $\hat{T}$ and $\hat{V}$.

Substituting $(\hat{1} + \hat{T}\hat{G}_0)\hat{V}$ for $\hat{T}$ in (18) and equating powers of $s$, we get

$$\hat{\Lambda}_g \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{\Lambda}_s = \hat{D}_0 \tag{22.a}$$

$$\hat{\Lambda}_g \hat{G}_0 \hat{V}_2 \hat{G}_0 \hat{\Lambda}_s = -\hat{\Lambda}_g \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{\Lambda}_s \tag{22.b}$$

$$\hat{\Lambda}_g \hat{G}_0 \hat{V}_3 \hat{G}_0 \hat{\Lambda}_s = -\hat{\Lambda}_g \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_2 \hat{G}_0 \hat{\Lambda}_s - \hat{\Lambda}_g \hat{G}_0 \hat{V}_2 \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{\Lambda}_s - \hat{\Lambda}_g \hat{G}_0 \hat{V}_1 \hat{G}_0 G1 \hat{G}_0 G1 \hat{G}_0 \hat{\Lambda}_s \tag{22.c}$$

$$\hat{\Lambda}_g \hat{G}_0 \hat{V}_m \hat{G}_0 \hat{\Lambda}_s = -\sum_{m'<m} \hat{\Lambda}_g \hat{G}_0 \hat{T}_{m'} \hat{G}_0 \hat{V}_{m-m'} \hat{G}_0 \hat{\Lambda}_s \tag{22.d}$$

By defining

$$\hat{D}_m(\omega) \equiv -\sum_{m'=1}^{m} \hat{\Lambda}_g \hat{G}_0 \hat{T}_{m'} \hat{G}_0 V_{m-m'+1} \hat{G}_0 \hat{\Lambda}_s \tag{23}$$

all of (22) can be put in the form

$$\hat{\Lambda}_g \hat{G}_0 \hat{V}_{m+1} \hat{G}_0 \hat{\Lambda}_s = \hat{D}_m \qquad , m=0,1,2,\ldots \tag{24}$$

Since the structure of $\hat{V}_m$ is the same for all $m$, the nature of the inversion for $\hat{V}_{m+1}$ in (24) does not change. If we can find $\hat{V}_1$ given $\hat{D}_0$, we can find $\hat{V}_{m+1}$ given $\hat{D}_m$. But can we find $\hat{D}_m$? From its definition, it is clear that finding $\hat{D}_m$ amounts to collecting a sum of terms of the form

$$-\hat{\Lambda}_g \hat{G}_0 \hat{V}_{n1} \hat{G}_0 \hat{V}_{n2} \cdots \hat{V}_{nm} \hat{G}_0 \hat{\Lambda}_s$$

It should also be clear that since $\hat{G}_0$ is calculable, these things are too, but as $m$ increases things are going to get very tedious. However, given a long, tranquil life span and a large computer, $\hat{D}_m$ is in principle calculable.

There is no guarantee at this point that the power series expansions (20) for $\hat{T}$ and $\hat{V}$ converge. However, assuming they do, and that at some level ($m = M$, say) we are satisfied with the result, this scheme has a very nice feature. At no point[5] in the computation did we need to know $s(\omega)$. It enters only when we sum the series of $\hat{V}_m s^{-m}$ to obtain $\hat{V}$. This is quite handy, since we probably didn't know $s(\omega)$ (very well, anyway) at the outset. Given the $\hat{V}_m$ sequence, we are now at liberty to "deconvolve" the data, picking $s(\omega)$ to yield the "best" $\hat{V}$.

[5] Well, we did need it to subtract $s\hat{\Lambda}_g \hat{G}_0 \hat{\Lambda}_s$ from $\hat{D}$. However, that term, for a one-way Green's function, is a direct arrival which would be filtered out anyway.

It is not likely that a full inversion by the $T$-matrix method will prove feasible. In $\hat{D}_{m'}$, each $\hat{V}$ acts as an effective source distributed over the volume of the earth, and each $\hat{G}_0 \hat{V}$ requires the equivalent of a volume integral to evaluate it. By contemporary standards, even in 2-D, the amount of computation required is prohibitive. Not only that, but many terms in the series will likely be required if travel times need to be adjusted, since in this case a polynomial is being used to approximate a phase.

The economics seem much more favorable, however, when we look at the problem of surface multiple removal. For an earth with a reflecting surface, the $\hat{D}_m$ do several things. They adjust propagation velocity, propagation and reflection amplitudes, describe intra-bed multiples, and describe surface multiples. Unlike the other phenomena, the surface multiples may be described by an effective source distributed on the surface. Hence, each successive surface multiple in $\hat{D}_m$ requires only a surface integral, which is more like it. Moreover, the series should converge relatively rapidly.

**The 2-D Constant Velocity Background**

To illustrate the ideas developed above, we look at the simple example of a 2-D acoustic earth whose (flat) upper boundary is a free surface located at $z = 0$. The constant parameter acoustic wave operator will be chosen for $\hat{L}_0$:

$$\hat{L}_0 \sim \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_0^2} \tag{25}$$

The exploding Green's function $\hat{G}_0$ associated with this operator satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_0^2}\right] G_0(\omega, x, z, z') = -\delta(x)\delta(z-z') \tag{26}$$

It is sometimes more convenient to express this equation in the $k_x$ representation by Fourier transforming over $x$. Then[6]

$$\left[\frac{d^2}{dz^2} + q^2\right] G_0(\omega, k_x, z, z') = -\frac{\delta(z-z')}{(2\pi)^{1/2}} \tag{27}$$

with

$$q^2 = \frac{\omega^2}{v^2} - k_x^2 \tag{28}$$

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[6] The convention for Fourier transforms adopted here is $F(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx F(x) e^{-ikx}$

which has the simple solution

$$G_0(\omega,k_x,z,z') = \frac{e^{+iq|z-z'|} - e^{-iq|z-z'|}}{-(2\pi)^{1/2}2iq} \tag{29}$$

The second exponent in $G_0$ would not appear in free space. Here it provides the reflections from the free surface at $z = 0$.

To keep things as simple as possible, the real wave operator $\hat{L}$ will be taken to be the constant density acoustic operator

$$\hat{L}_0 \sim \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2(x,z)} \tag{30}$$

The corresponding potential

$$\hat{V} = \hat{L} - \hat{L}_0 \sim \frac{\omega^2}{v_0^2}\left[\frac{v_0^2}{v^2} - 1\right] = \frac{\omega^2}{v_0^2}\,\alpha(x,z) \tag{31}$$

is frequency dependent but local, with a single local parameter $\alpha(x,z)$ embedded therein.

The data we will assume to have been generated by sources at depth $\varepsilon_s$ and geophones at depth $\varepsilon_g$, so that the relevant elements of $\hat{D}_0$ have the form

$$< x_g,\varepsilon_g\,|\,\hat{D}_0(\omega)\,|\,x_s,\varepsilon_s > = s(\omega) < x_g,\varepsilon_g\,|\,\hat{G}_0\,\hat{T}\hat{G}_0\,|\,x_s,\varepsilon_s > \tag{32}$$

If we Fourier transform over $x_g$ and $x_s$, we get[7] (provided $T = 0$ at depths less than $\varepsilon_g$ or $\varepsilon_s$)

$$< k_g,\varepsilon_g\,|\,\hat{D}_0(\omega)\,|\,k_s,\varepsilon_s > = 2\pi s(\omega)\frac{\sin q_g\varepsilon_g}{q_g}\,\frac{\sin q_s\varepsilon_s}{q_s} < k_g,-q_g\,|\,\hat{T}\,|\,k_s,q_s > \tag{33}$$

where $q_g$ and $q_s$ are vertical spatial frequencies, constrained to satisfy

$$q_g = \frac{\omega}{v_0}\left[1 - \frac{k_g^2 v_0^2}{\omega^2}\right]^{1/2}, \quad q_s = \frac{\omega}{v_0}\left[1 - \frac{k_s^2 v_0^2}{\omega^2}\right]^{1/2} \tag{34}$$

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[7] We use the physicists convention in which Fourier transforms of the parameters on the right hand side of an operator are taken with opposite sign (in the exponent) to those on the left; i.e.

$$<k_g,\varepsilon_g\,|\,\hat{D}_0\,|\,k_g,\varepsilon_s> = \frac{1}{2\pi}\int dx_g \int dx_x\, <x_g,\varepsilon_g\,|\,\hat{D}_0\,|\,x_g,\varepsilon_s>\, e^{i(k_s x_s - k_g \varepsilon_g)}$$

We also use the rather sloppy convention of replacing a coordinate by its spatial frequency in an expression, leaving the functional form intact, to indicate a Fourier transform. Thus the expression $< k_g,-q_g\,|\,\hat{T}\,|\,k_s,q_s >$ on the R.H.S. of (33) is a quadruple Fourier transform of $< x_g,z_g\,|\,\hat{T}\,|\,x_s,z_s >$. Equation (33) can almost be written down by inspection - however, a short derivation is included in an appendix.

The quantity in brackets in (33) is the "on-shell" $T$-matrix, so called because the only elements which appear are those such that

$$k_g^2 + q_g^2 = k_s^2 + q_s^2 = \frac{\omega^2}{v_0^2} \tag{35}$$

By expanding $\hat{T}$ and $\hat{V}$ as a power series in $s^{-1}$ according to equation (20), we get the following series of equations

$$\langle k_g, -q_g | \hat{V}_1 | k_s, q_s \rangle = \frac{1}{2\pi} \frac{q_g}{\sin q_g \varepsilon_g} \frac{q_s}{\sin q_s \varepsilon_s} \langle k_g, \varepsilon_g | \hat{D}_0(\omega) | k_s, \varepsilon_s \rangle \tag{36.a}$$

$$\langle k_g, \varepsilon_g | \hat{D}_1(\omega) | k_s, \varepsilon_s \rangle = -\langle k_g, \varepsilon_g | \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_1 \hat{G}_0 | k_s, \varepsilon_s \rangle$$

$$= \frac{-2\pi \sin q_g \varepsilon_g \sin q_s \varepsilon_s}{q_g q_s} \langle k_g, -q_s | \hat{V}_1 \hat{G}_0 \hat{V}_1 | k_s, q_s \rangle \tag{36.b}$$

$$\langle k_g, -q_g | \hat{V}_2 | k_s, q_s \rangle = \frac{1}{2\pi} \frac{q_g}{\sin q_g \varepsilon_g} \frac{q_s}{\sin q_s \varepsilon_s} \langle k_g, \varepsilon_g | \hat{D}_1(\omega) | k_s, \varepsilon_s \rangle$$

$$= -\langle k_g, -q_g | \hat{V}_1 \hat{G}_0 \hat{V}_1 | k_s, q_s \rangle \tag{36.c}$$

$$\langle k_g, \varepsilon_g | \hat{D}_2(\omega) | k_s, \varepsilon_s \rangle = -\langle k_g, \varepsilon_g | \hat{G}_0 \left[ \hat{V}_1 \hat{G}_0 \hat{V}_2 + \hat{V}_2 \hat{G}_0 \hat{V}_1 + \hat{V}_1 \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_1 \right] \hat{G}_0 | k_s, \varepsilon_s \rangle$$

$$= \frac{-2\pi \sin q_g \varepsilon_g \sin q_s \varepsilon_s}{q_g q_s} \langle k_g, -q_g | \left[ \hat{V}_1 \hat{G}_0 \hat{V}_2 + \hat{V}_2 \hat{G}_0 \hat{V}_1 + \hat{V}_1 \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_1 \right] | k_s, q_s \rangle$$

(36.d)

$$\langle k_g, -q_g | \hat{V}_3 | k_s, q_s \rangle = \langle k_g, -q_g | \left[ \hat{V}_1 \hat{G}_0 \hat{V}_2 + \hat{V}_2 \hat{G}_0 \hat{V}_1 + \hat{V}_1 \hat{G}_0 \hat{V}_1 \hat{G}_0 \hat{V}_1 \right] | k_s, q_s \rangle \tag{36.e}$$

etc.

The full calculation of the elements of $\hat{D}_1$, $\hat{D}_2$, etc. in equations (36.b), (36.d), etc. requires an evaluation of terms of the form

$$\langle k_g, -q_g | \hat{V}_{i1} \hat{G}_0 \hat{V}_{i2} \hat{G}_0 \cdots \hat{G}_0 \hat{V}_{in} | k_s, q_s \rangle$$

For every $\hat{G}_0$ in this expression a double integration (over $x$ and $z$ or $k_x$ and $k_z$ ) is required. Even for this simple 2-D example, the amount of computation required is formidable.

Suppose, however, that the only important contributors to this term are the surface multiples. We can then evaluate this expression with a single integral for each $\hat{G}_0$. We have

$$<k_g,-q_g\,|\,\hat{V}_{i1}\hat{G}_0\,\hat{V}_{i2}\cdots\hat{G}_0\,\hat{V}_{in}\,|\,k_s,q_s> = \int dx_1\cdots dx_n\int dz_1\cdots dz_n\,\frac{\omega^{2n}}{v_0^{2n}}\,\frac{e^{-ik_gx_1+iq_gz_1}}{2\pi}\cdot$$

$$\cdot a_{i1}(x_1,z_1)G_0(\omega,x_1-x_2,z_1,z_2)a_{i2}(x_2,z_2)\cdots G_0(\omega,x_{n-1}-x_n,z_{n-1},z_n)\cdot$$

$$\cdot a_{in}(x_n,z_n)\,\frac{e^{ik_sx_n+iq_sz_n}}{2\pi}$$

The $n$ integrals over $x$ can be replaced by $n-1$ integrals over $k_x$ by decomposing each $G_0$ as a Fourier transform

$$G_0(\omega,x_j-x_{j+1},z_j,z_{j+1}) \;=\; \frac{1}{(2\pi)^{1/2}}\int dk_j\; e^{ik_j(x_j-x_{j+1})}G_0(\omega,k_j,z_j,z_{j+1})$$

and noting that the $x$ integrals affect Fourier transforms of each $a_{ij}$:

$$<k_g,-q_g\,|\,\hat{V}_{i1}\hat{G}_0\,\hat{V}_{i2}\hat{G}_0\cdots\hat{G}_0\,\hat{V}_{in}\,|\,k_s,q_s> = \int dk_1\cdots dk_{n-1}\int dz_1\cdots dz_n\,\frac{1}{(2\pi)^{3/2}}\cdot$$

$$\cdot\frac{\omega^2}{v_0^2}\,e^{iq_gz_1}a_{i1}(k_g-k_1,z_1)G_0(\omega,k_1,z_1,z_2)a_{i2}(k_1-k_2,z_2)G_0(\omega,k_2,z_2,z_3)$$

$$\cdots G_0(\omega,k_{n-1},z_{n-1},z_n)a_{in}(k_{n-1}-k_s,z_n)e^{iq_sz_n} \qquad (37)$$

$G_0$ is given by equation (29). However, only the second exponent in (29) involves the free surface reflection. Neglecting the first exponent we get

$$<k_g,-q_g\,|\,\hat{V}_{i1}\hat{G}_0\,\hat{V}_{i2}\hat{G}_0\cdots\hat{G}_0\,\hat{V}_{in}\,|\,k_s,q_s> \approx \frac{\omega^{2n}}{v_0^{2n}}\,\frac{(2\pi)^{-n/2-1}}{(2i)^{n-1}}\int\frac{dk_1}{q_1}\cdots\frac{dk_{n-1}}{q_{n-1}}\int dz_1\cdots dz_n$$

$$\cdot e^{iq_gz_1}a_{i1}(k_g-k_1,z_1)e^{iq_1(z_1+z_2)}a_{i2}(k_1-k_2,z_2)e^{iq_2(z_2+z_3)}\cdots e^{iq_{n-1}(z_{n-1}+z_n)}$$

$$\cdot a_{in}(k_{n-1}-k_s,z_n)e^{iq_sz_n}$$

The $z$ integrals are also just Fourier transforms of the $a$'s, so

$$<k_g,-q_g\,|\,\hat{V}_{i1}\hat{G}_0\,\hat{V}_{i2}\hat{G}_0\cdots\hat{G}_0\,\hat{V}_{in}\,|\,k_s,q_s> = \frac{1}{(2\pi)}\,\frac{\omega^{2n}}{v_0^{2n}}\,\frac{1}{(2i)^{n-1}}\int\frac{dk_1}{q_1}\,a_{i1}(k_g-k_1,-q_g-q_1)$$

$$\cdot\int\frac{dk_2}{q_2}\,a_{i2}(k_1-k_2,-q_1-q_2)\int\frac{dk_3}{q_3}\,a_{i3}(k_2-k_3,-q_2-q_3)\cdots\int\frac{dk_{n-1}}{q_{n-1}}\cdot$$

$$\cdot a_{in-1}(k_{n-2}-k_{n-1},-q_{n-2}-q_n)a_{in}(k_{n-1}-k_s,-q_{n-1}-q_s) \qquad (38)$$

Thus, under these circumstances $\hat{D}_m$ can be evaluated with a single integral per $\hat{G}_0$.

It is hard to imagine a case where this simple constant background velocity model would be directly useful. Perhaps a slight modification could render it applicable to

the removal of deep water bottom multiples. To be generally useful, however, the model should be reformulated with a variable background velocity.

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# *APPENDIX*

**Derivation of the "On Shell" T Matrix Equation**

The right hand side of equation (32) can be expanded as a quadruple integral over $x$ and $z$, yielding

$$<x_g,\varepsilon_g|\hat{D}_0(\omega)|x_s,\varepsilon_s> = s(\omega)\int dx_1dx_2\int dz_1dz_2G_0(\omega,x_g-x_1,\varepsilon_g,z_1)\cdot$$
$$\cdot<x_1,z_1|\hat{T}|x_2,z_2>\,G_0(\omega,x_2-x_s,z_2,\varepsilon_s) \qquad (A.1)$$

The $x$ integrals are in fact convolutions, hence can be eliminated by Fourier transforming over $x_g$ and $x_s$. We get

$$<k_g,\varepsilon_g|\hat{D}_0(\omega)|k_s,\varepsilon_s> = s(\omega)(2\pi)\int dz_1dz_2G_0(\omega,k_g,\varepsilon_g,z_1)\cdot$$
$$\cdot<k_g,z_1|\hat{T}|k_s,z_2>G_0(\omega,k_s,z_2,\varepsilon_s) \qquad (A.2)$$

Another form of equation (29) for $G_0$ is

$$G_0(\omega,k_x,z,z') = \frac{\sin qz_<}{(2\pi)^{1/2}q}\,e^{iqz_>} \qquad (A.3)$$

where $z_>(z_<)$ is the larger (smaller) of $z$ and $z'$. Thus, if $<k_g,z_1|\hat{T}|k_s,z_2> \neq 0$ only when $z_1 > \varepsilon_g$ and $z_2 > \varepsilon_s$ (from the definition of $\hat{T}$ it should be clear that this amounts to requiring that velocity be constant ($a = 0$) above $\varepsilon_g$ *or* $\varepsilon_s$ the equation (A.2) becomes

$$<k_g,\varepsilon_g|\hat{D}_0(\omega)|k_s,\varepsilon_s> = s(\omega)\,\frac{\sin q_g\varepsilon_g}{q_g}\,\frac{\sin q_s\varepsilon_s}{q_s}\int dz_1dz_2\cdot$$
$$\cdot e^{iq_gz_1}<k_g,z_1|\hat{T}|k_s,z_2>\,e^{iq_sz_2} \qquad (A.4)$$

The $z$ integrals are now recognizable as Fourier transforms. With the sign and amplitude conventions in use here, we have, in fact, equation (33).